

Vibration of Strongly Nonlinear Discontinuous Systems

V. I. Babitsky · V. L. Krupenin

Translated by A. Veprik

With 124 Figures

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Preface to English Edition

Investigation of vibration in complex linear mechanical structures is a fundamental area of dynamical analysis. The regular methods, which are based on classical mathematical theory, exploit the general characteristic features of linear structures: modal representation and superposition. These allow to split a system of any complexity into a combination of simple oscillators (modal analysis), and to calculate the response of any of them to arbitrary excitation as the sum of the responses due to the action of the simple harmonic components (Fourier frequency analysis). The application of modal and frequency analysis is a standard practice in modern engineering. This is supported by a wide development of special data acquisition equipment, digital signal processing and simulation software.

The introduction of nonlinear factors into mechanical structures makes such analysis generally inapplicable. This problem has stimulated the development of some special mathematical methods based mainly on approximation of nonlinear structures with equivalent linear substitutes. Such methods make it possible to evaluate perturbations due to a small nonlinear deviation in the characteristics of structures.

However, engineering practice also needs the efficient analysis of mechanical structures with strongly nonlinear interactions causing the generation or transformation of impulsive and impact actions. Such severe processes in mechanical structures, which are known as vibro-impact regimes, can be the reason for many structural failures. On the other hand, vibro-impact processes are the most effective mechanical actions in testing, material processing and demolition. Their analysis and optimisation can significantly improve these technologies.

The investigation of structures with multiple impact interactions needs the development of a mathematical technique for the analysis of systems with discontinuities, where even a vibration of small intensity may lead to the generation of impulsive or impact processes. The typical mechanical examples of such interactions are various contact phenomena in the structures.

A special technique was elaborated for the analysis of vibro-impact phenomena based on the application of the structural characteristics of interacting linear subsystems. This allows obtaining an exact solution of many important problems in the dynamics of vibro-impact systems with the use of traditional structural and frequency characteristics. In addition, a natural route to the efficient approximation of solutions was developed. This also gave an opportunity to use the analytical technique mentioned in combination with preliminary experimental measurements even in the absence of total mathematical descriptions of the structures and without reference to differential equations.

The description and application of these general methods for the analysis of vibration in different vibro-impact structures are the main content of this book. After its publication in Russian, the authors and their colleagues undertook further investigations of some complex strongly nonlinear structures previously unexplored.

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As a result, some new mechanical phenomena were predicted and observed experimentally. This confirmed the effectiveness of the analytical procedures described. During these years there were also further developments in the technique of the methods, their mathematical substantiation and applications.

In order to acquaint the reader with these new results, we supply an English edition of the book with additional appendices 2-4 written together with our colleagues. They are devoted to the description of the new averaging technique for calculation of resonant processes in vibro-impact systems (VSh Burd, VL Krupenin, Appendix 2), review of the resonant vibro-impact phenomena in solids (VI Babitsky, VL Krupenin, Appendix 3), and demonstration of an engineering application of the analytical technique in combination with the modern simulating tools (VI Babitsky, AM Veprik, Appendix 4). The book is also upgraded with a new list of references on the development and application of the methods described and new books on vibration of strongly nonlinear mechanical systems.

We hope that the book will help applied mathematicians and research engineers to analyse strongly nonlinear systems with structural discontinuities and give teachers and advanced students a useful knowledge of strongly nonlinear dynamics. Some readers, who are strongly motivated by the engineering applications of the theory, may find it useful to read Appendices 3 and 4 before familiarisation with the main contents of the book. These appendices are specially prepared for such autonomous and introductory study.

We thank deeply Prof. VSh Burd for the contribution to the English edition, Dr AM Veprik, the translator of the book, for fruitful collaboration and efficient preparation of the English manuscript, Mr M James for final polishing of galley-proof, and the staff of Springer for the accomplishment of this project.

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Loughborough, Moscow.

Foreword

Among the wide diversity of nonlinear mechanical systems, it is possible to distinguish a representative class of the systems which may be characterised by the presence of threshold nonlinear positional forces. Under particular configurations, such systems demonstrate a sudden change in the behaviour of elastic and dissipative forces. Mathematical study of such systems involves an analysis of equations of motion containing large-factored nonlinear terms which are associated with the above threshold nonlinearity. Due to this, we distinguish such discontinuous systems from the much wider class of essentially nonlinear systems, and define them as *strongly nonlinear systems*.

The vibration occurring in strongly nonlinear systems may be characterised by a sudden and abrupt change of the velocity at particular time instants. Such a vibration is said to be non-smooth. The systems most studied from this class are those with relaxation (Van Der Pol, Andronov, Vitt, Khaikin, Teodorovich, etc. [5,65,70,71,98,171,181]), where the non-smooth vibration usually appears due to the presence of large nonconservative nonlinear forces. Equations of motion describing the vibration with relaxation may be written in such a manner that the highest derivative is accompanied by a small parameter. The methods of integration of these equations have been developed by Vasilieva and Butuzov [182], Volosov and Morgunov [190], Dorodnitsin [38], Zhelezisov [201], Mischenko and Kozov [115], Pontragin [137], Tichonov [174,175], etc.

In a system with threshold nonlinearity, the non-smooth vibration occurs due to the action of large conservative forces. This is distinct from a system with relaxation. This monograph addresses the systematic representation of the new methods of analysis recently developed by the authors as applied to the above-mentioned discontinued systems. The presentation is followed by the study of specific features of dynamic processes that are intrinsic for these systems. In particular, special attention is given to an analysis of different resonant phenomena which take unusual and diverse forms. The methods presented are applied widely to the analysis of mechanical systems which are designed for the generation and transformation of intensive processes of an impulsive nature. These are machines for rock fragmentation, impact processing, special types of shock testing machines and many other types of machinery.

Systems containing only small orders of magnitude in their nonlinear terms are said to be "weakly nonlinear" or "quasi-linear". The system is thought of as "essentially nonlinear" if, at least one nonlinear term in the equation of motion does not contain physically small parameter. Finally, the presence of at least one nonlinear term with a large parameter turns the above system into "strongly nonlinear". The use of quantitative characteristics for the parameters of the nonlinear terms for approximate analysis gives an indication of the main features of dynamic behaviour, and allows the influence of additional factors to be ranked.

In a majority of the above systems, the impulsive forces are developed through systematic collisions. Thus they are called *vibro-impact systems*.

In contrast to the machines which are designed to utilise the positive features of vibro-impact processes, parasitic vibro-impact processes may be developed by any mechanism with clearances and structures with discontinuities, thus generating excessive levels of vibration and acoustic pollution. This leads to harmful effects on health. The peak stresses, which are developed at vibro-impact interactions, often exceed the allowed level. This causes fast accumulation of the critical fatigue and, therefore, drastic reduction of the useful life of a machine and structure.

Models of vibro-impact systems along with specific properties are studied in numerous papers and systemised in the monographs by Kobrinsky and Kobrinsky [74,75] and Babitsky [17].

The rigorous method of integration of the equations of motion for vibro-impact systems is based on the *stitching of general solutions* corresponding to the time intervals between successive impacts. This method was successfully applied to the study of vibro-impact processes which take place in particular simplified models. For an analysis of more realistic models, the ideas of equivalent linearisation and frequency methods were applied. An application of the stitching method requires significant simplification of the model of the analysed object. The equivalent linearisation of the impact force leads to the neglect of higher-order harmonics and to a consequential loss of information.

At the same time, there exists a regular method for obtaining the exact solutions for a wide class of piecewise-linear systems, which relies solely on correspondent *particular solutions*. Therefore, it is no longer necessary to know the general solutions within the intervals of linearity. This concept, actively developed by Rosenwasser for the problems of the theory of automatic control [148-150], is very effective tool for the analysis of vibro-impact systems [11,16,17]. By implementing the "spectral" approach, the concept correlates the solutions obtained by the method of equivalent linearisation with exact solutions, and to some extent combines the advantages of both methods.

Consequently, essential progress was achieved in the understanding of the dynamics of basic models. In particular, the new approach became of fundamental importance for systems where the stitching of general solutions is problematic (or even impossible), and for systems where the solutions obtained by the method of equivalent linearisation are insufficiently representative. In addition, it became possible to avoid the use of the differential equations of motion within the framework of the developed approach. For example, an experimentally measured frequency response of the system (such as receptance) may be used to obtain the full and exact description of the vibro-impact process. The presence of the external forcing and/or parametric excitation easily may be taken into account within the above conceptual scheme.

The next step has been made to develop the analysis of vibro-impact processes in the presence of small additional nonlinearities based on the method described [14]. In so doing, the asymptotic methods of averaging in quasi-conservative systems which were

developed by Bogolubov and Mitropolsky [24,117,118], Volosov and Morgunov [190] and Moiseev [119] etc. were effectively applied. As a result of these efforts, the problems of build-up and stability of vibration were solved. It also became possible to analyse self-sustained and random vibration.

The developed approach allowed for the development of a generic scheme for the analysis of discontinuous strongly nonlinear systems. The schematic steps are:

- to "extract" the model of the generative vibro-impact system from a full model of the system in accordance with a special logical rule;
- to obtain the generative solution describing the dynamics of the generative system;
- to construct an asymptotic expansion in the vicinity of the generative solution.

The principal term of such an expansion is nonsmooth, therefore, a generative vibro-impact system which is related to the initial system has to be described in terms of generalised singular functions. Hence, the authors named the above procedure *singularisation* [89].

Some analogy may be established between the methods of linearisation of quasi-linear systems and singularisation of strongly nonlinear systems. In the first case, the quasi-linear system is substituted by its extreme case. This is a linear system with parameters which are obtained in accordance with the weak type of nonlinearity. In the second case, the strongly nonlinear system is substituted using its extreme case which is a vibro-impact system with parameters obtained in accordance with the features of threshold nonlinearity. As a result of the above singularisation, the solution obtained possesses a representative spectrum of harmonics and numerous tiny mechanical effects intrinsic to a strongly nonlinear system may be described even in the first approximation.

The approach developed makes it possible to obtain the corresponding solutions without referring to the differential equations of motion which contain a small parameter associated with the higher derivative, but by using the regular asymptotic expansion. The aspects of the theory of vibration of strongly nonlinear systems considered are not presented systematically in the modern literature; the separate articles can not clarify the problem completely. The present monograph systematises the recently obtained results and bridges the above gap. The general composition of the material is indicated in the contents. Therefore, only brief comments are necessary here. For the reader's convenience the book is arranged so that, along with the sequential reading of the monograph, the separate chapters can be studied (as sketched in Fig. 0.1, where the numbering corresponds to that of the chapters).

Averaging in vibro-impact and similar systems was discussed previously using a different approach by Krylov and Bogolubov [95], Samoilenko [152, 153], Zhuravlev [202-204], etc.

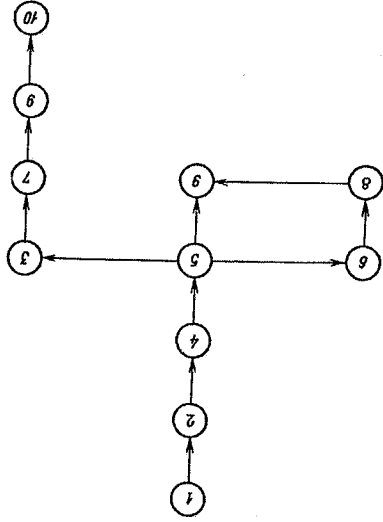


Fig. 0.1

The study of the dynamic behaviour of a vibro-impact single-degree-of-freedom (SDOF) system can sometimes give an insight into the main physical features of more complex strongly nonlinear systems. The reader who is familiar with the theory of periodic Green functions (PGF) may only make a cursory examination of the content of Chap. 1 before further reading. For understanding of Chap. 3 a more thorough knowledge of the above theory is required.

If the reader's fundamental interest is focused on the study of forced vibrations, the reading may be limited to §1,2,4,5,9 (the portion of material concerning the case of random vibration may be omitted). §3,7,10 are recommended for the study of parametric vibration.

The reader with interests in the study of integral equations and other analytical presentations of operators of mechanical systems as applied to the problems which are beyond the scope of the present monograph, may find the material of Chap. 1 to be valuable.

The authors particularly present the application of the method developed to the solution of different problems in the theory of vibration. All the solutions obtained are represented in a clear analytical form which allows for physical interpretation. The effectiveness of these procedures might be considerably increased with the use of computation facilities.

The solutions may also be constructed from the experimentally measured transfer functions of the linear subsystems involved. Therefore, by combining the experimental methods of linear frequency analysis with the results of the theory of strongly nonlinear systems, one can predict the behaviour of real-world machinery (bearing

assemblies, movable joints, bumpered vibro-isolation systems, impact dynamic dampers, etc.).

The application problems studied in this monograph are limited to the fields of vibration and vibro-impact systems, which are the areas of the current professional interests of the authors. For those who wish to read the above subjects more comprehensively, we strongly recommend the monograph of one of the authors [17]. The authors believe that the methods described can be applied to a wide range of nonlinear resonant phenomena.

The present monograph is intended for researchers and engineers involved in the research and design of mechanical vibratory systems. It can also be a good supplement for lecturers and advanced students concerned with the theory of nonlinear vibration. The presentation is aimed at the level of mathematical training of engineering researchers. All additional mathematical information is provided in the text or by direct references. A basic knowledge of the methods of nonlinear vibration analysis is desirable.

The list of references was limited to those used in the text together with some related general works. For this reason, many very important publications in the theory of nonlinear vibration were not included in the bibliography. An attempt to create a full list of references on nonlinear vibration is a separate problem of special interest.

During the preparation of the monograph, numerous discussions with MZ Kolovsky, MA Krasnoseiskiy, BI Morgunov, EN Rozenwasser and VF Zhuravlev were of great importance to the authors.

Significant contributions to the composition and completion of the monograph were made voluntarily by VSh Burd, the author of Appendix 1, using previously unpublished material and by AS Kovaleva (the author of Sect. 8.8 and 8.9 and coauthor of sections 8.5 and 8.7).

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Chapter 1 Operators of Linear Systems

§1 Dynamic Compliance

1.1. Operator of Mechanical System

The dynamic behaviour of a complex mechanical system may be described by either using the regular language of "differential equations" or the language of the "theory of systems".

Let $\mathbf{x} = (x_1, x_2, \dots, x_n)$ denote the vector of generalised coordinates of the system. It is

typically defined by the differential equations of motion, reflecting the inherent structure of the system, external force and some additional (initial, for example) conditions. We say that the mechanical system transforms the "input signal" (external

force) into the "output signal" (deflection, velocity or acceleration).

Consider further that the external forcing may be conceived as the vector

$\mathbf{g} = (g_1, g_2, \dots, g_n)$, which depends, in turn, upon time, generalised coordinates and

velocities.

Using the language of the theory of systems, the above force-to-displacement transform may be expressed symbolically in the form

$$\mathbf{x} = \mathbf{A}_0 \mathbf{g}$$

where $\mathbf{A}_0$ denotes the operator of the mechanical system, reflecting its inherent structure in combination with some additional conditions, and relating the force $\mathbf{g}$ to

the displacement $\mathbf{x}$. The above statement may also be used for systems with

distributed parameters. In these cases, a dynamic response will be a function of time

and spatial variables.

No additional assumptions have been made so far with regard to the specific features

of the functions $\mathbf{x}$ and $\mathbf{g}$. Moreover, the term "inherent structures" has to be specified

more precisely. Depending on the method of analysis, the same force (for example, the damping force $b_1 \dot{x}_1$ or nonlinear potential force $c_1 x_1^3$) may be treated as external or as

internal. Hence, the inherent features of the system may be specified in different ways.

In this chapter, we always use the linear operator $\mathbf{A}_0$ [49]:

$$\mathbf{A}_0(\alpha_1 g_1 + \alpha_2 g_2) = \alpha_1 \mathbf{A}_0(g_1) + \alpha_2 \mathbf{A}_0(g_2)$$

where α_1 and α_2 are constants.

The linear operator of the mechanical system corresponds to linear differential equations of motion reflecting the structure of a system. At the same time, the linear operator of the mechanical system often may be derived using experimental data, i.e.

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without original differential equations. In our representation, different analytical forms of the operator Λ_0 are required. In the interim, a brief introduction into the theory of generalised functions is necessary for effective and convenient understanding of the material. In the theory of automatic control, the description of operators of linear systems is typically based on the dynamic responses to some referenced inputs. The mechanical analogy of such input is impact. The Dirac δ -function may be treated as a suitable input. The impact force may be described using the δ -function.

1.2. Fundamental Features of the Generalised Dirac δ -function

Classical analysis studies functions, i.e. the relationship between members of numerical sets. The fundamental concept of analysis, the definite integral, allows for expansion of the concept of the above relationship. Two functions $f(t), g(t) \in C^0(\mathbf{R}^1)$ are given. Hereafter the following notation is used: $\mathbf{R}^1$ is the set of real numbers and $C^n(\mathbf{R}^1)$ is the set of n -time continuously differentiable at $t \in \mathbf{R}^1$ functions. In particular, C^0 and C^∞ denote the sets of continuous and infinitely differentiable functions. Consider the integral

$$I = \int_{-\infty}^{\infty} f(t)g(t)dt$$

In order to guarantee its existence, assume further that the function $g(t)$ is zero outside some finite time interval $[a, b]$. Such functions are known as *finite functions* and are finitely supported with the *support* $[a, b]$. If the function $g(t)$ is defined, the value I depends upon $f(t)$, i.e. $I = I[f]$. Therefore, the above integral assigns a particular value $I(f)$ to each particular function $f \in C^0(\mathbf{R}^1)$. It is evident that this integral is a function of a function, but not of a point in the numeric scale. The functional $I(f)$ is defined on the set of continuous bounded functions. The linearity of the integral I follows from its definition

$$I(\lambda_1 f_1 + \lambda_2 f_2) = \lambda_1 I(f_1) + \lambda_2 I(f_2)$$

where λ_1 and λ_2 are arbitrary constants.

In order to expand the concept of a functional consider the operator Λ_0 which relates the functions g and x in accordance with the rule

$$x(t) = \Lambda_0 g(t)$$

Consider here the sequence of continuous functions $\{f_n(t)\}$ which can not have an ordinary limit. The above sequence will converge and a weak limit of this sequence will exist if the limit of the following numerical sequence exists

$$\lim_{n \rightarrow \infty} I = \lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} f_n(t)g_n(t)dt = I_0 < \infty, \quad g \in C^0(\mathbf{R}^1)$$

The interval $[t_0 - \varepsilon_{1n}, t_0 + \varepsilon_{1n}]$ will be specified further. Supposing $\lim_{n \rightarrow \infty} \varepsilon_{1n} = \lim_{n \rightarrow \infty} \varepsilon_{2n} = 0$, consider the sequence of continuous functions $\{\delta_n(t)\}$ which possesses the features

$$\begin{aligned} \delta_n(t) &> 0 \text{ for } t \in [t_0 - \varepsilon_{1n}, t_0 + \varepsilon_{2n}] \\ \delta_n(t) &\equiv 0 \text{ for } t \notin [t_0 - \varepsilon_{1n}, t_0 + \varepsilon_{2n}] \\ \int_{-\infty}^{\infty} \delta_n(t) dt &= J_n > 0 \text{ and } \lim_{n \rightarrow \infty} J_n = J > 0 \end{aligned}$$

It is evident that the above sequence converges, since its terms tend to zero for $t \neq t_0$ and tend to infinity when $t = t_0$. Hence, the above integral is finite.

Despite the singularities, the *weak limit* of the sequence $\{\delta_n(t)\}$ exists. The mean-value theorem yields

$$\int_{-\varepsilon_{1n}}^{\infty} \delta_n(t)g(t)dt = g(\Theta_n) \int_{-\varepsilon_{1n}}^{\infty} \delta_n(t)dt = J_n g(\Theta_n)$$

where $\Theta_n \in [t_0 - \varepsilon_{1n}, t_0 + \varepsilon_{1n}]$. Considering further the continuity of the function $g(t)$ we find that

$$\lim_{n \rightarrow \infty} J_n g(\Theta_n) = \lim_{n \rightarrow \infty} J_n g(\lim_{n \rightarrow \infty} \Theta_n) = J g(t_0)$$

i.e. the desired limit exists and is equal to $J g(t_0)$.

The sequences $\{\delta_n(t)\}$ are said to be δ -type sequences. The weak limit of these δ -type sequences is defined by the functional, assigning to this limit the finite value $J g(t_0)$. The above limit does not depend upon the specifics of the sequence $\{\delta_n(t)\}$. All δ -type sequences are equivalent. It is denoted by the symbol $J\delta(t-t_0)$. This function is known as the *Dirac δ -function*. From the above,

$$\lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} \delta_n(t)g(t)dt = \int_{-\infty}^{\infty} J\delta(t-t_0)g(t)dt = J g(t_0)$$

The δ -function is not classical. It possesses new distinctions which are defined by the features of the definite integral and is known as a *generalised function*. A complete and rigorous interpretation of the theory of such functions (commented by numerous examples) is provided in [56, 69, 149, 157, 188]. In the present monograph we

describe some generalised functions which are directly associated with the δ - functions. The main features of the δ - functions are established here.

(A). Firstly, it must be established in which sense should we understand an equality of expressions containing the δ - function.

Consider two generalised functions $\alpha_1(t)$ and $\alpha_2(t)$ which are defined using the corresponding linear functionals $I_1(\alpha_1)$ and $I_2(\alpha_2)$ based on arbitrary finite functions $g(t) \in C^0(\mathbf{R}^1)$. By definition, the above functions are equivalent if

$$\int_{-\infty}^{+\infty} \alpha_1(t) g(t) dt = \int_{-\infty}^{+\infty} \alpha_2(t) g(t) dt$$

Hereafter the "integral" definition of the equality $\alpha_1(t) = \alpha_2(t)$ for generalised functions is assumed.

(B). Suppose that t_0 and $J = 1$ in the definition of the δ - function. Thus, a centred, normalised and even δ - function may be obtained

$$\int_{-\infty}^{+\infty} \delta(t) g(t) dt = g(0), \quad \int_{-\infty}^{+\infty} \delta(t) dt = 1, \quad \delta(t) = \delta(-t).$$

Clearly it can be seen that $\delta(at) = |a|^{-1} \delta(t)$, $a \in \mathbf{R}^1$. It follows from the computation

$$\int_{-\infty}^{+\infty} \delta(at) g(t) dt = \lim_{n \rightarrow \infty} \int_{-\infty}^{+\infty} \delta_n(at) g(t) dt$$

$$= |a|^{-1} \lim_{n \rightarrow \infty} \int_{-\infty}^{+\infty} \delta_n(at) g[a^{-1}t] dt = |a|^{-1} g(0)$$

Q.E.D. In the particular case, $a = -1$, we obtain $\delta(t) = \delta(-t)$. It can be shown readily that $g(a \pm t) \delta(t) = g(a) \delta(t)$.

The products of the generalised functions along with the products of generalised and discontinuous functions are not defined in the common sense of the word. For

example, the expressions $\delta^2(t)$ and $\delta(t) \operatorname{sgn}(t)$ are senseless. But the expression $\delta(t) \operatorname{sgn}(t - b)$ may be determined, that is:

$$\delta(t) \operatorname{sgn}(t - b) = \begin{cases} \delta(t) & \text{if } t > b \\ -\delta(t) & \text{if } t < b \end{cases}$$

Since $g(t)$ is a continuous function, its convolution with the δ - function may also be determined

$$\int_{-\infty}^{+\infty} g(\tau) \delta(t - \tau) d\tau = g(t)$$

In a similar manner the convolution of δ - functions may be found

$$\int_{-\infty}^{+\infty} \delta(t - \tau) \delta(s - \tau) d\tau = \delta(t - s)$$

A more detailed description of the features of a convolution may be found in the cited references.

(C) As mentioned above (see Sect. (A)), the generalised functions may be created by using the linear functionals prescribed on the set of the finite functions from $C^n(\mathbf{R}^1)$. Let $\mathbf{D}$ be a set of finite and infinitely differentiable functions. The linear functionals defined on $\mathbf{D}$ will be studied further.

The functional

$$\eta = \int_{-\infty}^{+\infty} 1 \cdot \varphi(t) dt, \quad \varphi(t) \in \mathbf{D}$$

is generated by the well-known Heaviside function

$$\eta(t) = \begin{cases} 0 & \text{for } t < 0 \\ 1 & \text{for } t \geq 0 \end{cases}$$

As stated above, any linear functional $I[f]$ might generate the generalised function;

the Heaviside function may be treated also as a generalised function.

The fundamental difference between the η - function and the δ - function is as follows. The η - function is generated by the locally integrable function, i.e. by the function that can be integrated on any finite interval. The δ - function can not be generated in a like manner.

The generalised functions, generated using the locally integrable functions are known as *regular* generalised functions. Particularly, all the functions on $C^0(\mathbf{R}^1)$ may be declared as regular generalised functions. The rest of the generalised functions are known as *singular* generalised functions. In particular, the δ - function is a *singular* generalised function.

The main advantage of approaching the generalised functions as functionals on $\mathbf{D}$ is the possibility of differentiating them (in a certain sense of the word).

Let $f(t) \in C^n(\mathbf{R}^1)$, $n > 1$ and $\varphi(t) \in \mathbf{D}$. Using integration by parts we evaluate the definite integral

$$\int_{-\infty}^{+\infty} \frac{d}{dt} \varphi(t) dt = f(t) \varphi(t) \Big|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} f(t) \varphi(t) dt$$

The function $\varphi(t)$ is a finite function, therefore $f(t) \varphi(t) \Big|_{-\infty}^{+\infty} = 0$ and for any $m \leq n$

$$\int_{-\infty}^{\infty} \frac{d_m}{d_m} \phi(t) dt = (-1)^m \int_{-\infty}^{\infty} f(t) \frac{d_m}{d_m} \phi(t) dt$$

This formula defines the derivative of the generalised function. For the derivatives of the δ -function we have

$$\int_{-\infty}^{\infty} \frac{d_m}{d_m} f \phi(t) dt = (-1)^m \int_{-\infty}^{\infty} f(t) \frac{d_m}{d_m} \phi(t) dt$$

The main features of the generalised derivatives may be obtained readily. Consider, for example, $\eta'(t) = \delta(t)$. Then,

$$\int_{-\infty}^{\infty} \eta'(t) \phi(t) dt = - \int_{-\infty}^{\infty} \phi'(t) dt = -\phi(t) \Big|_{-\infty}^{\infty} = \phi(0) = \int_{-\infty}^{\infty} \delta(t) \phi(t) dt$$

The latter can be rewritten as

$$\eta(t) = \int_{-\infty}^{\infty} \delta(t) dt$$

For the functions $f(t) = C^n(\mathbf{R}^1)$ the generalised derivative coincides with the classical derivative. Considering this fact in combination with the above proven equality $\eta(t) = \int_{-\infty}^{\infty} \delta(t) dt$, we obtain the differentiating rule for the piecewise-smooth function $g(t)$ with discontinuities g_k at points t_k . For this purpose the following function is mentioned:

$$g_1(t) = g(t) - \sum_k g_k \eta(t - t_k)$$

This function may be differentiated regularly on the continuity intervals. The obtained derivative, therefore, coincides with $g'(t)$ everywhere except at the points $\{t_k\}$. Hence, $g_1'(t)$ coincides with the generalised derivative, and

$$g'(t) = g_1'(t) + \sum_k g_k \delta(t - t_k)$$

where $g_1'(t)$ is the classical part of the derivative of the function $g(t)$. Hence, the derivative of the piecewise-smooth function may be constructed from the regular and generalised components. The classical component of the generalised function is sometimes denoted using curly bracket, $g_1'(t) = \{g'(t)\}$.
(D) Laplace and Fourier integral transformations play an important role in the theory of generalised functions.

Let $\phi(t)$ be a complex-valued function from D. The Fourier transformation of $\phi(t)$ is defined as:

$$\tilde{\phi}(i\omega) = \int_{-\infty}^{\infty} \phi(t) \exp(-i\omega t) dt$$

The Fourier transformation of a generalised function may be defined as a linear functional constructed on the complex functions $\tilde{\phi}(i\omega)$. The definition of the Fourier transformation of the generalised function $f(t)$ is then based on the Parseval-Plancherel equality, which is well known in the theory of classical Fourier transformations,

$$2\pi \int_{-\infty}^{\infty} \overline{f(t)} \phi(t) dt = \int_{-\infty}^{\infty} \overline{f(i\omega)} \tilde{\phi}(i\omega) d\omega$$

where the bar indicates the operation of the complex conjugation. By using the above reasoning, $\tilde{\delta}(i\omega)$ may be calculated,

$$\int_{-\infty}^{\infty} \tilde{\delta}(i\omega) \tilde{\phi}(i\omega) d\omega = 2\pi \int_{-\infty}^{\infty} \delta(t) \phi(t) dt = 2\pi \phi(0)$$

From the formula of inverse Fourier transformations we find

$$\phi(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{\phi}(i\omega) d\omega$$

Combining the two latter expressions we obtain $\tilde{\delta}(i\omega) = 1$ and, therefore,

$$\delta(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(i\omega t) d\omega$$

i.e. the δ -function possesses a uniform spectrum.

For the shifted δ -function we have

$$\delta(t - t_0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp[i\omega(t - t_0)] d\omega, \quad \tilde{\delta}(i\omega) = \exp(-i\omega t_0)$$

The Fourier transformation of the derivatives of the δ -function may be constructed

$$\tilde{\delta}_{(m)}(i\omega) = (i\omega)^m \exp(-i\omega t_0)$$

Similarly, the theory of Laplace transformation may be formulated for the generalised functions,

$$f(p) = \int_{-\infty}^0 f(t) \exp(-pt) dt$$

where p is a complex number and $f(t)$ is a generalised function in $\mathbf{D}$.

A well-known and rigorous presentation of the theory of the integral transformation of generalised functions may be found in reference [69].

(E) The physical meaning of the introduction of δ -functions into the theory of mechanical impact will be explained by using the following model. Consider that at a collision of two bodies the full time of contact τ , exceeds the duration of internal waves caused by the collision. A good example of this model may be a moving ball of mass m , which hits a stationary wall elastically. The force of the impact interaction, $f_1(t)$, will be a finite function with the finite support $[t_1, t_2]$, $t_1 - t_2 = \tau$. The impulse of the impact force, therefore, is

$$J = 2mV = \int_{t_1}^{t_2} f_1(t) dt$$

where V is the velocity of the ball prior to collision.

The generalised model of the momentary impact appears to be well behaved for the mathematical analysis. In the case of a conservative impact the graph of the function $f_1(t)$ will be symmetrical with respect to the line $t = t_0 = \frac{1}{2}(t_2 - t_1)$.

Collapse the finite support of the impact force onto the fixed point t_0 . At the same time, we keep the main measure of impact, impulse of impact force, invariable. In other words, the following sequence of force functions $\{f_n(t)\}$ will be constructed

$$f_n(t) = \begin{cases} > 0 & \text{for } t \in [t_n, t_{2n}] \\ = 0 & \text{for } t \notin [t_n, t_{2n}] \end{cases}$$

and $\lim_{n \rightarrow \infty} t_n = t_0$ if $t_1 = t_1$, $t_2 = t_2$. We can consider, for example, that

$$t_{1n} = t_0 - \frac{t_0 - t_1}{n}, \quad t_{2n} = t_0 - \frac{t_2 - t_0}{n}$$

and for all $n \geq 1$

$$\int_{t_1}^{t_2} f_n(t) dt = J$$

Therefore, the sequence $\{f_n(t)\}$ may be considered as a δ -shaped sequence (see Fig. 1.1) and as $n \rightarrow \infty$, we obtain a limit impact model (in the generalised sense),

$$f_\infty(t) = J\delta(t - t_0)$$

The choice of the point t_0 is arbitrary. Previously it was caused by the condition that impact is absolutely elastic. This is where the direct loading time equals the restitution time. Other situations are also possible, but considering the generalised impact model, we neglect further the specific shape of the functions $\{f_n(t)\}$ since all δ -type sequences are equivalent. It is worthwhile to assume that the mentioned curves are symmetric with respect to the line $t = t_0$ and close values of the factor n generate the "close" shapes of the mentioned above curves, etc. By expressing the impact force in terms of the δ -function, we characterise the impact using only two quantities: the moment of occurrence and the impulse of impact (the support and integral measure).

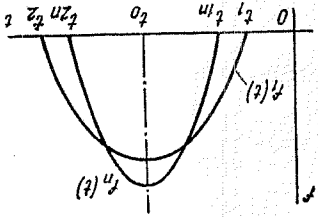


Fig. 1.1

It is common practice in physics and mathematics to use singular generalised functions whilst dealing with the calculations of densities of pointwise distributions of mass or charge. From the reasoning of $\mathbf{D}$ we derive a very important result. The spectrum of generalised impact is represented by a uniform function. As obtained by experiment, a collision of solid bodies generates a broadband frequency spectrum, the uniform approximation of which gives satisfactory results. That is why the impact excitation is so typical at experimental testing. In the analysis, such a choice simplifies the mathematical description of the operators of linear systems.

(G) If a differential equation of motion contains generalised functions, the corresponding solution must also be treated as a generalised function. For example, the solution of the differential equation

$$\ddot{x} + \Omega^2 x = J\delta(t - t_0)$$

is a function such that its substitution into the initial equation turns it into an identity in the sense of (A). The equations containing the generalised function play an important role in our further presentation.

(H) Concluding this brief introduction into the theory of generalised functions, we discuss the so-called left and right δ -functions (asymmetrical impulse functions, see

reference [81]). In order to indicate the "physical location" of the corresponding singularity the symbols $\delta_-(t)$ and $\delta_+(t)$ are used to denote the left and right δ -functions, respectively.

Let the function $g(t)$ be continuous in the right (left) half-space of the point $t = t_0$. The above mentioned symbols may be introduced using the following equations

$$\int_{t_0+0}^{t_0-0} g(t) \delta_+(t - t_0) dt = \frac{1}{2} g(t_0 + 0) \\ \int_{t_0+0}^{t_0-0} g(t) \delta_-(t - t_0) dt = \frac{1}{2} g(t_0 - 0)$$

Therefore, the introduction of the right and left δ -functions allows for the integration of the generalised functions which have discontinuities, but are continuous in the right (left) vicinity of the singularity point. Hence, we can define the products of the functions $\delta_-(t)$ and $\delta_+(t)$ with the discontinuous classical functions that are continuous in the right (left) vicinity of singularity point.

If $t = t_0$ is the point of singularity, then the right and left δ -functions may be treated as the generalised limits of the δ -shaped sequences, located all the time in the right (left) vicinity of the above-said point. For the function $g(t) \in \mathbf{D}$ we find, therefore, that $\delta_-(t) + \delta_+(t) = \delta(t)$.

Each of the functions $\delta_-(t)$ and $\delta_+(t)$ may be normalised by the unity, or speaking generally, by an arbitrary number. If, for instance, $p + q = s$ ($p, q > 0$), so the equality $p\delta_-(t) + q\delta_+(t) = \frac{2}{1}\delta(t)$ yields a measure of contribution of the "left and right parts" of the asymmetrical δ -shaped sequence.

1.3. Green Functions for Systems with Lumped Parameters

Consider a linear and scleronomic mechanical system with n degrees of freedom and full dissipation. The term "scleronomic" defines systems with stationary and

¹ The theory of right and left δ -functions is described in the monograph: Ljanze VE, Storog OG, The methods of theory of unbounded operators, Naukova Dumka, Kiev, 1983 (in Russian).

holonomic constraints. An assumption of full dissipation means that *any motion* of the system is associated with energy dissipation. The position of the system will be described by using the vector $\mathbf{x} \in \mathbf{R}^n$. Assume that under the external forcing $\mathbf{g}(t)$ the system vibrates about the position of asymptotically stable equilibrium. The differential equation of forced motion takes the form

$$\mathbf{A}\ddot{\mathbf{x}} + \mathbf{B}\dot{\mathbf{x}} + \mathbf{C}\mathbf{x} = \mathbf{g}(t) \quad (1.1)$$

where $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ are the matrices of symmetric and positively defined quadratic forms

$$\mathcal{K} = \frac{1}{2}(\mathbf{A}\mathbf{x}, \mathbf{x}), \quad \mathcal{D} = \frac{1}{2}(\mathbf{B}\dot{\mathbf{x}}, \dot{\mathbf{x}}), \quad \Pi = \frac{1}{2}(\mathbf{C}\mathbf{x}, \mathbf{x})$$

expressing, respectively, the kinetic energy, the dissipative Raleigh function and the potential energy of the system [28].

We introduce the basis in $\mathbf{R}^n$, $\{\mathbf{r}_k\}$, $k = 1, \dots, n$, $\mathbf{r}_k = (0, 0, \dots, 0, 1, 0, \dots, 0)$ where unity is positioned at the k th place. The matrix compiled of such vectors is said to be the unity matrix.

The Green function² of system (1.1) is the matrix $H(t, \tau)$ consisting of n particular solutions when it is taken consecutively $g(t - \tau) = \delta(t - \tau)\mathbf{r}_j$, $j = 1, \dots, n$, $t, \tau \in \mathbf{R}^1$. By virtue of the system stationarity,

$$H(t, \tau) = H(t - \tau)$$

In the following section the Green function will be constructed. Now we establish its main features based on the definition.

The Green function may be represented using the form: $\mathbf{H}(t) = \eta(t)\mathbf{H}_1(t)$, where $\mathbf{H}_1(t)$ is the solution of the matrix equation

$$\mathbf{A}\ddot{\mathbf{H}}_1 + \mathbf{B}\dot{\mathbf{H}}_1 + \mathbf{C}\mathbf{H}_1 = 0$$

with initial conditions $\mathbf{H}_1(0) = 0$, $\dot{\mathbf{H}}_1(0) = \mathbf{A}^{-1}$ (0 denotes a zero matrix).

Using the rules of generalised differentiation (see Sect. 1.2 (D)) we find

$$\dot{\mathbf{H}}(t) = \eta(t)\mathbf{H}_1(t), \quad \ddot{\mathbf{H}}(t) = \delta(t)\mathbf{A}^{-1} + \eta(t)\dot{\mathbf{H}}_1(t)$$

Therefore,

¹ Hereafter $\mathbf{R}^n$ means the Euclidean coordinate n -space with the scalar product $(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^n x_i y_i$, $\mathbf{x} = (x_1, \dots, x_n)$, $\mathbf{y} = (y_1, \dots, y_n)$.

² The term "Green function" was adopted in accordance with the traditions of mechanics. In the theory of automatic control, optics, etc, the Green function is known as a function of *weight* or as an *impulse function*, *dispersion function*, etc.

$$\begin{aligned} & \mathbf{A}\ddot{\mathbf{H}}(t) + \mathbf{B}\dot{\mathbf{H}}(t) + \mathbf{C}\mathbf{H}(t) \\ &= \eta(t)(\mathbf{A}\ddot{\mathbf{H}}_1(t) + \mathbf{B}\dot{\mathbf{H}}_1(t) + \mathbf{C}\mathbf{H}_1(t))\mathbf{I} + \delta(t)\mathbf{I} = \delta(t)\mathbf{I} \end{aligned}$$

Q.E.D.

The derivative of the Green function when $t = 0$ possesses the first-order discontinuity $\mathbf{A}^{-1}$, which may be considered as the reciprocal to the system inertia described by the matrix $\mathbf{A}$.

Given the Green function, the particular solution of Eq. (1.1) may be obtained as

$$\mathbf{x}(t) = \int_{t_0}^t \mathbf{H}(t-s)\mathbf{g}(s)ds \quad (1.2)$$

where t_0 is the initial time instant and $t_0 \neq 0$ in the general case.

In order to obtain the solution to the Cauchy problem for Eq. (1.1), the general solution of the homogeneous equation depending upon the initial conditions

$$\mathbf{x}(t_0) = \mathbf{x}_0, \quad \dot{\mathbf{x}}(t_0) = \mathbf{V}_0$$

has to be added to the above-mentioned particular solution.

For the left semiplane of the point $t = t_0$, the solution $\mathbf{x}(t)$ and the external force $\mathbf{g}(t)$ are complemented by zeros. The above extensions are denoted $\tilde{\mathbf{x}}(t)$ and $\tilde{\mathbf{g}}(t)$.

At $t = t_0$ the extension $\tilde{\mathbf{x}}(t)$ and its derivative possess the increments $\mathbf{x}_0$ and $\mathbf{V}_0$. In accordance with the rules of differentiation of generalised functions (see section 1.2

(D)) we find

$$\dot{\mathbf{x}}(t) = \tilde{\mathbf{x}}(t) - \mathbf{x}_0\delta(t-t_0), \quad \ddot{\mathbf{x}}(t) = \tilde{\mathbf{x}}(t) - \mathbf{x}_0\delta(t-t_0) - \mathbf{V}_0\delta(t-t_0)$$

The desired process is a function defined for $-\infty < t < \infty$ and, therefore, it must satisfy Eq. (1.1) with a modified right-hand-side term,

$$\mathbf{A}\ddot{\tilde{\mathbf{x}}} + \mathbf{B}\dot{\tilde{\mathbf{x}}} + \mathbf{C}\tilde{\mathbf{x}} = \tilde{\mathbf{g}}(t) + (\mathbf{A}\mathbf{V}_0 + \mathbf{B}\mathbf{x}_0)\delta(t-t_0) + \mathbf{A}\mathbf{x}_0\delta'(t-t_0)$$

Therefore, the initial conditions may be considered as trivial. The right-hand side of the above equation depends on the generalised functions. The corresponding solution for the Cauchy problem using (1.2) is obtained in the form

$$\mathbf{x}(t) = \int_{t_0}^t \mathbf{H}(t-s)\mathbf{g}(s)ds + \mathbf{H}(t-t_0)(\mathbf{A}\mathbf{V}_0 + \mathbf{B}\mathbf{x}_0) + \mathbf{A}\mathbf{H}(t-t_0)\mathbf{x}_0 \quad (1.3)$$

$$\mathbf{x}(t) = \int_{-\infty}^t \mathbf{H}(t-s)\mathbf{g}(s)ds \quad (1.4)$$

assumption, instead of (1.2) we find that

In the following representation we primarily consider the steady-state regime. Due to the presence of full dissipation in the mentioned systems, the corresponding components of the matrices $\mathbf{H}$ and $\mathbf{H}$ tend to zero as $t \rightarrow \infty$. Hence, we may consider that $t_0 \rightarrow -\infty$ for the analysis of the steady-state regimes. As a result of this

or, in component-wise notation

$$x_k(t) = \sum_{j=1}^n \int_{-\infty}^t h_{kj}(t-s)g_j(s)ds \quad k=1, \dots, n$$

Here the components of the matrix $\mathbf{H}(t) = [h_{kj}(t)]$ define the response of the k -

indexed point as a result of the unit force application in the j -indexed point.

The diagonal elements of the above matrix $h_{kk}(t)$ are known as the local Green functions in the points $x_1, x_2, \dots, x_n$ and the nondiagonal elements $h_{kj}(t)$ are known as the transfer Green functions from the points x_k to the points x_j .

The physical sense of the formula (1.4) and the equality $\mathbf{H}(t) = \eta(t)\mathbf{H}_1(t)$ may be cleared up with the help of the following reasoning. The external force $\mathbf{g}(t)$ may be expressed in terms of the infinite "sum of the instantaneous forces" $\mathbf{g}(t)\delta(t-s)$ in the form

$$\mathbf{g}(t) = \int_{-\infty}^{\infty} \mathbf{g}(s)\delta(t-s)ds$$

The decomposition of the instantaneous force with respect to the basis $\{\mathbf{r}_k\}$ takes the form

$$\mathbf{g}(s)\delta(t-s) = \sum_{k=1}^n g_k(s)\delta(t-s)\mathbf{r}_k$$

where $g_k(s)$ is the k -indexed component of the force $\mathbf{g}(s)$. Therefore, the force

$$\mathbf{g}(t) = \sum_{k=1}^n \left[\int_{-\infty}^{\infty} g_k(s)\delta(t-s)ds \right] \mathbf{r}_k$$

By the definition of the Green function, any instantaneous force $g_k(s)\delta(t-s)\mathbf{r}_k$ causes the corresponding response $g_k(s)h_{kk}(t-s)$, whilst $h_{kj}(t)$ denotes the k -indexed column of the matrix $\mathbf{H}(t)$. Therefore, the solution is constructed in terms of the summation of the actions of each instantaneous force factor,

$$\mathbf{x}(t) = \sum_{k=1}^n \int_{-\infty}^t h_{kj}(t-s)g_k(s)ds = \int_{-\infty}^t \mathbf{H}(t-s)\mathbf{g}(s)ds$$

In the latter integral the upper limit may be considered to be ∞ because $\mathbf{H}(t-s) = 0$

for $t < s$.

Actually, each of the columns of the matrix $\mathbf{H}$ is constructed from the solutions obtained as a response to the unit impulse application to the point x_k in the time

instant t_0 . Assuming that $H(t) \neq 0$ for $t < t_0$ we allow for a response (effect) to appear before an impulse application (cause). The causality principle forbids this and we obtain (1.4).

The Green function may be created using the very general recipes from the theory of ordinary differential equations [138,173]. We apply operational calculus for this purpose.

Relationship (1.4) or (1.3) gives the possibility of the analytical description of the operator Δ_0 for a linear and stationary mechanical system.

Considering (1.4), let us denote the Volterra linear integral operator in the form

$$\hat{H}g = \int_{-\infty}^t H(t-s)g(s)ds$$

The operator $\hat{H}$ realises the relationship between an input signal (the force g) and an output (the deflection x) and, therefore, defines the operator Δ_0 of a linear system in full. The Green function $H(t)$ is the kernel of the corresponding integral operator. The calculations required for its definition are made below.

1.4. Operator of Dynamic Compliance

Relationship (1.4) defines an operator of the linear system (1.1) as the Volterra integral operator, which, in turn, may be completely defined by the kernel, the Green function $H(t)$. This function is defined here using symbolic calculus. Considering zero initial conditions, the Laplace transformation of both sides of (1.1) with $g(t) = d(t)r_k$ is to be evaluated. For the k -indexed column of the Green function we find

$$[Ap^2 + Bp + C]\tilde{x}(p) = r_k$$

where p is a complex number, an argument of the Laplace image of a function. We also considered that $\operatorname{Re}(p) > 0$.

For the components of the vector $\tilde{x}(p)$, the latter equality may be treated as a system of linear algebraic equations, the principal determinant

$$D_{2n}(p) = \begin{vmatrix} a_{n1}p^2 + b_{n1}p + c_{n1} & a_{n2}p^2 + b_{n2}p + c_{n2} & \dots & a_{nn}p^2 + b_{nn}p + c_{nn} \\ a_{21}p^2 + b_{21}p + c_{21} & a_{22}p^2 + b_{22}p + c_{22} & \dots & a_{2n}p^2 + b_{2n}p + c_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1}p^2 + b_{n1}p + c_{n1} & a_{n2}p^2 + b_{n2}p + c_{n2} & \dots & a_{nn}p^2 + b_{nn}p + c_{nn} \end{vmatrix}$$

of which is a polynomial of degree $2n$. The determinants corresponding to the components $x_j(p)$, $j = 1, \dots, n$, may be constructed by replacing the j -indexed

column with the corresponding components of the vector r_k . In the replaced column, the single nonzero component is k -indexed. Therefore, the desired determinants may be resolved using the above column. In doing so, we find the determinants corresponding to $\tilde{x}_j(p)$

$$D_{2n-2}^{(k,j)}(p) = (-1)^{k+j} D_{kj}(p)$$

where $D_{kj}(p)$ is the determinant obtained by removing the k -indexed row and j -

indexed column from $D_{2n}(p)$. It can be shown readily that $D_{2n-2}^{(k,j)}(p)$ is a polynomial of degree $2n-2$.

Hence, each component of the Green function may be represented in the form

$$(1.5) \quad \tilde{h}_{jk}(p) = \frac{D_{2n-2}^{(k,j)}(p)}{D_{2n}(p)}$$

By virtue of symmetry of the matrices defining the kinetic energy, the dissipative function and the potential energy of the system, $\tilde{h}_{jk}(p) = \tilde{h}_{kj}(p)$. Hence, the

corresponding originals, transfer Green functions, are also symmetric, $h_{jk}(t) = h_{kj}(t)$.

That means that the displacement obtained by the j -indexed point as a result of the force application at the k -indexed point equals that of k -indexed point as a result of the application of the same force at the j -indexed point (*reciprocity principle*).

Expression (1.5) is a fractionally rational function. In mechanics the following cases are usually considered. The above fractions are irreducible and the characteristic equation $D_{2n}(p) = 0$ does not possess multiple roots. Then (1.5) may be expanded

easily into the partial fractions

$$\tilde{h}_{jk}(p) = \sum_{m=1}^{2n} \frac{D_{2n-2}^{(k,j)}(p)}{D_{2n}(p)} \frac{1}{p - p_m}$$

where p_m are the roots of the equation $D_{2n}(p) = 0$ and (*) means differentiation with

respect to the argument. By using the relation $(p - p_m)^{-1} \Leftrightarrow \exp(-p_m t)$ we find, after simple manipulations,

$$(1.6) \quad h_{jk}(t) = \sum_{m=1}^{2n} \frac{D_{2n-2}^{(k,j)}(p_m)}{D_{2n}'(p_m)} \exp(p_m t), \quad t \geq 0$$

In a case with repetitive roots, the decomposition formula will be more complicated since terms like $(p - p_m)^{-N}$, $N > 1$, appear. Hence, the power members in the inversion formula take the form

$$h_{jk}(t) = \sum_{l=1}^{N_q} \sum_{m=1}^q \frac{C_{(k,j),l}^{(q,m)}}{(m-1)!} \exp(p_l^q t), \quad t \geq 0$$

where

$$C_{(j,k)}^{q_m} = \frac{1}{\left[\frac{d^{N_q-m}}{p^{N_q-m}} (p - p^q)^{N_q} D_{2n-2}^{(k,j)}(p) \right]} \left[\frac{d^{N_q-m}}{p^{N_q-m}} (N^q - m)! \right] \int_{p=p_q}^{\infty} D_{2n}^q(p)$$

where l is the number of different roots p_q of the characteristic equation $D_{2n}(p) = 0$, N_q is the order of the q -indexed root, $N_1 + N_2 + \dots + N_l = 2n$.

Hence, the Green function $\mathbf{H}(t)$ is calculated and the operator $\hat{\mathbf{H}}$ of linear system (1.1) is already defined.

The symbol p may be treated not only as an argument of functions in the space of complex variables, but also as a differentiation operator. It is known that for originals at zero initial conditions, $p\tilde{x}(p) \Leftrightarrow \dot{x}(t)$. Therefore, $p\dot{x}(t) = \dot{x}(t)$ and $p \equiv \frac{d}{dt}$. This is fundamental in the so-called *Heaviside operator method*. Similarly, the symbol p^{-1}

may be thought of as the integration operator, e^{-pt_0} as the lag operator, etc. Relationship (1.4) may be expressed in the operator form $\mathbf{x}(t) = \mathbf{L}(p)\mathbf{g}(t)$, where the matrix $\mathbf{L}(p) \equiv \hat{\mathbf{H}}(p)$.

This expression may be obtained formally by applying the convolution theorem with a zero lower limit in (1.4) for the vectors $\mathbf{x}, \mathbf{g}$ and the corresponding originals $\eta(t)\mathbf{x}, \eta(t)\mathbf{g}$. Hereafter, the operator $\mathbf{L}(p)$ is considered as some formal sequence of operations on the function. The above sequence is defined by the structure of the above operator and by the identity $p \equiv \frac{d}{dt}$.

The matrix $\mathbf{L}(p)$ is known as the *dynamic compliance (receptance)* operator of system (1.1).¹ The diagonal components of the above matrix, $L_{jj} \equiv \tilde{h}_{jj}(p)$, are the *local dynamic compliance operators*. The nondiagonal elements, $L_{kj} \equiv \tilde{h}_{kj}(p)$, represent the *transfer operators* from the points x_k into the points x_j , $L_{kj} \equiv L_{jk}$. The operator $\mathbf{L}(p)$ is one of the possible analytical representations of the operator $\mathbf{A}_0$ of the linear system (1.1).

The inverse operator $\mathbf{L}^{-1}(p) = \mathbf{A}p^2 + \mathbf{B}p + \mathbf{C}$ is known as the *dynamic stiffness operator*. Its diagonal and nondiagonal elements are known as the *local and transfer dynamic stiffness operators*.

The operator $\mathbf{L}^{-1}(p)$ defines the equation of motion, and correlates the deflections. In the theory of automatic control, the term *transfer function* is used instead of this designation.

1.5. The Eigenmode Decomposition of the Dynamic Compliance Operator

System (1.1) may be rewritten in a component-wise form using the eigenfunctions $q_1, \dots, q_n$

$$x_j(t) = \int_{-\infty}^t h_{jk}(t-s)e^{ps}ds = \int_{-\infty}^t h_{jk}(\tau)e^{-p\tau}d\tau \left[e^{pt} \right]$$

or

$$x_j(t) = L_{jk}(p)e^{pt}, \quad j = 1, \dots, n$$

Therefore, the components of the matrix $\mathbf{L}(p)$ may be treated as a response of a linear system to the forces $e^{pt}\mathbf{r}_k$, $k = 1, \dots, n$, where p is the complex parameter. An analytical expression of the operator $\mathbf{L}(p)$ may be treated as a response of a system to a standard excitation. Consider the force $\mathbf{g}(t) = e^{pt}\mathbf{r}_k$, where p is not an operator, but a complex parameter. From (1.4), the components of the vector of deflections are

expressed using the forces $\mathbf{g}_m(t)$ and the matrix $[\Theta_{mj}] = [\Theta_{jm}]^T$, by

$$\mathcal{K} = \frac{1}{2} \sum_{m=1}^n q_m^2, \quad \Pi = \frac{1}{2} \sum_{m=1}^n \Omega_m^2 q_m^2$$

potential energy into the form

$$x_j = \sum_{m=1}^n \Theta_{jm} q_m$$

where the eigenfunctions q_m and generalised coordinates are related

$$\ddot{q}_k + \sum_{j=1}^n \beta_{kj} \dot{q}_j + \Omega_k^2 q_k = \ddot{Q}_k(t), \quad k = 1, \dots, n$$

and the matrix $[\Theta_{jm}]$ performs a simultaneous reduction of the matrices of kinetic and

vector $\mathbf{x}(t)$ with the force vector $\mathbf{g}(t)$. An analytical expression for the Green

function of the operator $\mathbf{L}^{-1}(p)$ may be obtained using the theory of generalised functions. It is not needed here.

An analytical expression of the operator $\mathbf{L}(p)$ may be treated as a response of a system to a standard excitation. Consider the force $\mathbf{g}(t) = e^{pt}\mathbf{r}_k$, where p is not an operator, but a complex parameter. From (1.4), the components of the vector of

The quantities Θ_{jm} are the normalising factors of the j -indexed eigenmode with eigenfrequency $\Omega_j > 0$. In this case, $\beta_{kj} = \sum_{m=1}^n b_{km} \Theta_{mj}$.

Suppose that all the parameters β_{kj} are small quantities, therefore we neglect further their products and squares in comparison with the other terms. Then the characteristic equation of the above system in normal coordinates may be approached in the form

$$D_{2n}(p) = \begin{vmatrix} p^2 + \beta_{11}p + \Omega_1^2 & \beta_{12}p & \dots & \beta_{1n}p \\ \beta_{21}p & p^2 + \beta_{22}p + \Omega_2^2 & \dots & \beta_{2n}p \\ \dots & \dots & \dots & \dots \\ \beta_{n1}p & \beta_{n2}p & \dots & p^2 + \beta_{nn}p + \Omega_n^2 \end{vmatrix} = 0$$

where $\Gamma_m = \frac{1}{2} \beta_{mm} \Omega_m^{-1}$.

As a result of these manipulations, the roots of the above equation will be obtained as complex conjugate pairs

$$(1.7) \quad p_m = -\Gamma_m \Omega_m \pm i\Omega_m$$

With allowance made for the above assumption, the expressions of the dynamic compliance operators in terms of normal coordinates take the form

$$L_{(1)}^{jk}(p) \equiv (p^2 + 2\Gamma_j \Omega_j p + \Omega_j^2)^{-1}$$

The expressions for the normal coordinates yield the solution in the original coordinates,

$$q_j(t) = \frac{p^2 + 2\Gamma_j \Omega_j p + \Omega_j^2}{\mathcal{O}_j(t)}$$

After simple substitutions we find

$$x_t(t) = \sum_n \Theta_{nj} q_j(t) = \sum_n \sum_{m=1}^f \frac{p^2 + 2\Gamma_j \Omega_j p + \Omega_j^2}{\Theta_{mj} \Theta_{mj}} g_m(t)$$

The standard excitation $g(t)$ was selected to be in the form $e^{pt} \Gamma_m$. With respect to the relationship given at the end of the previous section, the components of the operator of dynamic compliance may be found in the form

(1.8)

$$L_{km}(p) = \sum_n \frac{\Theta_{kj} \Theta_{mj}}{p^2 + 2\Gamma_j \Omega_j p + \Omega_j^2}$$

This formula becomes exact in the absence of damping and represents the *eigenmode decomposition* of dynamic compliance operator.

It can be clearly seen that the approximate expression (1.8) is symmetric, $L_{km}(p) = L_{mk}(p)$.

In the presence of damping, the roots of Eq. (1.7) have negative real parts. Therefore, each term in the decomposition of the Green function, which is obtained in accordance with (1.8) and using (1.6), contains the multiplier $e^{(-\Gamma_m \Omega_m t)}$, showing exponential decay.

It can be shown that for the exact solution of the characteristic equation in the case of simple poles of the operator $L(p)$, the situation is not altered. An increase in the damping factor may only cause an increase in the decay rate. In the presence of multiple roots the situation is even more complicated, but in any case the functions $h_{km}(t)$ are damped.

If there is no damping at all, the roots of (1.7) are purely imaginary quantities. The corresponding functions $h_{km}(t)$ are oscillating.

1.6. Linear System as a Low-pass Filter

Consider again relationship (1.4). As shown above, the presence of damping reflects in an exponential decay of the components of the matrix $H(t)$. That means that these components are absolutely integrable for $t \in \mathbb{R}^1$

$$\int_{-\infty}^{\infty} |h_{km}(t)| dt = \int_{-\infty}^{\infty} |h_{km}(t)| dt < \infty$$

and, therefore, the Fourier transform $\tilde{H}(i\omega)$ exists. If the Fourier transform of the external force exists, then, using the convolution theorem, we find from (1.4)

$$\tilde{x}(i\omega) = \tilde{H}(i\omega) \tilde{g}(i\omega), \quad \tilde{g}(i\omega) = \int_{-\infty}^{\infty} g(t) e^{-i\omega t} dt$$

The function $\tilde{H}(i\omega) \equiv L(i\omega)$ (the substitution $p = i\omega$ is correct in this case) is said to be the *dynamic compliance* (receptance) of system (1.1).

The concepts of *local* or *transfer dynamic compliance* and *local* or *transfer dynamic*

¹ In the theory of automatic control the term *frequency response characteristic* is used as an equivalent.

stiffness $L^{-1}(i\omega)$ may be considered accordingly.

The difference between dynamic compliance and dynamic compliance operator is illusory. In this case the notation $p = i\omega$ and $p = i\omega$ are absolutely equivalent, and, therefore

$$x(t) = L(i\omega)g(t)$$

However, it is important here that the substitution $p = i\omega$, allows study of the spectral characteristics of the Green function

$$\tilde{h}_{jk}(i\omega) = L_{jk}(i\omega) = |L_{jk}(i\omega)| e^{i \arg L_{jk}(i\omega)}$$

where $|L_{jk}(i\omega)| = O(\omega^{-2})$ as $\omega \rightarrow \infty$, due to the fact that (1.5) represents the rational fraction function. The power of the numerator in (1.5) is lower by two orders than that of the denominator. For example, the eigenfunction decomposition of dynamic compliance (1.8) yields

$$L_{km}(i\omega) = \sum_n \frac{\Theta_{kj} \Theta_{mj}}{(\Omega_j^2 - \omega^2 - i2\Omega_j \Gamma_j \omega)} \quad \left| L_{km}(i\omega) \right| = \sum_n \frac{\Theta_{kj} \Theta_{mj}}{\sqrt{(\Omega_j^2 - \omega^2)^2 + 4\Gamma_j^2 \Omega_j^2 \omega^2}}$$

The systems for which the modulus of dynamic compliance tends to zero as $\omega \rightarrow \infty$, are said to be *filtering systems*. This reflects a specific feature of linear systems. They enhance the low-frequency components at the transformation of the external force into the dynamic response. The components of the Green function possess a descending spectrum, whilst the impact force described with the help of the δ -function possesses a uniform spectrum.

The filtration feature also explains the transformation of the discontinuous input (impact) into the continuous response. It can be seen directly from system (1.1) that the accelerations will be proportional to the δ -functions, but, due to the smoothing, the evaluation of the velocities and deflections produces continuous functions. An operator of dynamic compliance may be represented in terms of the sum of n operators of resonant filters with the natural frequencies Ω_k . The dynamic compliance modulus yields amplification when $\omega = \Omega_k$ due to the appearance of the

The symbol $\beta = O(\alpha)$ signifies the quantities of the same order. For instance, $\beta(t) = O(\alpha(t))$ as $t \rightarrow t_0$ if $\lim_{t \rightarrow t_0} \beta(t) \alpha^{-1}(t) = \text{const}$. Another notation may be used: $\beta \sim \alpha$ (see [124, 128]), in this case $\text{const} = 1$.

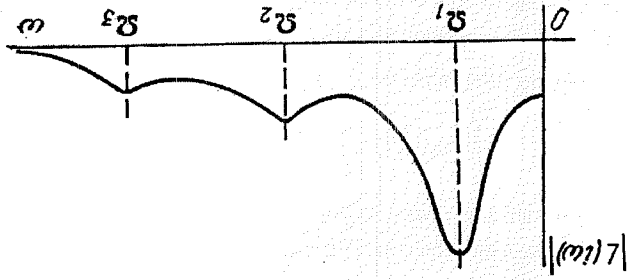


Fig. 1.2

small term $2\Gamma_k \Omega_k^2$ in the denominator of the k -indexed addend. The width of the spike above the quantity $\Gamma_k \Omega_k^2$. With an increase in the eigenfunction order, the height of the local dynamic compliance as a function of frequency. The detailed analysis of the features of the linear systems is presented in the monograph by Kolovsky [78]. Consider the case of the "undamped" system where $\text{Im } L_{km}(i\omega) = 0$, $k, m = 1, \dots, n$.

As already shown, the Green function $H(t)$ contains oscillating components that are not absolutely integrable functions for $t \in \mathbb{R}^1$. The ordinary Fourier transformation $\tilde{H}(i\omega)$ in this case is not defined and the formal substitution $p = i\omega$ is not correct. The above substitution becomes correct when applied to the generalised Fourier transformations (see [23]). Therefore, we suppose that the transformation $p = i\omega$ will be admissible for the functions considered in this monograph. The manipulations with the obtained expressions $L_{km}(i\omega)$ will be carried out in a regular manner with respect to the fact that $\lim_{\omega \rightarrow \Omega_j} |L_{km}(i\omega)| = \infty$ if $\text{Im } L_{km}(i\omega) = 0$.

It can be proven that if the operator of dynamic compliance is a response of the linear system to the standard force $e^{i\omega t} r_k$, then the dynamic compliance is the response of the linear system to the standard force factor, given by $e^{i\omega t} r_k$, $k = 1, \dots, n$. Let the system be exposed to the action of such a force. We may obtain from (1.4) for the components of the deflection vector

It is reasonable to use the *bilateral Laplace transformation* [148, 180] while using the operators of dynamic compliance.

$$x_f(t) = \int_{-\infty}^t h_{fs}(t-s) e^{s} ds = \left| L_{fs} \right| e^{i(\omega t + \arg L(i\omega))} \left| L(i\omega) \right|$$

Therefore, the modulus of dynamic compliance indicates the magnitude amplification/attenuation, and its argument, the phase lag at the transformation of the input force $e^{i\omega t}$ into the deflection performed by a linear system. The dynamic compliance, therefore, is the supplementary form of a representation of the operator A_0 of a linear system.

1.7. Linear Single-Degree-of-Freedom System

Despite the fact that in the present monograph we repeatedly come across the operators of complex linear systems, we start our study with an analysis of a SDOF system. The equation of motion takes the form

$$\ddot{x} + 2bx + \Omega^2 x = g(t)$$

where $b \geq 0$ is the damping factor, Ω is the natural undamped frequency and $g(t)$ is the external forcing.

We review the possible analytical presentations of the operator of the above system. With the use of the identity $p \equiv \frac{d}{dt}$ we find

$$(p^2 + 2bp + \Omega^2)x = g(t)$$

Therefore, the operators of dynamic compliance and stiffness take the form

$$L^{-1}(p) = p^2 + 2bp + \Omega^2, \quad L(p) = (p^2 + 2bp + \Omega^2)^{-1}$$

By substituting $p = i\omega$, we obtain the dynamic compliance

$$L(i\omega) = \left[(\Omega^2 - \omega^2) + i2b\omega \right]^{-1}$$

with modulus and argument

$$\left| L(i\omega) \right| = \frac{1}{\sqrt{(\Omega^2 - \omega^2)^2 + 4b^2\omega^2}}, \quad \arg L(i\omega) = \text{atan} \frac{-2b\omega}{\Omega^2 - \omega^2}$$

Therefore, the magnitude and the phase lag of the system response to the standard excitation $e^{i\omega t}$ is fully defined.

Using expression (1.6), the Green function for $\Omega^2 > b^2$ may be found in the form

$$h(t) = \frac{e^{-bt} \eta(t)}{\sqrt{\Omega^2 - b^2}} \sin \sqrt{\Omega^2 - b^2} t$$

Also, the same representation may be derived readily from a general solution of the

equation of motion with the initial conditions $x(0) = 0$, $\dot{x}(0) = 1$.

In the case $b \rightarrow 0$, the expression for the operator $L(p)$ of an undamped system may be obtained; the corresponding dynamic compliance $L(i\omega) = \frac{1}{\Omega^2 - \omega^2}$ possesses a singularity when $\omega = \Omega$, and the Green function takes the form

$$h(t) = \frac{\Omega}{\eta(t)} \sin \Omega t$$

It is interesting to note the possibility of the passage to a limit at $\Omega \rightarrow 0$; that is equivalent to the case $\dot{x} = g(t)$. The Green function of such a system is $h(t) = t$, $t \geq 0$. In the case of the overdamped system $\Omega^2 < b^2$ we have the rational roots of the characteristic equation and the Green function does not oscillate

$$h(t) = \eta(t) \frac{e^{p_1 t} - e^{p_2 t}}{p_1 - p_2}, \quad p_{1,2} = -b \pm \sqrt{b^2 - \Omega^2}$$

Therefore, we have obtained a nonoscillating process which dies out in time. The situation will be the same in the case of critical damping, $\Omega^2 = b^2$.

1.8. Operators of Rod Systems

The deflections of continuous systems are described using the vector-function $u(x, t)$ with $u, x \in \mathbb{R}^3$, where x is the vector of the instantaneous coordinates of the points of a distributed media.

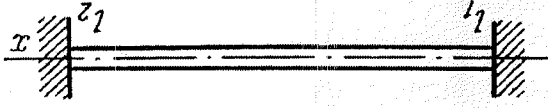


Fig. 1.3

For such a system the operators of dynamic compliance and the Green functions may be obtained using the methods of [30,31]. Here single-dimensional rod systems are concerned: $u, x \in \mathbb{R}^1$. Consider the longitudinal or transverse vibration $u(x, t)$ of the thin linear bar (see Fig. 1.3). Here $x \in [l_1, l_2]$; with the possibility that $l_1 = -\infty$ and/or $l_2 = \infty$. In a very general case, a distribution of the deflections $u(x, t)$ obeys the equation in partial derivatives

$$(1.9) \quad \mathcal{L}_q^q \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x} \right) u(x, t) = f(x, t)$$

where $f(x, t)$ is a distribution of the external force along the bar and $\mathcal{L}_q^q(p, p_1)$ is a q -order polynomial in the variables p and p_1 with time-independent coefficients

($q > 1$).

For the wave equation

$$\left(\frac{\partial^2}{\partial t^2} - c^2 \frac{\partial^2}{\partial x^2} \right) u(x, t) = f(x, t)$$

the above polynomial takes the form $\mathcal{L}_2(p, p_1) = p^2 - c^2 p_1^2$.

Using the above operator language, the polynomial $\mathcal{L}_q$ may be treated as an operator

of dynamic stiffness defining the force in accordance with the deflection. In order to define this operator uniquely the boundary and initial conditions have to be applied to

(1.9). Depending upon these conditions different solutions $u(x, t)$ may be obtained for the same external force $f(x, t)$.

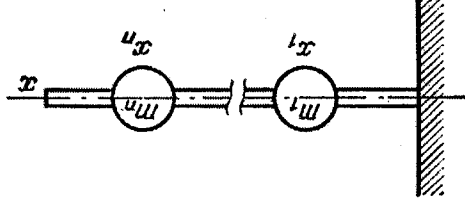


Fig. 1. 4

The steady-state solution will be considered further. Therefore, it is reasonable to consider now only zero initial conditions.

In general, the function $f(x, t)$ is not necessarily continuous. Suppose that at the points $x = x_k$ of the rod, some lumped masses m_k , $k = 1, 2, \dots, n$, are disposed (see Fig. 1.4). That causes the appearance of the concentrated forces of inertia

$-m_k \frac{\partial^2 u(x_k, t)}{\partial t^2}$ at the above points. Using the δ -functions, Eq. (1.9) may be written in the form

¹ Instead of the term "distributed force" we use the term "force". We are going to use it everywhere whilst it does not lead to misunderstanding.

$$\mathcal{L}_q^q \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x} \right) u(x, t) = f(x, t) - \sum_{k=1}^n m_k \frac{\partial^2 u(x_k, t)}{\partial t^2} \delta(x - x_k)$$

Similarly, the transformation of the boundary and initial conditions into external forces yields singular functions to appear in the right-hand side of (1.9).

Here we can repeat the reasoning of Sect. 1.3 concerning the insertion of the initial conditions into the right-hand side of Eq. (1.1). The procedure of the reduction of the initial and boundary conditions to the right-hand side of equation is described in details in [30, 31].

Consider further that equation of motion (1.9) may be uniquely solved in a generalised form, for example, with the considered functions (including the boundary conditions and concentrated factors).

The Green function $H(x, t, \tau)$ of system (1.9) is the partial solution at $f(x, t) = \delta(x - y)\delta(t - \tau)$, $x, y \in [l_1, l_2]$, $t, \tau \in \mathbf{R}^1$, and under zero initial conditions.

Due to the assumption made, the unique Green function exists. Also, since the system is invariant, it can be considered that $\tau = 0$. But even the presence of the x -invariant coefficients at $l_2 \neq \pm\infty$ does not guarantee the x -invariance of the system. The coordinate shift may put us outside the rod's boundaries.

The steady-state solution of (1.9) may be constructed as a "sum of disturbances" from all pointwise forces (compare with Sect. 1.3)

$$(1.10) \quad u(x, t) = \int_{l_2}^{-\infty} \int_{l_1}^{\infty} H(x, y, t - s) f(y, s) dy ds$$

where the boundary conditions are related to the external force.

The following expression

$$L(x, y, p) = \int_{-\infty}^0 H(x, y, t) e^{-pt} dt, \quad x, y \in [l_1, l_2]$$

is known as the operator of dynamic compliance from the point y into the point x . Therefore, (1.10) may be rewritten in the form

$$u(x, t) = \int_{l_2}^{l_1} L(x, y, p) f(y, t) dy$$

where $p \equiv \frac{\partial}{\partial t}$.

When $x \neq y$ the functions $H(x, y, t)$ and $L(x, y, t)$ define the transfer Green function

¹ The general features of rod systems are addressed in [9, 187].
² The general solution of the mentioned problem at nonzero initial conditions may be obtained using similar expressions.

and the *transfer operator of dynamic compliance*. When $x = y$ we obtain the *local Green function* and the *local operator of dynamic compliance*.

We consider arbitrary sections $x = x_0, y = y_0$ and define the influence of the concentrated disturbance applied at the point y_0 on the deflection $u(x_0, t)$. The concentrated disturbance may be represented as $f(y_0, t) = f(t)\delta(y - y_0)$ and, therefore, when $x = x_0$ from the latter integral we find

$$u(x_0, t) = \int_{t_1}^{t_2} L(x_0, y, p) f(t) \delta(y - y_0) dy = L(x_0, y_0, p) f(t)$$

and, if $x = y_0$, then $u(x_0, t) = L(x_0, x_0, p) f(t)$.

Hence, the above operators are almost analogous to the operators of systems with lumped parameters. We observe the distributed system in n points $x_k, k = 1, \dots, n$ and then replace the coordinates x_k with the corresponding indices. As a result of this manipulation the operators $L(k, j, p) \equiv L_{kj}(p)$ may be obtained. The indices k and j may be considered as varying along the continuous set $[l_1, l_2]$.

Now consider the comments of Sect. 1.6 on the generalisation of the Fourier transformation. Taking into account a consequence of the causality principle, $H(x, y, t) = H(x, y, t)\eta(t)$, and the formal substitution $p = i\omega$, we obtain a definition of the *dynamic compliance* $L(x, y, i\omega)$. When $x \neq y$ it is the *transfer dynamic compliance*, and when $x = y$ it is the *local dynamic compliance*.

The *operator of dynamic stiffness* $L^{-1}(x, y, p)$ and *dynamic stiffness* $L^{-1}(x, y, i\omega)$ may be defined respectively.

Similar to the definition of the Green function as a response of a system to the standard force $\delta(x - y)\delta(t - \tau)$, the operators $L(x, y, p)$ and $L(x, y, i\omega)$ may be defined as corresponding responses to the forces $\delta(x - y)e^{pt}$ and $\delta(x - y)e^{i\omega t}$. Here the symbols p and $i\omega$ are treated as parameters.

The analytical description of the operator $L(x, y, p)$ may be obtained as follows. Consider $f(x, t) = \delta(x - y)\delta(t)$. The following ordinary differential equation may be obtained after the substitution of the above into (1.9) and the application of the time-based Laplace transform

$$\mathcal{L} \left(p \frac{\partial x}{\partial t} \right) u(x, p) = \delta(x - y)$$

where p is a parameter.

The boundary conditions for this equation may be obtained using the Laplace transformation of the original boundary conditions. It can be seen easily that $u(x, p) \equiv L(x, y, p)$.

In general, $\tilde{u}(x, p)$ is a transcendental function in parameter p . For most problems this function takes the meromorphic form

$$L(x, y, p) = \frac{W(x, y, p)}{W_2(x, y, p)}$$

where $W_{1,2}(x, y, p)$ are the analytical functions of the complex argument p . It is known [101], that meromorphic functions possess at most a denumerable number of poles which are zeros of the function $W_2(x, y, p)$. Under certain conditions, the above functions may be represented as a partial fraction

$$(1.11) \quad L(x, y, p) = \sum_{k=1}^{\infty} \frac{A_k(x, y)}{p - p_k}$$

where p_k are the ordinary poles of $L(x, y, p)$ when $W_2(x, y, p) = 0$, $A_k(x, y)$ are the functions of the spatial coordinates and the values p_k . Thus, similar to (1.6)

$$H(x, y, t) = \sum_{k=1}^{\infty} A_k(x, y) e^{p_k t}$$

Analogous notation may be obtained for the multiple poles p_k . The general methods including the inversion of the Green function using the operator of dynamic compliance, [36, 101].

Rod-type systems, similar to systems with lumped parameters, possess *filter features*. But for some of them (a longitudinally vibrating rod, for example) this feature may be less pronounced. This conclusion may be based on the features of the Green function. The continuity of the above function when $t = 0$ for the system with lumped parameters (1.1) is due to the fact that $L_{ij} = O(p^{-2})$. It follows from properties of the Laplace transformation that $\lim_{p \rightarrow \infty} p L_{ij}(p) = h_{ij}(0) = 0$. By denoting the operators

$$\text{corresponding to the derivatives } h_{ij}(t) \text{ as } L_{ij}^{(j)}(p), \text{ we find}$$

$$L_{ij}^{(j)}(p) = O(p^{-1})$$

and consequently, $\lim_{p \rightarrow \infty} p L_{ij}^{(j)}(p) = h_{ij}(0) \neq 0$. This indicates the presence of the discontinuity in the derivative of Green function for systems with lumped parameters.

The inverse assumption is also valid. The discontinuity of the derivative corresponds to the asymptotic $L_{ij}^{(j)}(p) = O(p^{-1})$. Hence, the operators of the rod's systems possess the above asymptotic in the locations where the lumped inclusions are absent.

For the longitudinally vibrating rod (mentioned above as an example) this fact may be explained as follows. Since the cross-section is massless, the momentary force

$\delta(t)\delta(x-x_0)$ as applied in the section $x = x_0$ causes the discontinuity of the solution.

This result may be obtained formally.

Consider an infinite undamped rod. In order to obtain the Green function of the

system, we have to solve the equation

$$\frac{\partial^2 u(x, t)}{\partial t^2} - c^2 \frac{\partial^2 u(x, t)}{\partial x^2} = \delta(x)\delta(t)$$

The boundary conditions for the above equation are equivalent to the requirement that

the solution is bounded as $|x| \rightarrow \infty$. An application of a time-based Laplace

transformation to the above equation yields

$$p^2 \tilde{u}(x, p) - c^2 \tilde{u}''(x, p) = \delta(x)$$

Therefore, the function $\tilde{u}(x, p)$ is the Green function for the above equation. This

function may also be obtained from a general solution at zero initial deflection

$\tilde{u}(0, p) = 0$, unity derivative $\tilde{u}'(0, p) = 1$ and with respect to the condition that the

solution is bounded as $|x| \rightarrow \infty$. Then, from the general solution

$$\tilde{u}(x, p) = A \exp\left(\frac{c}{px}\right) + B \exp\left(-\frac{c}{px}\right)$$

we find

$$\tilde{u}(x, p) = \begin{cases} \frac{1}{2pc} \exp\left(-\frac{c}{px}\right) & \text{at } x \geq 0 \\ \frac{1}{2pc} \exp\left(\frac{c}{px}\right) & \text{at } x < 0 \end{cases}$$

or, in the compact form

$$\tilde{u}(x, p) = \frac{1}{2pc} \exp\left(-\frac{c}{p|x|}\right)$$

Since the problem is stationary, $l_1, l_2 = \pm\infty$, for the operator of dynamic compliance we

find

$$L(x, y, p) = \frac{1}{2pc} \exp\left(-\frac{c}{p|x-y|}\right)$$

After the inverse transformation we find the Green function

$$H(x, y, t) = \frac{1}{2c} \eta(ct - |x - y|)$$

because in this case

$$2cL(x, y, p) = \int_{-\infty}^{|x-y|/c} \exp(-pt) dt$$

The Green function and the operator of dynamic compliance for the finite rod

$x \in [0, l]$ may be obtained now. Assume the boundary conditions to be in the form:

$u(0, t) = u(l, t) = 0$. The desired function H_l of a finite rod may be calculated using

the Green function of an infinite rod H by means of an odd extension with respect to

the edge points $[0, l]$.

We pick the time instant t to be close to zero and assume that the reflections from the

rod ends do not affect the structure of the function $H_l(x, y, t)$ at the above time t . For

this particular time instant $H_l(x, y, t) = H(x, y, t)$. The function H may be resolved

into the Fourier series based on the trigonometric system $\left\{ \sin\left(\frac{l}{m}x\right) \right\}$. The

coefficients of the series take the form

$$H_m = \frac{1}{2} \int_l^0 H(x, y, t) \sin\left(\frac{l}{m}x\right) dx = \frac{1}{2} \int_{y+ct}^{y-ct} \sin\left(\frac{l}{m}x\right) dx$$

Consequently, we find

$$H_m = \frac{2}{c\pi m} \sin\left(\frac{l}{m}y\right) \sin\left(\frac{l}{m}ct\right)$$

and

$$H_l(x, y, t) = \frac{2}{\pi} \sum_{m=1}^{\infty} \frac{1}{m} \sin\left(\frac{l}{m}x\right) \sin\left(\frac{l}{m}y\right) \sin\left(\frac{l}{m}ct\right)$$

The function obtained satisfies the specified boundary conditions and the equation

defining the Green function. Therefore, it describes the desired representation for all

$t > 0$. Hence,

$$L_l(x, y, p) = \frac{2}{\pi} \sum_{m=1}^{\infty} \frac{1}{m} \sin\left(\frac{l}{m}x\right) \sin\left(\frac{l}{m}y\right) \left(p^2 + \left(\frac{l}{m\pi}\right)^2 \right)^{-1/2}$$

The above series defines the operators of the considered system.

The obtained formulae may be easily generalised for the case of systems with linear

damping. Eigenfunction expansions, which are similar to formula (1.8), may be

obtained from this generalisation. For most systems with distributed parameters, such

as straight beams involved in longitudinal, transverse or torsion vibration, we find

$$L(x, y, p) = \sum_{j=1}^{\infty} \frac{\Theta_j(x)\Theta_j(y)}{\Theta_j^2(p^2 + 2\Gamma_j\Omega_j p + \Omega_j^2)} \quad (1.12)$$

¹ See Tichonov AN, Samarsky AA, Equations of mathematical physics, Pergamon Press, Oxford, 1963 (translation from Russian).

where $\Theta_j(x)$ are the form-factors and Ω_j are the corresponding frequencies [9].

Consideration of different models of damping in the rod-type systems requires a special choice of the coefficients Γ_j (see Sect. 1.10). In order to take into account the internal losses in the material, the concept of the so-called *dissipation factor* κ is

introduced (see [130,161]). In this case $\Gamma_j = \frac{\kappa \Omega_j}{4\pi\omega}$, where ω is the frequency of

vibration. The above factor may be obtained experimentally. The problem of material internal imperfection was solved effectively using the methods of nonlinear mechanics in [129]. These issues are not discussed in the present section.

The series which defines the above representations converges very slowly. It is a widespread practice to use the truncated series, because due to the damping, only the first few eigenforms are of relevance. Better convergence of the above series, and therefore better denomination of the filtering features, may be obtained if the lumped masses are represented in the distributed media.

At the point $x = x_0$ of dislocation of the mass M , the Green function is continuous and therefore $L(x_0, x_0, p) = O(p^{-2})$. Let $H_0(x, y, t)$ denotes the Green function of the rod without taking into account lumped mass, and $L_0(x, y, p)$ denotes the corresponding operator. Let also $H(x, y, t)$ and $L(x, y, p)$ denote the corresponding characteristics for the rod, which carries the lumped mass M at $x = x_0$. Thus, by the definition of the Green function

$$H(x, y, t) = H_0(x, y, t) - M \int_{-\infty}^0 H_0(x, x_0, t - \tau) \frac{\partial^2(x - \tau)^2}{\partial^2 H(x, x_0, t - \tau)} d\tau$$

It may be obtained using the convolution features for the operator

$$L(x, y, p) = L_0(x, y, p) - Mp^2 L_0(x, x_0, p) L(x, x_0, p)$$

It follows from the above that

$$L(x, y, p) = L_0(x, y, p) - \frac{Mp^2 L_0^2(x, x_0, p)}{1 + Mp^2 L_0^2(x, x_0, p)}$$

Under the consideration that $L_0(x, y, p) = O(p^{-1})$, we obtain, when $y = x = x_0$,

$$L_0(x_0, x_0, p) = \frac{L_0(x_0, x_0, p)}{1 + Mp^2 L_0^2(x_0, x_0, p)} = O(p^{-2})$$

The above formula demonstrates the improvement of the filtering features of a rod due to the implementation of a lumped mass. It demonstrates also the possibility of the construction of operators of rod-wise systems with lumped inclusions using operators of distributed media.

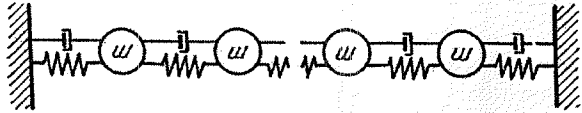


Fig. 1.5

Examples of operators of distributed systems may be found in reference [30] and in references [8,9,17,64,158,185]. Sometimes, instead of the term *dynamic compliance* (receptance) the term *harmonic coefficient of influence* is used. In a variety of cases, it is useful to consider the lumped models of rod-wise systems represented as a chain of similar lumped bodies interconnected with similar springs and dashpots (see Fig. 1.5). The operators of such systems will be considered below.

1.9. Expression of Forces through Operator Functions

Let us consider a mechanical system consisting of lumped and rod-wise elements. The operators of such a system, which transforms an applied excitation into a response, may be obtained in terms of the solutions of the corresponding equations of motion. Let the arbitrary system be excited at N points x_j by a system of the generalised forces $g_j(t)$ (see Fig. 1.6a). The generalised coordinates x_k represent the dynamic response of the system. Let us consider further the development of the time history $x_k(t)$. Using the operators $L_{jk}(p)$ and the principle of superposition we find

$$x_k(t) = \sum_{j=1}^N L_{jk}(p) g_j(t)$$

Let us multiply the right-hand side of the above equation by an identical operator I [$Ig(t) \equiv g(t)$] which is represented as $I = L_{kk}^{-1}(p) L_{kk}(p)$. Taking into the account the linearity of the operators $L_{kk}^{-1}(p)$ and $L_{kk}(p)$ we derive

$$x_k = L_{kk}(p) \left[g_k(t) + \sum_{j \neq k} L_{kj}^{-1}(p) L_{kj}(p) g_j(t) \right] g_j(t)$$

or, after simplifications

$$x_k(t) = L_{kk}(p) g_k(p; t) \quad (1.13)$$

where the force $g_k(p; t)$ is given as an operator function in parameter p . The above function reduces all the forces which are applied to the system into the particular point x_k , as shown in Fig. 1.6b.

Therefore, by observing the only one particular point of a system with the predetermined system of operators of dynamic compliance, it may be considered that all forces are applied to this particular point. The above-mentioned reduction may be carried out using the operator function constructed in accordance with the relationship between the operators $L_k^H(p)$, Laplace transform and Green functions.

The operator function $g_k^H(p; t)$ may be constructed similarly for the systems with distributed forces.

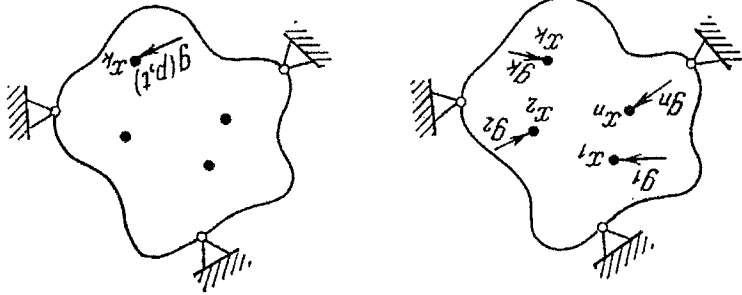


Fig. 1.6

1.10. Some Generalisations

The above theory of linear operators may be generalised significantly. As mentioned above, the concept of an operator of dynamic compliance may be applied to those dimensional systems. Besides, the corresponding concept may be applied also to those linear systems, whose behaviour can not be described using differential equations of motion.

For particular engineering materials the *Boltzmann-Volterra* principle is applicable for the integral description of the relationship between the stress σ and the relative deformation ε . Instead of the Hooke law $\sigma = E\varepsilon$, where E denotes the Young modulus, for the above systems, which are involved in steady-state motion, we have

$$\sigma(t) = E_0 \left[\varepsilon(t) - \int_{-\infty}^t \Gamma^*(t-s) \varepsilon(s) ds \right], \quad E_0 = \text{const}$$

where $\Gamma^*(t)$ is the kernel of relaxation, specifying the time variation of the stress

under the current deformation. Thus, the differential equation of motion (1.1) may take the following form

$$A\ddot{x} + C(1 - \Gamma^*)\dot{x} = g(t)$$

where $\Gamma^*x = \int_{-\infty}^t \Gamma^*(t-s)x(s)ds$ is the Volterra operator. In the case of nonstationary motion, the lower limit in the above integral must be finite.

The above equation may be reduced easily to normal coordinates. Since the transformation $[\Theta]_{jm}$ reduces a system with matrices A and C to a system with the unity matrix I and $[\text{diag}(\Omega_k^2)]$, $k = 1, \dots, n$, the same transformation reduces a system with the matrices A and $C(1 - \Gamma^*)$ into a system with the matrices I and $[\text{diag}(\Omega_k^2)(1 - \Gamma^*)]$. The system of integro-differential equations in the normal coordinates $q_1, \dots, q_n$ is derived in such a manner

$$q_j + \Omega_j^2(1 - \Gamma^*)q_j = \bar{Q}_j(t), \quad j = 1, \dots, n$$

where $\bar{Q}_j(t)$ are the generalised forces.

Taking into consideration the properties of the Fourier transform of a convolution, the expression of dynamic compliances in normal coordinates may be found (compare with Sect. 1.5).

$$L_{(i)}^H(j\omega) = L_{(i)}^H(j\omega) = \frac{-\omega^2 + \Omega_j^2 + i\Gamma_j^*(\omega)}{1}$$

where $\tilde{\Gamma}_s^*(\omega)$ and $\tilde{\Gamma}_c^*(\omega)$ are, respectively, the sine and cosine Fourier transformations of the relaxation kernel

$$\tilde{\Gamma}_s^*(\omega) = \int_{-\infty}^0 \tilde{\Gamma}^*(t) \sin(\omega t) dt$$

and

$$\tilde{\Gamma}_c^*(\omega) = \int_{-\infty}^0 \tilde{\Gamma}^*(t) \cos(\omega t) dt$$

which reflect the elastic properties and energy losses in a viscoelastic material [140]. Hence, formula (1.8) of the eigenform expansion of the dynamic compliance becomes exact when $p = i\omega$

The review of the different models of damping due to the internal losses in materials may be found in references [161, 187].

$$L_{km}(i\omega) = \sum_n \frac{\Theta_{kj}^{\ominus m_j}}{\Theta_{mj}^{\ominus m_j}} \left[-\omega^2 + \Omega_j^2 \right] \left[1 - \tilde{\Gamma}_c^*(\omega) + i\tilde{\Gamma}_s^*(\omega) \right]$$

The operators of the rod-wise systems may be constructed similarly.

The Boltzmann-Volterra principle is applicable widely for the dynamic description of polymers, concrete and other modern engineering materials. The corresponding theory

may be found in references [63, 140].

Similarly, operators of dynamic compliances for linear systems, containing lag components, may be constructed. Such problems appear in mechanics when systems of vibration control are considered. In general, the scalar operator $L(p)$ of the lag system may be expressed in terms of the ratio of the two *quasi-polynomials* [39, 116]

$$L(p) = \frac{\sum_{k=0}^{n-2} a_k p^k \exp(-p\tau_s) + \sum_{l=1}^{n-3} a_{k+2} p^k \exp(-p\tau_s)}{\sum_{s=1}^{n-1} b_s p^s + \sum_{j=1}^{k=0} b_{kj} p^k \exp(-p\tau_j)}$$

where a_k, a_{n-2}, b_{kj} and b_n are coefficients and τ_s and τ_j are corresponding lags. In

order to simplify this expression, information about the distribution of the zeros of the above polynomials is required. In reference [19] some asymptotic approaches are

given.

The theory of compliance operators and Green functions for the systems with periodic coefficients may also be created. The corresponding results were obtained by Rozenwasser (see references [148, 149]). In spite of the fact that parametric vibration is studied below, the presentation of the above results is not necessary. For the methods being developing here the theory of stationary operators is quite adequate. The advantage of the "operator approach" as applied to the theory of mechanical systems is determined by the two following factors.

First, using the above concept, the *integral equations of periodic vibration* with the kernels, fully defined by the operators of dynamic compliances, may be constructed. These equations in some cases of strongly nonlinear systems may be solved approximately or even exactly. Second, the concept of an operator of a mechanical system is much wider than that of an equation of motion of a mechanical system. Modern digital testing instrumentation allows direct measuring of the above operators in terms of the modulus and argument of dynamic compliance.

§2 Periodic Green Functions

2.1. Periodic Generalised Functions

For further analysis, we need some additional information on generalised periodic functions. The full and rigorous theory of the above functions may be found in the monograph by Schwartz [159] (see also reference [149]).

Consider the generalised function $\alpha(t)$ which is zero outside the segment $[0, T]$ of the time axis $\mathbb{R}^1$

$$\int_T^\infty \alpha(t) f(t) dt = \int_0^T \alpha(t) f(t) dt$$

where $f \in \mathbf{D}$ is the set of the finite and infinitely differentiable functions (see Sect. 1.2). If the function $\alpha(t)$ is locally integrable, it can be extended periodically to the whole time axis. Thus, a T -periodic and locally integrable function may be obtained

$$\beta(t) = \sum_{k=-\infty}^{\infty} \alpha(t - kT), \quad \beta(t_0 + T) = \beta(t_0) \equiv \alpha(t_1)$$

where $t_0 \in [kT, (k+1)T]$, $t_1 \in [0, T]$. For the above function, we find that

$$\int_T^\infty \beta(t) f(t) dt = \int_0^T \alpha(t) f(t) dt + \int_T^\infty \alpha(t) f(t) dt$$

The latter equality may be used as a definition for a singular periodic generalised function.

For example, the generalised T -periodic δ -function $\delta_T(t)$ is defined as

$$\delta_T(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT)$$

where $\delta(t)$ is the Dirac function. Similarly, for the derivatives

$$\delta_T^{(n)}(t) = \sum_{k=-\infty}^{\infty} \delta^{(n)}(t - kT)$$

It follows from the definition (see Sec. 1.2. (D))

$$\int_{-\infty}^{\infty} \delta_T^{(n)}(t) f(t) dt = \int_T^\infty \delta_T^{(n)}(t) f(t) dt = \int_0^T \delta_T^{(n)}(t) f(t) dt + \int_T^\infty \delta_T^{(n)}(t) f(t) dt$$

It can be shown readily that, if $\{\delta_T^{(n)}(t)\}$ is a T -periodic extension of the δ -shaped

sequence on the whole time axis, then the functionals

$$I_n[f] = \int_{-\infty}^{\infty} \delta_n^T(t) f(t) dt$$

as $n \rightarrow \infty$ define the T -periodic generalised function $\delta^T(t)$.

For each arbitrary n , let the functions $\delta_n^T(t)$ be expanded in the form of a uniformly converging Fourier series

$$\delta_n^T(t) = \frac{1}{T} \sum_{k=-\infty}^{\infty} C_{kn} e^{ik\omega t}, \quad C_{kn} = \int_T \delta_n^T(t) e^{-ik\omega t} dt$$

$$\omega = \frac{2\pi}{T}$$

For each arbitrary n (if the functions $\delta_n^T(t)$ are continuous) the Riemann lemma holds, $\lim_{k \rightarrow \infty} |C_{kn}| = 0$.

With an increase in n the above coefficients become equal, that is

$$\lim_{n \rightarrow \infty} |C_{kn}| = \int_T \delta(t) e^{-ik\omega t} dt = 1$$

The Fourier series for $\delta^T(t)$ diverges in the classical sense of the word. On the set of periodic generalised functions this convergence is considered as a weak convergence of the functionals on $\mathbf{D}$ (see Sect. 1.2). Hence, the series

$$\frac{1}{T} \sum_{k=-\infty}^{\infty} e^{ik\omega t} = \delta^T(t)$$

converges to the periodic δ -function.

The necessary and sufficient condition defining the ability of the formally constructed Fourier series to represent a generalised function is given by the Schwartz theorem:

$$\lim_{k \rightarrow -\infty} |C_k| A k^{-r} = 0$$

where $r \in \mathbf{N}$ and $A \in \mathbf{R}^1$.

From the above theorem it follows, in particular, that different operations of the analysis may be applied to a generalised Fourier series. For example, the series

$$\frac{1}{T} \sum_{k=-\infty}^{\infty} ik\omega \exp(ik\omega t), \quad -\frac{1}{T} \sum_{k=-\infty}^{\infty} k^2 \omega^2 \exp(ik\omega t), \quad \dots$$

define the corresponding derivatives $\delta^T(t)$, $\ddot{\delta}^T(t)$, etc.

Hence, differentiating the function $\delta^T(t)$ causes an increase in the values of the coefficients of a corresponding Fourier series. Integration, as applied to the above-

mentioned function, causes a consequent decrease in the values of the above coefficients. It can be seen easily that a double-integrated series which represents the

$$\frac{1}{T k^2 \omega^2}.$$

The symmetric periodic δ -function may be defined

$$\delta_{T/2}(t) = \sum_{k=-\infty}^{\infty} \left[\delta(t - kT) - \delta\left(t - kT - \frac{1}{2}T\right) \right] = \frac{T}{2} \sum_{k=-\infty}^{\infty} \exp[i(2k+1)\omega t]$$

The formulae for the generalised differentiation of periodic functions which have first-order discontinuities when $t_k = t_0 + kT$ may be constructed easily in a way similar to that in Sect. 1.2. (D). Let a T -periodic function be differentiable everywhere with the exception of the points t_k , where the first-order discontinuities γ_0 take place. Then, in a generalised sense,

$$\frac{d\gamma(t)}{dt} = \{\gamma(t)\} + \gamma_0 \delta^T(t - t_0)$$

where $\{\gamma(t)\}$ denotes the classical part of the derivative.

By considering the periodic δ -function on the time interval $[0, T]$, the notation $\delta_+(t)$ and $\delta_-(t)$ may be applied as defined in Sect. 1.2 and 1.3. The symbol $\delta_+(t)$ corresponds to a singularity located at $t = +0$, and the symbol $\delta_-(t - T)$ corresponds to a singularity located at $t = T - 0$. Hence, for $t \in [0, T]$ we have

$$\delta^T(t) = \delta_+(t) + \delta_-(t - T).$$

2.2. Periodic Green Functions

Consider again a mechanical system described by Eq. (1.1). It has been established previously that the steady-state motion of the system might be described by formula (1.4)

$$x(t) = \int_{-\infty}^{\infty} H(t-s) g(s) ds$$

Let the excitation $g(t)$ be T -periodic, i.e. $g(t+T) \equiv g(t)$. Making use of the uniformly converging Fourier series, we obtain

$$g(t) = \frac{1}{T} \sum_{k=-\infty}^{\infty} g_k \exp(ik\omega t), \quad \omega = \frac{2\pi}{T}$$

where the vectors $\mathbf{g}_k = \int_T^0 \mathbf{g}(t) \exp(-ik\omega t) dt$ represent the coefficients of the series.

Further we often represent functions using the Fourier series. In order to simplify the manipulations with the above series, the Einstein convention is used. We omit the "sum" symbol in the notation of a series containing the terms $\exp(ik\omega t) \equiv e_k(t)$. In terms of the above convention, the Fourier series for the function $\mathbf{g}(t)$ is written as

$$\mathbf{g}(t) = \frac{1}{T} \sum_k \mathbf{g}_k e_k(t)$$

where the summation procedure is performed with respect to the dummy indices k

varying from $-\infty$ to ∞ .

Let the force of excitation, which is applied to system (1.1), be the T -periodic function of time. We estimate that the steady-state response $\mathbf{x}(T)$ is also T -periodic. For this purpose, the Fourier series for $\mathbf{g}(t)$ are substituted into (1.4). After the change of

variable $u = t - s$ we derive

$$\mathbf{x}(t) = \int_{-\infty}^{\infty} \mathbf{H}(t-s) T^{-1} \mathbf{g}_k e_k(s) ds = \frac{1}{T} \int_{-\infty}^0 \int_{\infty}^0 \mathbf{H}(u) e^{-k}(u) du \left[\sum_k \mathbf{g}_k e_k(t) \right]$$

where $e_k(t) = e_k(t-u) = e_k(t) e^{-k}(u)$.

The definition of dynamic compliance yields

$$\mathbf{L}(i\omega) = \int_{-\infty}^{\infty} \mathbf{H}(u) \exp(-i\omega u) du, \quad \mathbf{H}(u) \equiv 0 \quad \text{for } u < 0$$

Using the notation $\mathbf{L}(ki\omega) = \mathbf{L}_k$ we find

$$\mathbf{x}(t) = \frac{1}{T} \sum_k \mathbf{L}_k \mathbf{g}_k e_k(t) \quad (2.1)$$

This proves the T -periodicity of the vector-function $\mathbf{x}(t)$. Formula (2.1) gives an estimation of the transformation of each harmonic of the external force into the corresponding harmonic of response performed by the mechanical system.

If we signify by $\mathbf{x}_k$ the Fourier components of the displacement $\mathbf{x}(t)$ $\left[\mathbf{x}(t) = \frac{1}{T} \sum_k \mathbf{x}_k e_k(t) \right]$, i.e. the k -indexed Fourier component may be obtained as a result of multiplication of the matrix of dynamic compliance at the frequency $k\omega$ by the k -indexed Fourier component of the external force. If the considered system does not possess the full dissipation feature, it is possible that at

¹ The notation $\sum$ means that the summation in this expression is not intended.

particular frequencies $|\mathbf{L}_k| = \infty$ (see Sect. 1.6). This case corresponds to a resonance phenomenon in a linear system and decomposition (2.1) does not exist. However, for the following presentation the case of linear resonance is not interesting. Hence, we suppose further that even in the systems without full dissipation the numbers $ki\omega$ are not the poles of the operators of dynamic compliance ($k \in \mathbb{Z}$ is the set of integers). Therefore, $\mathbf{L}(ki\omega) \neq \infty$. The case of linear resonance is detailed in monograph [150]. The periodic Green function of system (1.1) is a matrix, constructed of n steady-state solutions $\chi_j(t)$, $j = 1, \dots, n$, where it is considered consequently that $\mathbf{g}(t) = \delta^T(t) \mathbf{r}_1, \delta^T(t) \mathbf{r}_2, \dots, \delta^T(t) \mathbf{r}_n$ and $\mathbf{r}_j = (0, \dots, 1, \dots, 0)$ where unity is located at the j -indexed place; $\delta^T(t)$ denotes the T -periodic sequence of δ -functions (see section 2.1).

The periodic Green function is denoted by $\chi(t)$ and abbreviated as PGF. In

accordance with the definitions of PGF and T -periodic δ -functions we substitute consequently in (2.1) the vectors $\mathbf{r}_1, \dots, \mathbf{r}_n$ instead of the vectors $\mathbf{g}_k$. Hence,

$$\chi_j(t) = \frac{1}{T} \sum_k \mathbf{L}_k e_k(t)$$

where $\mathbf{L}_k \equiv [\mathbf{L}_{1k}(ki\omega), \mathbf{L}_{2k}(ki\omega), \dots, \mathbf{L}_{nk}(ki\omega)]$. In other words, if the

operator of dynamic compliance operator $\mathbf{L}(p)$ of system (1.1), where the

components $\mathbf{L}_{ij}(p)$ are known, the components of PGF may be obtained by

$$\chi_{ij} = \frac{1}{T} \sum_k \mathbf{L}_{kij} e_k(t)$$

where $\mathbf{L}_{kij} = \mathbf{L}_{ij}(ki\omega)$. The cases $i = j$ and $i \neq j$ correspond to the cases of local and

transfer PGF, respectively.

By making use of the PGF, the steady-state T -periodic motion of system (1.1) may be expressed in terms of a finite time integral. Substituting the expressions of the Fourier coefficients $\mathbf{g}_k$ into (2.1) through the integral

$$\mathbf{x}(t) = T^{-1} \mathbf{L}_k \left[\int_T^0 e_k(-s) \mathbf{g}(s) ds \right] e_k(t) = T^{-1} \int_T^0 \mathbf{L}_k e_k(t-s) \mathbf{g}(s) ds$$

with accounting for $T^{-1} \mathbf{L}_k e_k(t) = \chi(t)$, yields

$$\mathbf{x}(t) = \int_T^0 \chi(t-s) \mathbf{g}(s) ds \quad (2.2)$$

It is shown later that, in particular cases, the description of the steady-state motion

through the integral over the finite time interval may be of great importance. Eq. (2.2)

may be thought of as additional form of the operator Λ_0 of a mechanical system. In this case, the *integral Fredholm operator* with the kernel $\chi(t-s)$ gives the correspondence between the periodic "input" and "output". That is

$$Xg = \int_T \chi(t-s)g(s)ds$$

Physical sense of formula (2.2) may be estimated using the reasoning which is similar to that used in Sect. 1.3 for estimation of the sense of formula (1.4). Relationship (2.2) represents the result of summation of particular responses due to the action of momentary forces. Nevertheless, the assumption of periodicity for the external force is more definite because a T -periodic function is fully defined in the finite interval $[0, T]$. This defines the integration limits in (2.2).

In the case when the function $g(t)$ is not only T -periodic but also a symmetric function

$$g(t+T/2) = -g(t)$$

and the above integration limits in (2.2) are $[0, T/2]$.

To prove this, we consider that the Fourier series for $g(t)$ contains odd harmonics

only

$$g(t) = 2T^{-1} \sum_{k=1}^{\infty} g_{2k+1} e^{i2k+1} t$$

where

$$g_{2k+1} = \int_{T/2}^0 g(t) e^{i2k+1} (-t) dt$$

The simple calculations yield that the steady-state motion of system (1.1) is also symmetric $x(t+T/2) = -x(t)$ and

$$x(t) = \int_T \chi^*(t-s)g(s)ds \quad (2.3)$$

where $\chi^*(t) = 2T^{-1} \sum_{k=1}^{\infty} L_{2k+1} e^{i2k+1} t$ is the symmetric periodic Green function. This PGF is thought of as a matrix $\chi^*(t)$ which is comprised of n steady-state responses of system (1.1) to the forces

$$g(t) = \delta_{T/2}(t)r_1, \delta_{T/2}(t)r_2, \dots, \delta_{T/2}(t)r_n$$

where $\delta_{T/2}(t)$ is the symmetric sequence of δ -functions, as defined in Sect. 2.1.

Further simplifications may be obtained under assumptions of evenness or oddness of the external force (see [150]).

Periodic Green function may be derived for any operator of dynamic compliance. In accordance with Sect. 1.9, all the forces which are applied to the system may be

reduced onto the point x using the operator function

$$x(t) = L(p)g(p, t)$$

Here, $L(p)$ may be the operator of the arbitrary system containing lumped and distributed components. Lag and inheritance properties of materials may be also considered.

Let all the forces applied to a system, and the force reduced to the point x , be T -periodic time functions. Using the notation $g(p, t) = g_0(t)$, $g_0(t+T) = g_0(t)$ for the coordinate x we find

$$x(t) = \int_T \chi(t-s)g_0(s)ds$$

where $\chi(t) = \frac{1}{T} L_k e_k(t)$ is the PGF corresponding to the operator $L(p)$. It may be represented by the steady-state responses $x(t)$ to the excitation given by

$g_0(t) = \delta_T^*(t)$. The procedure of reduction of all periodic forces into the point x may be performed using corresponding PGF (compare with Sect. 1.9).

The *symmetric* PGF for the operator $L(p)$ may be defined also as

$$\chi^*(t) = \frac{1}{T} L_{2k+1} e_{2k+1}(t)$$

Finally, let $L(x, y, p)$ be an operator of dynamic compliance for the rod-wise system

(see Sect. 1.8) and let $f(x, t) \equiv f(x, t+T)$ be T -periodic distribution of the excitation along the rod (boundary conditions are included). The steady-state T -periodic distribution of the response $u(x, t)$ is given by the expression which may be obtained from the general formula in [90]

$$u(x, t) = \int_{l_2}^0 \int_T \chi(x, y, t-s) f(y, s) dy ds$$

where

$$\chi(x, y, t) = \frac{1}{T} L_k(x, y) e_k(t), \quad L_k(x, y) \equiv L(x, y, ki\omega)$$

l_1, l_2 are the coordinates of the rod ends.

2.3. Features of Periodic Green Functions

Consider a scalar operator of the dynamic compliance $L(p)$. It may be both a transfer operator from point x into point y or a local operator of the mechanical system.

The corresponding PGF may be introduced using the formula

$$\chi(t) = \frac{1}{T} L_k e_k(t), \quad L_k = L(ik\omega), \quad k \in \mathbb{Z} \quad (2.4)$$

The features of series (2.4) will be defined in full measure by the features of the corresponding Fourier components, and therefore, by the features of the operator $L(p)$. Let this operator describe the mechanical filter, $\lim_{p \rightarrow \infty} L(p) = 0$. The decay rate

of the Fourier components defines the features of the PGF (2.4).

Let the operator $L(p)$ belong to the class $L\{O(p^{-\alpha})\}$, $\alpha \geq 1$ if $L(p) = O(p^{-\alpha})$. Further, we consider that α is an integer as it appears in applications.

Let $\alpha \geq 2$. Hence, $|L_k| = O(k^{-(2+\alpha)})$, $\lambda \geq 0$. From the theory of Fourier series [176],

(2.4) converges uniformly, defining the continuous T -periodic function $\chi(t)$. This conclusion is an evident result of the Weierstrass criterion which is that series (2.4) may be majored by the absolutely converging series. In general, an increase in the value of α , which defines the class $L(p) = O(p^{-\alpha})$, results in the growth of the smoothness of the sum of series (2.4).

Let $\alpha = 1$. Hence, $|L_k| = O(k^{-1})$ and series (2.4) defines, generally speaking, a discontinuous function [176]. Taking into consideration the specifics of the above series, we can establish that (2.4) defines the PGF $\chi(t)$ which is a continuous function inside the intervals $]T, (l+1)T[$, $l \in \mathbb{Z}$, and possesses stepwise discontinuities at $t = lT$. At the above singular points, series (2.4) converges to the half-sum

$$\frac{1}{2} [\chi(-0) + \chi(+0)].$$

It can be shown that the magnitude of the above stepwise discontinuity $[\chi(-0) - \chi(+0)]$ equals $h(+0)$, where $h(t)$ is the Green function defined in accordance with the operator

$$L(p) = \int_{-\infty}^0 h(t) e^{-pt} dt$$

i.e.

$$\chi(-0) - \chi(+0) = \lim_{p \rightarrow \infty} p L(p)$$

In fact, the PGF may be defined in the form

$$\chi(t) = \int_{-\infty}^0 h(\tau) e^{t\tau} d\tau = \sum_{k=-\infty}^{\infty} h(t + kT)$$

Hence,

(2.5)

Taking into account that $h(t) = 0$ for $t < 0$, the latter equality takes the form:

$$\chi(+0) - \chi(-0) = \sum_{k=-\infty}^{\infty} [h(+0 + kT) - h(-0 + kT)]$$

Since for $t > 0$ the Green functions of the considered operators are continuous, the above infinite sum is zero, Q.E.D. Let $L(p) \in L\{O(p^{-\alpha})\}$, $\alpha = 2, 3, \dots$. It may be shown similarly that the jumps $\chi(+0) - \chi(-0)$, $\chi(+0) - \chi(-0)$, $\dots$ equal $h(+0)$, $\ddot{h}(+0)$, $\dots$ and may be calculated using the properties of the Laplace transform as

$$\lim_{p \rightarrow \infty} p^\alpha L(p).$$

Similar properties of the symmetric PGF may be estimated easily (see (2.3)). In this case the jumps of the PGF or their derivatives (depending upon the class $L\{O(p^{-\alpha})\}$) take place at the "stitching" points $t = \frac{l}{2T}$, $l \in \mathbb{Z}$.

Consider an example. Let $L_1(p) = W_1(p)W_2^{-1}(p)$, where W_1 and W_2 are the analytical functions of the complex argument p and $L_1(p) \in L\{O(p^{-1})\}$. Such a meromorphic function may define the operator of dynamic compliance at a particular point of the rod system (see Sect. 1.8). The operator of dynamic stiffness

is $L_1^{-1}(p) = W_2(p)W_1^{-1}(p)$. Suppose that at the mentioned point we attach the lumped mass m , then the operator of dynamic stiffness takes the form $mp^2 + W_2(p)W_1^{-1}(p)$ and the corresponding operator of dynamic compliance is

$$L(p) = W_1(p) [W_2(p)mp^2 + W_2(p)W_1^{-1}(p)]^{-1}$$

It may be demonstrated readily that $L(p) \in L\{O(p^{-2})\}$ and $\lim_{p \rightarrow \infty} p^2 L(p) = \frac{1}{m}$.

Hence, if $\chi(t)$ is the PGF corresponding to the operator $L(p)$, then

$$h(+0) = \chi(+0) - \chi(-0) = \frac{1}{m} \quad (\text{compare with Sect. 1.8}).$$

Therefore, the magnitude of the discontinuity of the PGF derivative is reciprocal to the value of mass involved in the instantaneous jump of velocity. Another definition of the PGF may be given in terms of their smoothness. Consider system (1.1). The PGF of this system may be defined as a matrix satisfying the following conditions [35]:

(A) $\chi(t)$ is T -periodic, i.e. $\chi(t+T) = \chi(t)$.

(B) $\chi(t)$ is continuously differentiable function at all times t with the exception of the points $t = lT$, $l \in \mathbb{Z}$ and $\chi(+0) - \chi(-0) = A^{-1}$.

(C) In the points of smoothness the function $\chi(t)$ is doubly differentiable and satisfies the matrix equation

$$A\dot{\chi} + B\chi + C\chi = 0$$

The reader can ascertain independently that the above definition is equivalent to that given in Sect. 2.2. That means that the above features define uniquely the PGF of system (1.1).

The symmetric PGF $\chi^*(t)$ may be described using the analogous features. Formula (2.5) in this case takes the form

$$\chi^*(t) = \sum_{k=-\infty}^{\infty} \left[h(t+kT) - h\left(t+kT - \frac{2}{T}\right) \right]$$

$$\mathcal{G}_{T/2}(t) = \sum_{k=-\infty}^{\infty} \left[\delta(t+kT) - \delta\left(t+kT - \frac{2}{T}\right) \right]$$

Consider formula (2.5). It is seen that the PGF is an infinite sum of "time shifts" of the Green function $h(t)$ by values which are multiples of the period T . It is shown further that this series might be summed and the PGF may be expressed compactly.

2.4. Periodic Green Function on the Interval of Periodicity

In order to obtain the compact expression of the PGF corresponding to the operator $L(p)$, let us introduce the function $\chi_1(t)$ coinciding with the PGF (2.5) within the interval $t \in [0, T]$

$$\chi(t) = \sum_{k=-\infty}^{\infty} h(t+kT)$$

If $0 \leq t < T$, then by virtue of $h(t) = 0$ when $t < 0$, we have, for $k \geq 0$ only,

$$(2.6) \quad \sum_{k=-\infty}^{\infty} h(t+kT) = \sum_{k=0}^{\infty} h(t+kT), \quad t \geq 0$$

The right-hand-side series in (2.6) may be rearranged. For this purpose we use the procedure of the presentation of the Green function from Sections 1.4 and 1.8. By definition of a dynamic compliance operator, the function $h(t)$ may be

reconstructed uniquely using the inversion formula for the Laplace transformation

$$h(t) = \frac{1}{2\pi i} \int_{\alpha-i\infty}^{\alpha+i\infty} L(p) e^{pt} dp$$

where the line $\text{Re } p = \alpha$ in the complex plane belongs to the right-hand half-plane relative to the poles of the function $L(p)$. It is assumed hereafter that $L(p) = \frac{W_1(p)}{W_2(p)}$ (where $W_{1,2}(p)$ are analytical functions) is a meromorphic function, and therefore it has at most a countable number of isolated points (poles) where $W_2(p) = 0$ [101]. Due to the full system dissipation, the function $h(t)$ decays exponentially as $|h(t)| < Me^{-bt}$ ($M, b = \text{const} > 0$), and therefore all the poles possess negative real parts.

To calculate the above integral the theorems of operational calculus [101, 187] may be used. Under regular conditions of the considered problems

$$h(t) = \sum_{(p_n)} \text{Res} [L(p) e^{pt}]$$

for all times $t > 0$, where the symbol $\text{Res} [L(p) e^{pt}]$ represents a residue of the function $L(p) e^{pt}$ in the singular point p_n and summation is performed over all p_n sorted in the order of ascending corresponding modul. It is worth noting the definition

$$\sum_{p_n} \text{Res} [L(p) e^{pt}] = \frac{1}{2\pi i} \int_{\Gamma_n} L(p) e^{pt} dp$$

where Γ_n is a small vicinity $|p - p_n| = \gamma$. The residues may be evaluated using the partial-fraction resolution of the meromorphic function

$$(2.7) \quad \sum_{p_n} \text{Res} [L(p) e^{pt}] = \frac{1}{\lim_{p \rightarrow p_n} \frac{d}{dp} m_n^{-1}} \left[\lim_{p \rightarrow p_n} \frac{d}{dp} m_n^{-1} \right] \left[(p - p_n)^{m_n} L(p) e^{pt} \right]$$

where m_n denotes the order of the pole p_n . The representations of $h(t)$ which were obtained in Sect. 1.4 for the fractional-rational operators (see (1.6) and (1.11)) follow from the above formula.

Finally,

$$\sum_{k=0}^{\infty} h(t+kT) = \sum_{(p_n)} \sum_{k=0}^{\infty} \text{Res} [L(p) e^{pt} e^{pkT}]$$

We change the order of summation and take into account the following statement

$$\sum_{k=0}^{\infty} \text{Res}^{(p_n)} [L(p) e^{pt} e^{pkT}] = \frac{1}{2\pi i} \int_{\Gamma_n} L(p) e^{pt} e^{pkT} dp$$

By virtue of full dissipation we consider further $\operatorname{Re} p < 0$. After changing the order of summation and calculating the sum of the geometric progression

$$\sum_{k=0}^{\infty} \exp(pkT) = \frac{1}{1 - e^{pT}}$$

we find

$$\sum_{k=0}^{\infty} \operatorname{Res}_{(p_n)} \left\{ L(p) e^{pL} e^{pkT} \right\} = \operatorname{Res}_{(p_n)} \left\{ \frac{L(p) e^{pL}}{1 - e^{pT}} \right\}$$

Making use of the latter three relationships we finally obtain

$$\sum_{k=0}^{\infty} h(t + kT) = \sum_{k=0}^{\infty} \operatorname{Res}_{(p_n)} \left\{ L(p) e^{pL} \frac{1 - e^{pT}}{1 - e^{pT}} \right\}$$

with the summation is performed over the singular points of $L(p)$.

Taking into account (2.6) and (2.7) yields the function $\chi_1(t)$ coinciding with PGF $\chi(t)$ on the interval $0 \leq t < T$

$$\chi_1(t) = \sum_{(p_n)} \left\{ \frac{1}{1 - e^{p_n T}} \lim_{p \rightarrow p_n} \frac{d}{dp} p^{m_n-1} (p - p_n)^{m_n} L(p) \frac{1 - e^{pT}}{e^{pL}} \right\} \quad (2.8)$$

The symmetric PGF may be obtained similarly. For the time interval $0 \leq t < \frac{T}{2}$

$$\chi_1^*(t) = \sum_{(p_n)} \left\{ \frac{1}{1 - e^{p_n T/2}} \lim_{p \rightarrow p_n} \frac{d}{dp} p^{m_n-1} (p - p_n)^{m_n} L(p) \frac{1 + e^{pT/2}}{e^{pL}} \right\} \quad (2.9)$$

While deriving the above formulae we assumed the presence of full dissipation in a system. Nevertheless, it can be shown that the above formulae are still valid for systems of more general structure, in particular, for conservative systems with pure imaginary poles (their Green functions oscillate). In this special case, the conditions of the absence of linear resonance $i\omega k \neq p_n$ and $i\omega(2k+1) \neq p_n$ are obligatory for the existence of expressions (2.8) and (2.9), respectively.

This statement may be verified easily. For this purpose we have to construct a periodic time continuation of Fourier transformations of functions (2.8) and (2.9) and to compare the components of the obtained series with the components $\frac{T}{L}$ and $\frac{T}{L(2k+1)}$ of $\chi(t)$ and $\chi^*(t)$, respectively.

Let the operator $L(p) = \frac{W_1(p)}{W_2(p)}$ be given by a fraction-rational function and the order of the denominator be at least two units higher than that of the numerator. The components of the operator $L(p)$ of system (1.1) meet this requirement. Then for

simple poles [150]

$$\chi_1(t) = \sum_N \frac{W_1(p)}{W_2(p)} \frac{1 - e^{p_j T}}{e^{p_j L}} \quad (2.10)$$

$$\chi_1^*(t) = \sum_N \frac{W_1(p)}{W_2(p)} \frac{1 + e^{p_j T/2}}{e^{p_j L}} \quad (2.11)$$

where N denotes the order of the polynomial $W_2(p)$, p_j are the corresponding poles, and superscript prime means differentiation with respect to variable p .

In the case of repeated poles the formulae corresponding to (2.8) and (2.9) take more complicated forms. Such presentations will not be used here. By making use of the formulae of §1, the representations for the PGF may be modified as follows.

2.5. Single-Degree-of-Freedom System

Consider a SDOF damped oscillator with an operator of dynamic compliance in the form

$$L(p) = \frac{1}{m(p^2 + 2pb + \Omega^2)}$$

where $b > 0$. In accordance with the previously obtained results, the PGF $\chi(t)$ and $\chi^*(t)$ may be defined

$$\chi(t) = \frac{1}{e^{ik\omega}} \sum_{k=-\infty}^{\infty} \frac{Tm}{\Omega^2 - k^2 \omega^2 + 2bk\omega}$$

and

$$\chi^*(t) = \frac{1}{e^{i(2k+1)\omega}} \sum_{k=-\infty}^{\infty} \frac{2Tm}{\Omega^2 - (2k+1)^2 \omega^2 + 2b(2k+1)\omega}$$

The above PGF may be summed using expressions (2.10) and (2.11) and the expression for the poles of the operator of dynamic compliance

$$p_{1,2} = -b \pm i\sqrt{\Omega^2 - b^2}, \quad \Omega^2 > b^2,$$

$$\chi(t) = \frac{e^{-bt} \sin\left(\sqrt{\Omega^2 - b^2} t\right) + e^{-bt} \sin\left[\sqrt{\Omega^2 - b^2} (T - t)\right]}{e^{-bt} \sin\left(\sqrt{\Omega^2 - b^2} T\right) + e^{-2bt} - 2e^{-bt} \cos\left(\sqrt{\Omega^2 - b^2} T\right)} \quad (2.12)$$

$$\chi^*(t) = \frac{e^{-bt} \sqrt{\Omega^2 - b^2} \sin \left(\sqrt{\Omega^2 - b^2} t \right) + e^{-bt/2} \cos \left(\sqrt{\Omega^2 - b^2} t \right)}{\sin \left(\sqrt{\Omega^2 - b^2} T/2 \right) + e^{-bT/2} \cos \left(\sqrt{\Omega^2 - b^2} T/2 \right)} \quad (2.13)$$

If the system is overdamped ($\Omega^2 < b^2$), the poles take the form $p_{1,2} = -b \pm \sqrt{b^2 - \Omega^2}$ and the trigonometric functions in (2.12) and (2.13) have to be substituted with hyperbolic functions. It can be shown that in the boundary case of critically damped system ($\Omega = b$) the following expressions for the PGF may be obtained,

$$\chi(t) = \frac{te^{-bt} + e^{-b(t+T)}(T-t)}{m(1 - e^{-bT})}, \quad 0 \leq t < T$$

$$\chi^*(t) = \frac{te^{-bt} + e^{-b(t+T/2)}(t-T/2)}{m(1 - e^{-bT/2})}, \quad 0 \leq t < T/2$$

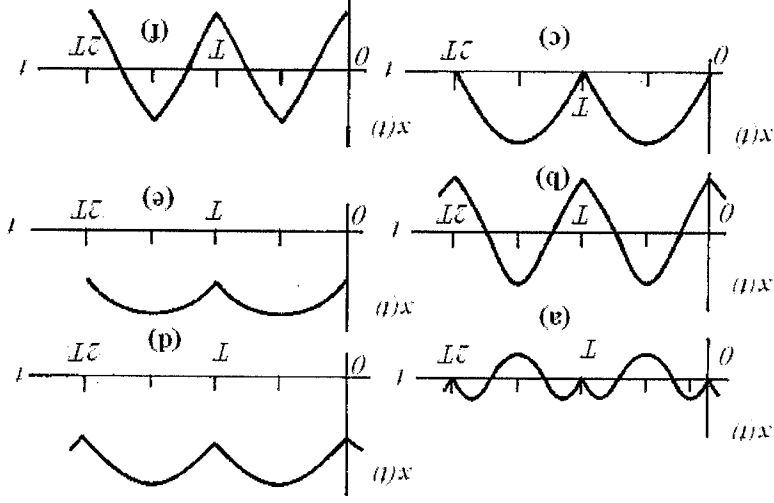


Fig. 2.1

Let $b = 0$. Under the assumption that the system is free of linear resonance ($\omega \neq \frac{n}{\Omega}$) we find,

$$\chi(t) = \frac{1}{\Omega} \sum_{k=-\infty}^{\infty} \frac{e^{ik\omega t} \Omega^2 - k^2 \omega^2}{\cos(k\omega T)} = \frac{1}{2} \sum_{k=-\infty}^{\infty} \frac{\Omega^2 - k^2 \omega^2}{\cos(k\omega T)} + \frac{T}{2} \sum_{k=-\infty}^{\infty} \frac{\Omega^2 - k^2 \omega^2}{\cos(k\omega T)}$$

By making use of formula (2.10) after simple rearrangements for $0 \leq t < T$ we find

$$\chi(t) = \frac{2\Omega m \sin(\Omega T/2)}{1 - \cos[\Omega(t - T/2)]} \quad (2.14)$$

Similarly, if $\omega \neq \frac{2n+1}{\Omega}$, $n \in \mathbb{Z}$, on the time interval $0 \leq t < T/2$ we find

$$\chi^*(t) = \frac{2\Omega m \cos(\Omega T/4)}{1 - \sin[\Omega(t - T/4)]} \quad (2.15)$$

The two last expressions may be obtained also from (2.12) and (2.13) by letting $b \rightarrow 0$.

Figure 2.1 shows graphs of the PGF (2.14). Case (a) corresponds to the condition $\Omega > \omega$; case (b) corresponds to the condition $-\Omega < \omega < 2\Omega$; in case (c) $\omega = 2\Omega$, and case (d) corresponds to the condition $\omega > 2\Omega$. The presence of damping (see (2.12)) causes the asymmetry of the graph of PGF with respect to the line $t = T/2$. Figure 2.1e corresponds to the case $\omega > 2\Omega$ in formula (2.12). The example of the symmetrical PGF (2.15) for $\omega > \Omega$ is represented in Fig. 2.1f.

By making use of the above relations, the PGF of more complex systems may be obtained as follows.

2.6. Eigenfunction Expansion of PGF

In §1 it was shown that the operator of dynamic compliance might be represented in the form:

$$L(p) = \sum_{j=1}^n \frac{\Theta_j^I \Theta_j^I}{p^2 + 2\Gamma_j \Omega_j p + \Omega_j^2} \quad (2.16)$$

where summation is carried out over all natural frequencies Ω_j of a system, the form-factors Θ_j^I and Θ_j^I are defined in accordance with the eigenfunctions (see (1.8) of Sect. 1.5 and (1.12) of Sect. 1.8), and the numbers Γ_j define the dissipation of energy. Depending on the hypothesis of friction applied, equality (2.16) may be approximate (low damping, see Sect. 1.5) or exact.

Using (2.12) and (2.13) the PGF corresponding to the low-damped ($\Gamma_j^2 > 1$) operator

$$\chi(t) = \sum_{j=1}^n \frac{\Theta_j^I \Theta_j^I e^{-\Gamma_j \Omega_j t} \left[\sin \gamma_j t + e^{-\Gamma_j \Omega_j T} \sin \gamma_j (T-t) \right] + \gamma_j \left(1 + e^{-2\Gamma_j \Omega_j T} - 2e^{-\Gamma_j \Omega_j T} \cos \gamma_j T \right)}{\Omega_j^2 - \Gamma_j^2 \Omega_j^2}, \quad 0 \leq t < T \quad (2.17)$$

$$\chi^*(t) = \sum_{(j)} \frac{\Theta_I^j \Theta_{II}^j e^{-T_I \Omega_j t} \left[\sin \gamma_j t + e^{-T_I \Omega_j T/2} \cos \gamma_j T \right]}{\gamma_j \left(1 + e^{-T_I \Omega_j T} + 2e^{-T_I \Omega_j T/2} \cos \gamma_j T \right)}, \quad 0 \leq t < T/2, \quad (2.18)$$

where $\gamma_j = \Omega_j \sqrt{1 - T_j^2}$.

Expressions (2.17) and (2.18) represent the *eigenfunction expansion* of the PGF. In particular, let $T_j = 0$ for all j . From (2.14), (2.15) and (2.16) we find

$$\chi(t) = \sum_{(j)} \frac{2\Omega_j \Theta_I^j \Theta_{II}^j \cos \Omega_j (t - T/2)}{\sin \frac{\Omega_j T}{2}}, \quad 0 \leq t < T$$

$$\chi^*(t) = \sum_{(j)} \frac{\Theta_I^j \Theta_{II}^j \sin \Omega_j (t - T/4)}{2\Omega_j \cos \frac{\Omega_j T}{4}}, \quad 0 \leq t < T/2$$

2.7. Steady-state Motion

Consider expressions (2.2) and (2.3) defining the steady-state solution of Eq. (1.1) due to T -periodic excitation, or even a more general operation written for some specific point x of a mechanical system

$$x(t) = L(p)g(t)$$

where $g(t)$ is the external force reduced to the point x , $g(t+T) = g(t)$. As shown

$$x(t) = \int_T \chi(t-s)g(s)ds$$

where $\chi(t) = \frac{L e^{pt}}{T}$. It can be shown readily that the expressions for the PGF on the interval of periodicity can not be used directly. The above expressions were obtained, in general, for the time interval $0 \leq t < T$. If $0 \leq s < T$ then $-T \leq t-s < T$. Hence, the quantity $t-s$, strictly speaking, exceeds the above periodicity interval.

However, the function $\chi(t-s)$ may be constructed as a combination of the functions $\chi_1(t)$ representing the PGF for $0 \leq t < T$ and the Green function $h(t)$ [149]. Assume that the time instant t_0 does not belong to the half-interval $[0, T]$.

Then, there exists a number $N \in \mathbb{Z}$ such that $0 \leq t_0 + NT < T$ and, if $\chi(t) = \chi_1(t)$ for $t \in [0, T]$, then $\chi(t_0) = \chi(t_0 + NT) = \chi_1(t_0 + NT)$. For the function $\chi(t-s)$ we find consequently

$$\chi(t-s) = \begin{cases} \chi_1(t-s), & 0 \leq t-s < T \\ \chi_1(t-s+T), & -T < t-s \leq 0 \end{cases} \quad (2.20)$$

As above, in formula (2.6), $\chi(t) = \sum_{k=0}^{\infty} h(t+kT)$ for $t > 0$. Thus, relationship (2.20) may be rewritten in the form

$$\chi(t-s) = \begin{cases} \chi_1(t-s+T) + h(t-s), & 0 \leq t-s < T \\ \chi_1(t-s+T), & -T < t-s \leq 0 \end{cases} \quad (2.21)$$

In accordance with (2.21)

$$x(t) = \int_T \left[\chi_1(t-s+T) + h(t-s) \right] g(s) ds + \int_T \chi_1(t-s+T) ds$$

In the case of symmetrical forcing $g(t+T/2) = -g(t)$

$$x(t) = \int_T \chi^*(t-s)g(s)ds, \quad \chi^*(t) = \frac{T}{2} L_{2k+1} e^{2k+1} (t)$$

and similarly

$$\chi^*(t-s) = \begin{cases} \chi_1^*(t-s), & 0 \leq t-s < T/2 \\ -\chi_1^*(t-s+T/2), & -T/2 \leq t-s < 0 \end{cases}$$

$$\chi^*(t-s) = \begin{cases} \chi_1^*(t-s+T/2) + h(t-s), & 0 \leq t-s < T/2 \\ -\chi_1^*(t-s+T/2), & -T/2 \leq t-s < 0 \end{cases}$$

Formulae (2.22) define the above integral. In general, it is possible to obviate the need for formulae (2.20)-(2.22). By virtue of the features of periodic functions it is obvious that

$$x(t) = \int_T \chi(s)g(t-s)ds$$

and $x(t) = \int_{T/2}^0 \chi^*(s)g(t-s)ds$ for the symmetric force. However, expressions (2.20)-(2.22) will be useful in our following presentation.

2.8. Representation of PGF in the Form of Fast Convergent Fourier Series

It is not always possible to obtain simple formulae defining the PGF on the periodicity

interval. The structure of the operator $L(p)$ may be too complex (see Sect. 1.10) and/or it may be difficult to find its poles. In the case of an infinite number of poles, the corresponding representations also take the form of an infinite series. It may be even more advantageous to use a PGF given by the full Fourier series

$$\chi(t) = \frac{1}{t} L^k e^k(t) \text{ and then to truncate the above series to the few relevant harmonics.}$$

Let $L(p) \in \mathbf{L}\{O(p^{-k})\}$, $k = 1, 2, \dots$. It was noticed in Sect. 1.3, that the corresponding series representing the function $\chi(t)$ possesses $k-1$ continuous derivatives. In the

fundamental cases, such as $k = 1$ and $k = 2$, the first-order discontinuities of $\chi(t)$ take place at time instants $t = lT$, $l \in \mathbf{Z}$. Such a series converges slowly since its remainder behaves as $O(n^{-(k-1)})$ [176], where n denotes the number of the last harmonic considered. That is why the "truncation" causes an inaccuracy.

An increase in the accuracy of the approximate expressions for the PGF, given in the form of Fourier series, may be achieved using the classical method by Krylov [95] as follows [1, 149]. Let $L(p) \in \mathbf{L}\{O(p^{-k})\}$. As shown above, a discontinuity of the k th

derivative of the PGF $\chi(t)$ behaves as $h^{(k)}(+0) = \lim_{p \rightarrow \infty} p^k L(p)$. Denote $h^{(k)}(+0) \equiv h_{k0}$

and consider the operator

$$L^k(p) = \frac{p}{h_{k0}}$$

We set up the series

$$B_k(h_{k0}, t) = \frac{1}{t} \sum_{n=0}^{\infty} \frac{(n\omega)^k}{h_{k0}^k} e_n(t)$$

The functions $B_k(h_{k0}, t) = h_{k0} B_k(1, t)$, $t \in [0, T]$ denote *Bernoulli polynomials* [51].

The above polynomials are determined by the recurrence equation

$$B_k(1, t) = \int_0^1 B_{k-1}(1, t) dt + c_k$$

where the constant c_k is to be found from the condition $B_k(1, 0) = 0$ for odd $k > 1$,

and from the condition $B_{k+1}(1, T) = 0$ for even k .

In addition, it can be shown that for $0 \leq t < T$ [95]

$$B_1(h_{10}, t) = h_{10} \left(\frac{1}{t} - \frac{2}{T} \right)$$

Therefore, for $0 \leq t < T$, we find, in succession

The Bernoulli polynomials are of fundamental importance in the theory of finite differences and are treated in detail in monograph [51].

The operator $L(p) \in \mathbf{L}\{O(p^{-k})\}$ corresponds to the PGF

$$\chi(t) = B_k(h_{k0}, t) + \frac{T}{L_0} + \frac{1}{t} \sum_{n=0}^{\infty} \left[L_n - \frac{T}{L_0} - \frac{1}{h_{k0}^k} (n\omega)^k \right] e_n(t), \quad t \in [0, T]$$

The first two addends here are already defined. The third addend may be defined by considering the corresponding operator

$$L_{(1)}^k(p) = L(p) - \frac{p}{h_{k0}}$$

but $\lim_{p \rightarrow \infty} p^k L_{(1)}^k(p) = 0$, and therefore, this term defines the series with continuous k th

derivative, and with a remainder which behaves as $O(n^{-k})$. This process may be

continued as follows. First, introduce the number $h_{k+1,0}$ in accordance with the

formula $h_{k+1,0} = \lim_{p \rightarrow \infty} p^{k+1} L_{(1)}^k(p)$. Then consider the polynomial $B_{k+1}(h_{k+1,0}, t)$ and

treat as above. As a result of N consecutive steps, the following expansion is found,

$$\chi(t) = \sum_{N=0}^{\infty} B_n(h_{n0}, t) + \frac{T}{L_0} + \frac{1}{t} \sum_{n=0}^{\infty} \left[L_n - \frac{T}{L_0} - \frac{1}{h_{j0}^j} (n\omega)^j \right] e_n(t), \quad t \in [0, T]$$

(2.23)

where $k = 1, 2, \dots$ depends upon the class $\mathbf{L}\{O(p^{-k})\}$. The numbers h_{j0} are defined

$$h_{k0} = \lim_{p \rightarrow \infty} p^k L(p)$$

$$h_{k+1,0} = \lim_{p \rightarrow \infty} p^{k+1} \left[L(p) - \frac{p}{h_{k0}} \right]$$

$$h_{k+2,0} = \lim_{p \rightarrow \infty} p^{k+2} \left[L(p) - \frac{p}{h_{k0}} - \frac{p^2}{h_{k+1,0}^{k+1}} \right]$$

Expansion (2.23) allows the creation of effective approximations for the PGF. By making use of the first two addends and by increasing the number N , the third addend

may be obtained as an arbitrary fast convergent Fourier series with a remainder which behaves as $O(n^{-(k+N)})$.

Similar results may be obtained for the symmetric PGF $\chi^*(t) = \frac{T}{2} L_{2k+1} e^{2k+1}(t)$.

Consider the polynomials $B_k^*(h_{k0}, t)$, which are the symmetrical PGF for the operators

$$L_k(p) = \frac{p^k}{h_{k0}}. \text{ It is evident that}$$

$$B_k^*(h_{k0}, t) = h_{k0} B_k^*(1, t) \text{ and } B_k^*(1, t) = B_{k-1}^*(1, t)$$

in force of

$$B_k^*(h_{k0}, t) = \frac{T}{2} \sum_{n=-\infty}^{\infty} \frac{h_{k0} e^{2n+1}(t)}{[i(2n+1)\omega]^k}$$

Hence,

$$B_k^*(1, t) = \int_0^T B_{k-1}^*(1, t) dt + c_k, \quad 0 \leq t < T/2$$

where the coefficients c_k are defined from the conditions of symmetry

$$(B_k^*(h_{k0}, 0) = -B_k^*(h_{k0}, T/2)) \text{ and the continuity of the functions } B_k^* \text{ (the coefficients$$

of the above series behave as $\frac{1}{(n\omega)^2}$ whilst $k > 1$).

Hence,

$$c_k = -\frac{1}{2} \int_{T/2}^0 B_{k-1}^*(1, t) dt$$

It can be verified directly that $B_k^*(h_{k0}, t) = \frac{1}{2} h_{k0}$ for $0 \leq t < T/2$ (it corresponds to the

case of the "rectangular sine"). We may obtain recurrently that for $0 \leq t < T/2$

$$B_2^*(h_{20}, t) = h_{20} \left(t - \frac{2}{T} \right) \\ B_3^*(h_{30}, t) = h_{30} \left(t^2 - \frac{4}{T} t + \frac{8}{T^2} \right) \dots$$

By analogy with (2.23), the corresponding representation of the symmetric PGF for the operator $L\{O(p^{-k})\}$ may be obtained

§3 Parametric Periodic Green Functions

3.1. Integral Equations of Periodic Vibration

Up to now, the considered linear systems were stationary. Their parameters were time-invariant values. If A_0 denotes the operator of the system $\dot{x}(t) = A_0 g(t)$ (see Sect. 1.1), then for the arbitrary time instant $\tau = \text{const}$, $x(t - \tau) = A_0 g(t - \tau)$. The stationarity of the system is reflected in the following feature. Both the Green function $H(t, \tau)$ and the periodic Green function $\chi(t, \tau)$ of the system are functions of the difference of arguments only, $H(t, \tau) = H(t - \tau)$, $\chi(t, \tau) = \chi(t - \tau)$.

$$\chi^*(t) = \sum_{n=k}^{\infty} B_n^*(h_{n0}, t) + \frac{T}{2} \left[\sum_{n=-\infty}^{\infty} L_{2n+1} - \sum_{j=k}^j \frac{h_{j0}}{h_{j0}} i(2n+1)\omega \right] e^{2n+1}(t) \quad (2.24)$$

An analytical presentation of the operator A_0 becomes complicated if the system parameters vary with time. Consider a particular case where the system parameters are T -periodic functions of time, and this period coincides with that of the external force. Mechanical systems where the parameters are described by periodic time functions are known as *periodic nonstationary systems*. There exists the possibility of developing the theory of nonstationary Green functions and nonstationary operators of dynamic compliance [149, 195]. *Parametric periodic Green functions* (PPGF) are considered further [149, 150] in the framework of the abridged analysis of periodic processes.

Let the linear parametric system be given by the equation

$$A \ddot{x} + B \dot{x} + [C - \mu C_0(t)] x = g(t) \quad (3.1)$$

where the matrices A , B and C possess the same features¹ as the corresponding matrices of (1.1). $C_0(t)$ is a T -periodic matrix, $C_0(t+T) = C_0(t)$ and its components may be represented as a convergent Fourier series. The force $g(t)$ is

¹ More complicated cases of the variable matrices A and B may be reduced to the present case [66].

supposed to be T -periodic and expandable in terms of a similar series. The so-called *modulation level* $\mu \geq 0$ defines an intensity of parametric excitation.

A linear parametric system may be defined in the operator form similar to (3.1), as

$$L^{-1}(p)x(t) = \mu c(t)x(t) + g(t) \quad (3.2)$$

where c and g are T -periodic, and $L(p)$ is the operator of dynamic compliance (see §1).

Consider that T -periodic regimes exist in the considered systems. The forces corresponding to the variation of parameters may be treated as an external excitation. In accordance with the formulae of the previous section, the following may be obtained for the T -periodic regimes

$$x(t) = \mu \int_T^0 \chi(t-s)C_0(s)x(s)ds + \int_T^0 \chi(t-s)g(s)ds \quad (3.3)$$

$$x(t) = \mu \int_T^0 \chi(t-s)c(s)x(s)ds + \int_T^0 \chi(t-s)g(s)ds \quad (3.4)$$

Equations (3.3) and (3.4) are matrix and scalar *integral Fredholm equations*¹. The functions $K(t,s) \equiv \chi(t-s)C_0(s)$, $K(t,s) \equiv \chi(t-s)c(s)$ are known as the kernels of the integral equation, and the functions

$$f(t) = \int_T^0 \chi(t-s)g(s)ds, \quad f(t) = \int_T^0 \chi(t-s)g(s)ds$$

represent the free terms. They are the solutions of the corresponding stationary problems when $\mu = 0$.

Equations (3.3) and (3.4) may be used to define the periodic regimes. The first indication of the absence of the above periodic regimes is the unsolvability of the above equations.

Let $g(t+T/2) = -g(t)$ and $g(t+T/2) = -g(t)$. If, additionally, $C_0(t+T/2) = C_0(t)$ and $c(t+T/2) = c(t)$, i.e. the parametric excitation is $T/2$ -periodic, we can expect the existence of the T -periodic symmetric regime $x(t+T/2) = -x(t)$ and

$$x(t) = \mu \int_{T/2}^0 \chi^*(t-s)C_0(s)x(s)ds + \int_{T/2}^0 \chi^*(t-s)g(s)ds \quad (3.5)$$

The following Fredholm equations may be obtained for these processes instead of (3.3) and (3.4)

¹ More precisely, second order integral Fredholm equations.

3.2. Integral Fredholm Equations

The corresponding kernels $K^*(t,s) \equiv \chi^*(t-s)C_0(s)$, $K^*(t,s) \equiv \chi^*(t-s)c(s)$ are obtained in accordance with the symmetric PGF and free terms represent the symmetric regimes in the corresponding stationary systems. As discussed above, the periodic regimes in the case of parametric excitation may be defined as a solution of the corresponding integral equations. The general features of such equations are discussed here briefly¹. Scalar equations like (3.4) and (3.6) are mainly studied. It was noticed in §1 that the motion of each observable point of a mechanical system might be described using a scalar operator equation. For the following examples, such a reduction will be carried out readily in a nonstationary case. It is worth noting that the solutions to certain problems require convergence, in the generalised sense, at any rate of corresponding Fourier series defining the functions $c(t)$, $C_0(t)$ and $g(t)$ (see Sect. 2.1). Such assumptions are admissible for the further examples.

$$x(t) = \mu \int_{T/2}^0 \chi^*(t-s)c(s)x(s)ds + \int_{T/2}^0 \chi^*(t-s)g(s)ds \quad (3.6)$$

Consider the Fredholm equation

$$x(t) = \mu \int_T^0 K(t,s)x(s)ds + f(t) \quad (3.7)$$

where the parameter μ may be complex. Suppose further that the kernel $K(t,s)$ is continuous for $t,s \in [0,T]$ and the free term $f(t)$ is continuous for $t \in [0,T]$. Under such assumptions we can expect continuity of the solution $x(t)$ (if it exists). In the considered case, the kernel $K(t,s)$ is T -periodic with respect to both variables t and s . By making use of the Einstein convention, the double Fourier series is found, as follows

$$K(t,s) = \frac{1}{T^2} a_{km} e_k(t) e_m(s)$$

¹ The material represented here might be adequate for the reader who is not familiar with the theory of integral equations.

where $\frac{J^2}{2}$ are the Fourier coefficients ($k, n \in \mathbb{Z}$) and $e_k(t) \equiv \exp(ik\omega t)$.

The function $K(t, s)$ is continuous. Let its Fourier series converge uniformly. For the sufficiently large numbers N , we find that

$$K(t, s) \approx K_0(t, s) = \frac{1}{J^2} \sum_N \sum_{k=-N}^N a_{kn} e_k(t) e_n(s)$$

Introduce the following functions

$$\phi_k(t) = \frac{1}{J} e^{\frac{J}{2} e_k(t)}, \quad \psi_k(s) = \frac{1}{J} \sum_N a_{kn} e_n(s)$$

Due to the features of a trigonometric system, the functions $\{\phi_k(t)\}$ and $\{\psi_k(t)\}$ are linearly independent.

An approximate expansion for the kernel $K(t, s)$ may be expressed as

$$K_0(t, s) = \sum_N \phi_k(t) \psi_k(s)$$

The above expression represents a *degenerated kernel*. In Eq. (3.7) let $K(t, s) = K_0(t, s)$. By introducing the presentation for $K_0(t, s)$ into (3.7), the following may be obtained

$$x(t) = f(t) + \mu \sum_N \phi_k(t) \int_T \psi_k(s) x(s) ds$$

Denote

$$\alpha_k = \int_T \psi_k(s) x(s) ds$$

and rewrite $x(t)$ in the form

$$(3.8) \quad x(t) = f(t) + \mu \sum_N \alpha_k \phi_k(t)$$

Definition of the numbers α_k yields the solution to Eq. (3.7) with the kernel $K_0(t, s)$.

Substituting (3.8) into (3.7) yields

$$\left\{ \sum_N \phi_k(t) \left(\alpha_k - \int_T \psi_k(s) f(s) ds + \mu \sum_N \alpha_n \phi_n(s) \right) \right\} = 0$$

The functions $\{\phi_k(t)\}$ are linearly independent, therefore

$$\alpha_k - \int_T \psi_k(s) f(s) ds + \mu \sum_N \alpha_n \phi_n(s) ds = 0, \quad k = -N, \dots, 0, \dots, N$$

Denote

$$\int_T \psi_k(s) f(s) ds = f_k, \quad \int_T \phi_n(s) \psi_k(s) ds = K_{kn}$$

and rewrite the last equality as a system of linear equations in the unknown α_k

$$(3.9) \quad \alpha_k - \mu \sum_N K_{kn} \alpha_n = f_k, \quad k = -N, \dots, 0, \dots, N$$

The systems of linear algebraic equations may be solved using the Cramer's rule; i.e. the analysis of (3.7) with the kernel $K_0(t, s)$ may be conducted using the methods of linear algebra. The principal determinant of system (3.9),

$$D(\mu) = \begin{vmatrix} 1 - \mu K_{-N, -N} & -\mu K_{-N, -N-1} & \dots & -\mu K_{-N, N} \\ -\mu K_{-N-1, -N} & 1 - \mu K_{-N-1, -N-1} & \dots & -\mu K_{-N-1, N} \\ \dots & \dots & \dots & \dots \\ -\mu K_{N, -N} & -\mu K_{N, -N-1} & \dots & 1 - \mu K_{N, N} \end{vmatrix}$$

is a polynomial in μ of order $(2N+1)$ which has $(2N+1)$ roots because μ is a

complex number. The numbers μ_j , satisfying the equation $D(\mu_j) = 0$, are known as *characteristic numbers* or *eigenvalues* of the kernel K_0 . The other values of μ are referred to as the regular values. For $\mu \neq \mu_j$ an application of Cramer's rule and successive free-components column expansion of the auxiliary determinants yields

$$\alpha_k = \frac{1}{D(\mu)} \sum_N D_{kn}(\mu) f_n, \quad k = -N, \dots, 0, \dots, N$$

where $D_{kn}(\mu)$ are polynomials of degree $2N$ at most. The above expressions may be substituted into (3.8). By making use of the assumed notation we find

$$(3.10) \quad x(t) = f(t) + \mu \int_T R(t, s; \mu) f(s) ds$$

where

$$(3.11) \quad R(t, s; \mu) = \frac{1}{D(\mu)} \sum_N \sum_{k=-N}^N D_{kn}(\mu) \phi_k(t) \psi_n(s)$$

The function $R(t, s; \mu)$ is known as a *resolvent* of integral equation (3.7). Through this resolvent the above equation is solvable in a single quadrature. The resolvent is defined if and only if μ is not an eigenvalue of the kernel. In this

case the homogeneous equation (3.7) with $f \equiv 0$ only has the trivial solution. If $\mu = \mu_j$, then, depending upon the structure, system (3.9) may be unsolvable, or may possess an infinite number of solutions. In this particular case, the homogeneous equation (3.7) when $f \equiv 0$ possesses a nontrivial solution. All these conclusions follow from well-known basics of linear algebra and may be formulated in terms of the so-called *Fredholm alternative* which is that either the nonhomogeneous equation with the arbitrary right-hand term is solvable, or the corresponding homogeneous equation possesses nontrivial solutions. Up to now it was considered that the kernel $K(t, s)$ in Eq. (3.7) is substituted with the degenerative kernel $K_0(t, s)$. Fredholm proved that all the above conclusions might be generalised for the case of the nondegenerative kernel $K(t, s)$, and, therefore, this alternative also holds in the general case. He presented the resolvent as the fraction of two series,

$$R(t, s; \mu) = \frac{D(t, s; \mu)}{D(\mu)} \quad (3.12)$$

$$D(t, s; \mu) = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} D_j(t, s) \mu^j, \quad D(\mu) = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} B_j \mu^j$$

Here

$$B_0 = 1, \quad D_0(t, s) = K(t, s), \quad B_n = \int_T D_{n-1}(t, \tau) d\tau$$

$$D_n(t, s) = B_n K(t, s) - n \int_T K(t, \tau) D_{n-1}(\tau, s) d\tau$$

The numerator in expression (3.12) is known as the *first Fredholm minor (numerator)* and the denominator is known as the *Fredholm determinant (denominator)*. The solution of integral equation (3.7) may be obtained using the substitution of (3.12) into (3.10). Therefore, the above Fredholm series generalises the representation (3.11) for the case of the nondegenerative kernel $K(t, s)$. The series which represents $D(t, s; \mu)$ and $D(\mu)$ defines analytical functions in μ . The resolvent $R(t, s; \mu)$ is a meromorphic function. It is not defined if and only if $D(\mu) = 0$. The following theorem may be proven: Eq. (3.7) has at most a numerable quantity of the characteristic numbers $\{\mu_j\}$ which might cluster only as $j \rightarrow \infty$ only; each root of the equation $D(\mu) = 0$ is a *pole* of a resolvent. Since it is impossible to sum series (3.12) in general, the well-developed approximate

methods are used for the resolvent construction [67]. We introduce the presentation of a resolvent in terms of the *Neiman series*. Let μ_1 denote the first eigenvalue of Eq. (3.7). For $|\mu| < |\mu_1|$ this equation may be solved using the method of *successive approximations* by the scheme

$$x_n(t) = f(t) + \mu \int_T K(t, s) x_{n-1}(s) ds, \quad n = 1, 2, \dots$$

whilst the first approximation x_0 may be taken to be the form $f(t)$. It may be shown that

$$\lim_{n \rightarrow \infty} x_n = x(t) \quad \text{and} \quad x(t) = f(t) + \sum_{k=1}^{\infty} \mu^k \int_T K_k(t, s) f(s) ds$$

where the functions $K_k(t, s)$ are so-called *repeated (iterated) kernels* for the kernel $K(t, s)$,

$$K_k(t, s) = \int_T K(t, \tau) K_{k-1}(\tau, s) d\tau, \quad K_1(t, s) \equiv K(t, s)$$

Hence, for $|\mu| < |\mu_1|$, from (3.10), the representation for the resolvent may be found in the form

$$R(t, s; \mu) = \sum_{k=1}^{\infty} \mu^{k-1} K_k(t, s) \quad (3.13)$$

Series (3.13) is known as the *Neiman series*. If $|K(t, s)| \leq M_0 = \text{const}$ the above series converges uniformly for $|\mu| < \frac{1}{M_0 T}$.

From (3.12) it is seen that the representation for the resolvent may be in the general form obtained using the integrals defining the *repeated kernels* $K_k(t, s)$. Series (3.12) defines the meromorphic extension of the Neiman series in the complex plane μ . The following feature of the resolvent is required later in the presentation

$$R(t, s; \mu) = K(t, s) + \mu \int_T K(t, \tau) R(\tau, s; \mu) d\tau \quad (3.14)$$

$$R(t, s; \mu) = K(t, s) + \mu \int_T K(\tau, s) R(t, \tau; \mu) d\tau$$

These relations may be proven using representation (3.13). Equalities (3.14) mean the following. If a free term of an integral equation coincides with the corresponding

kernel, then a resolvent is a solution to the above equation. A kernel and a resolvent define two mutually inverse integral operators.

The analogous results may be obtained after considering the *matrix* integral equation

$$\mathbf{x}(t) = \mathbf{f}(t) + \mu \int_T \mathbf{K}(t, s) \mathbf{x}(s) ds$$

which is equivalent to the system of scalar equations

$$x_k(t) = f_k(t) + \mu \sum_{j=1}^N \int_T K_{kj}(t, s) x_j(s) ds, \quad k = 1, \dots, N$$

where $\mathbf{x} = (x_1, \dots, x_N)$, $\mathbf{f} = (f_1, \dots, f_N)$, $\mathbf{K}(t, s) \equiv [K_{kj}(t, s)]$ is a square $N \times N$ matrix; series (3.12) defines the matrix resolvent $\mathbf{R}(t, s; \mu)$.

Eq. (3.5) and (3.6) may be studied similarly. It is worth noting that most of the above results may be rewritten for the case of discontinuous kernels [150, 194].

3.3. Description of Parametric Periodic Green Functions

Using a resolvent, the solutions of Eq. (3.3)–(3.6) may be readily obtained. As an example, Eq. (3.4) possesses the solution

$$x(t) = f(t) + \mu \int_T R(t, s; \mu) f(s) ds, \quad f(t) = \int_T \chi(t-s) g(s) ds \quad (3.15)$$

where $R(t, s; \mu)$ is the resolvent of the kernel $\chi(t-s)c(s)$. This solution defines a T -periodic regime which exists in system (3.2) under the condition that the Fredholm denominator $D(\mu)$ does not tend to zero at the given modulation index. In agreement with the Fredholm alternative, the eigenvalues μ_j define the nontrivial periodic regimes in the homogeneous parametric system.

Consider further that $L^{-1}(p) = p^2 + \Omega^2$ and $c(t) = c_0 \cos 2\omega t$. Nontrivial solutions of the homogeneous version of equation (3.4) at $g \equiv 0$ may be represented through the Mathieu functions which define the boundaries of the stability zones for a SDOF system [187].

Some mechanical systems are studied later using integral equations. For this purpose some simplifications of representation (3.15) may be done.

The T -periodic solution $\kappa(t, s; \mu)$ of Eq. (3.2) at $g(t) = \delta^T(t-s)$ defines the *parametric periodic Green function* (PPGF) of the system which is described by the above equation. Under the assumption that such a function exists, by analogy with (1.4) and (2.2) we obtain

$$x(t) = \int_T \kappa(t, s; \mu) g(s) ds \quad (3.16)$$

instead of (3.15).

The substitution of $g(t) = \delta^T(t-s)$ into (3.4) yields the relationship which defines $\kappa(t, s; \mu)$,

$$x(t) = \mu \int_T \chi(t-s) c(s) x(s) ds + \chi(t-s)$$

Multiplying both parts of the above equation by $c(t)$ and the substitution $z(t) = c(t)x(t)$ yields

$$z(t) = \mu \int_T c(t) \chi(t-s) z(s) ds + c(t) \chi(t-s)$$

Using the notation $K_1(t, \tau) = c(t) \chi(t-\tau)$, the above expression may be rewritten in the form

$$z(t) = \mu \int_T K_1(t, \tau) z(\tau) d\tau + K_1(t, s)$$

Comparing the obtained equation with the first integral equation of resolvent (3.14) we conclude that

$$z = R_1(t, s; \mu)$$

where R_1 is a resolvent of the kernel $K_1(t, s)$. The internal structure of the above resolvent may be defined using expression (3.12). It may be shown readily that while doing a calculation of each function $D_n(t, s)$ for the kernel $K_1(t, s) = c(t) \chi(t-\tau)$ the parametric excitation $c(t)$ can be brought out from under the integral sign. Thus,

$$D_n(t, s) = c(t) D_n^*(t, s), \quad \text{where}$$

$$D_n^*(t, s) = B_n \chi(t-s) - n \int_T \chi(t-\tau) D_{n-1}(\tau, s) ds$$

The Fredholm numerator is of a similar structure, $D(t, s; \mu) = c(t) D^*(t, s; \mu)$, where D^* is some integer function in μ . Hence, the resolvent R_1 is of a similar form and may be represented as a product of $c(t)$ and some meromorphic function $\kappa(t, s; \mu)$. Therefore,

$$z = c(t) \kappa(t, s; \mu)$$

and, since $z = c(t)x$, we find

$$x = \kappa(t, s; \mu)$$

Hence, the response of system (3.2) to the periodic sequence of δ -functions

$$\delta^T(t-s) \text{ is given in the form}$$

$$(3.17) \quad \kappa(t, s; \mu) = \frac{D^*(t, s; \mu)}{D(\mu)}$$

where $D(\mu)$ is the Fredholm determinant of the kernel $c(t)\chi(t-s)$, and the function $D^*(t, s; \mu)$ is related to the first Fredholm minor as follows:

$$D^*(t, s; \mu) = D(t, s; \mu)/c(t). \text{ Any periodic solution of Eq. (3.2) (for } D(\mu) \neq 0) \text{ may}$$

be obtained using convolution (3.16). The PPGF for system (3.1) may be derived similarly. The matrix function $\kappa(t, s; \mu)$ is the PPGF of system (3.1). Its columns are constructed of n T -periodic solutions $\kappa_j(t, s; \mu)$, $j = 1, \dots, n$, where we use in

succession $g(t) = \delta^T(t-s)r_1$, $\delta^T(t-s)r_2$, $\dots$, $\delta^T(t-s)r_n$, $r_j = (0, \dots, 1, \dots, 0)$, and the unity is j -indexed. For the matrix PPGF, relationships (3.16) and (3.17) take a matrix form and are still valid.

The same definitions and representations may be written for the symmetric PPGF $\kappa^*(t, s; \mu)$ which corresponds to Eq. (3.5) and (3.6). The PPGF which is based on function (3.17) may be expressed as a quotient of two series in terms of expansions in the powers of μ . With respect to (3.12) we find

$$(3.18) \quad \kappa(t, s; \mu) = \frac{1 - \mu B_1 + \frac{1}{2}\mu^2 B_2 - \frac{1}{6}\mu^3 B_3 + \dots}{1 - \mu[\chi(t-s)B_1 - \kappa_2(t, s)]}$$

where $\kappa_n(t, s)c(t) = K_n(t, s)$, $K_n(t, s)$ are the repetitive kernels (see (3.13)). By making use of a definition for the kernels we find

$$\kappa_n(t, s) = \int_0^T \chi(t-\tau)c(\tau)\kappa_{n-1}(\tau, s)d\tau, \quad \kappa_1(t, s) = \chi(t-s)$$

The obtained expressions may be rewritten in terms of a trigonometric series. There exist other forms of representations of the function $\kappa(t, s; \mu)$ which are based on formulae (3.17) and (3.12) (see reference [67], where the accuracy of the above representations is assessed). The calculation of the kernels $\kappa_n(t, s)$ may be simplified if some sort of compact notation for the PGF $\chi(t)$ is known, see (2.17) for instance. However, the expressions for the PPGF $\kappa(t, s; \mu)$ are still awkward.

3.4. Excitation of Parametric Vibration by Impacts

In general, linear differential equations with periodic coefficients are not integrable in terms of elementary functions. Hence, many problems are not solved in an explicit form which allows for simple analysis of the zones of parametric instability. Usually, parametric vibration is studied using approximate methods, see references [25, 33, 39, 117, 155, 187, 191, 193], etc. These methods are suitably effective for obtaining reliable information about parametric vibration without knowledge of the exact solutions. However, the exact solutions (if obtained in explicit and compact form) are very valuable, not only because of uniqueness, but also as a reference for approximate calculations. Here we study one particular class of linear parametric systems where the parametric term is expressed in the form of a linear combination of Dirac δ -functions. The general method of solution is similar to that developed by Rosenwasser, see reference [149]. Using this approach, two mechanical problems will be solved. Note that these problems may be studied by using the methods developed by Su [168], Nagae and Hodjaev [121] (see also [153]). The present results are based on paper [91].

(A) Let the observable coordinate x of a mechanical system be described by the operator equation

$$(3.19) \quad x(t) = \mu L(p)c(t)x(t) + hL(p)g(t)$$

For practical calculations, the manageable results may be obtained by resolving the above kernels in the form of polynomials in powers of μ and by further truncating this series to low-degree terms.

From the structure of formula (3.18) it can be seen that accounting for the additional terms gives additional information about the PPGF which is defined by the kernels $\kappa_n(t, s)$ in the form of increasingly smooth functions. The singular (for important mechanical applications) PGF $\chi(t-s)$ is reintegrated many times whilst calculating the kernels $\kappa_n(t, s)$. Therefore, the Fourier series which represent these kernels converge increasingly quicker with an increase in the integer n , and, as a matter of fact, the functions $\kappa_n(t, s)$ contribute far less to the representation of the PPGF $\kappa(t, s; \mu)$.

An analysis of nonlinear systems is often based on the application of the PPGF as calculated below. It is worth noting that by using integral equations of periodic vibration it is possible to perform a full analysis of linear parametric systems (including the stability problems). The corresponding theory is very elegant and instructive; see references [149, 150]. In this paragraph the above method is applied for the analysis of one particular class of parametric systems.

where $c(t)$ and $g(t)$ are the parametric and force excitations, $L(p) \in L\{O(p^{-2})\}$ is the fractionally rational operator of dynamic compliance, μ , $h \geq 0$ are the known functions of the system parameters. Let the parametric excitation be

$$c(t) = \sum_{n=1}^{\infty} \sum_{k=1}^p c_k \delta(t - t_k - qT) \equiv \delta_T^c(t)$$

where $\delta(t)$ is Dirac function, $0 = t_1 < t_2 < \dots < t_n = T$ and T is the period of excitation. The function $g(t)$ is a T -periodic function of time and it is expandable into a Fourier series (at least in a generalised sense). Now we define the conditions of existence of nontrivial periodic vibration in the system, which is described by homogeneous

($h = 0$) equation (3.19), boundaries of the stability domains and the forced vibration of the nonhomogeneous system under the force $g(t)$. Let $h = 0$ in Eq. (3.19), and then there exists the PGF $\chi(t) = T^{-1} L_k e_k(t)$ which corresponds to the operator $L(p)$.

Then, the equation of the T -periodic vibration (3.4) takes the form

$$x(t) = \mu \int_T^0 \chi(t-s) \delta_T^c(s) x(s) ds \equiv \mu \sum_{n=1}^{\infty} c_k \chi(t - t_k) x(t_k)$$

Successively considering $t = t_j$, we obtain the system of linear equations in $x(t_j)$

$$x(t_j) = \mu \sum_{n=1}^{k-1} c_k \chi(t_j - t_k) x(t_k), \quad j = 1, \dots, n-1$$

Nontrivial periodic solutions will exist if and only if the principal determinant of this system is zero, that is

$$\det[X_{l,m}] = 0, \quad l, m = 1, \dots, n-1$$

$$X_{l,m} = -\mu c_m \chi(t_l - t_m), \quad l \neq m; \quad X_{l,l} = 1 - \mu c_l \chi(0)$$

In accordance with the general theory, arbitrary solution of Eq. (3.19) at $h = 0$ can be found in the form $x(t) = e^{\mu t} y(t)$, where $y(t+T) = y(t)$ and χ is the characteristic exponent. If for all such exponents, $\operatorname{Re} \lambda_m \leq \alpha < 0$, $m = 1, 2, \dots$, then system (3.19)

will be asymptotically stable. The substitution of the assumed solution $x(t)$ into the homogeneous equation (3.19) yields

$$e^{\mu t} y(t) = \mu L(p) c(t) e^{\mu t} y(t)$$

¹ The general theory of parametric systems, which is represented in reference [193], assumes a higher smoothness of periodic coefficients of equation. The present calculations may be considered as a formal generalisation of this theory for systems with impulsive coefficients.

Using the correspondence between the operators and the Laplace transformations (see Sect. 1.4) and the shift theorem, we find $\tilde{y}(p - \lambda) = \mu L(p) \tilde{z}(p - \lambda)$, where $\tilde{z}(p)$ is the Laplace transform of the product $c(t)y(t)$. The substitution $p_1 + \lambda = p$ yields for the latter equality, $\tilde{y}(p_1) = \mu L(p_1 + \lambda) \tilde{z}(p_1)$, or using the operator notation

$$y(t) = \mu L(p_1 + \lambda) z(t), \quad z(t) = c(t) y(t)$$

Here p_1 is treated as the operator $\frac{d}{dt}$, hence the subscript index may be omitted. Finally, for the T -periodic function $y(t)$, we find

$$y(t) = \mu L(p + \lambda) c(t) y(t)$$

Let $\chi(\lambda; t) = T^{-1} L_k e_k(t)$ be the PGF which corresponds to the operator $L(p + \lambda)$ $(L_k(\lambda) \equiv L(ki\omega + \lambda))$. For $y(t)$ we obtain the homogeneous integral equation

$$y(t) = \mu \int_T^0 \chi(\lambda; t-s) c(s) y(s) ds$$

Using the definition for $c(t)$ we obtain the following formula

$$y(t) = \mu \sum_{n=1}^{k-1} c_k \chi(\lambda; t - t_k) y(t_k)$$

Subsequently letting $t = t_j$ yields the system of algebraic linear equations in $y(t_j)$

$$y(t_j) = \mu \sum_{n=1}^{k-1} c_k \chi(\lambda; t_j - t_k) y(t_k), \quad j = 1, \dots, n-1$$

Since we are looking for nontrivial solutions of the above system, it is necessary that

$$\det[X_{l,m}(\lambda)] = 0, \quad l, m = 1, \dots, n-1$$

where $X_{l,m}(\lambda) = -\mu c_m \chi(\lambda; t_l - t_m)$, $l \neq m$, $X_{l,l}(\lambda) = 1 - \mu c_l \chi(\lambda; 0)$. The above condition may be thought of as the characteristic equation for the considered system. If $\lambda_j = \lambda_j(\mu; \omega)$, $j = 1, 2, \dots$, represent the solutions, where at least for a single j $\operatorname{Re} \lambda_j > 0$, the parametric resonance may arise. The exponential growth in the amplitude of $x(t)$ will be given by $\exp[(\operatorname{Re} \lambda_j)t]$.

The equations $\det[X_{l,m}(\lambda)] = 0$ and $\det[X_{l,m}(\lambda)] = 0$ fully define periodic regimes and zones of parametric instability. The first equation may be obtained from the second one when $\lambda = 0$. For the description of the forced vibration we obtain the PPGF. Suppose $g(t) = \delta_T^c(t-s)$. Then from (3.19), for T -periodic processes $x \equiv x(t, s)$, we find

$$\kappa(t, s) = \mu \int_0^T \chi(t - \tau) c(\tau, s) \kappa(\tau, s) d\tau + h \int_0^T \chi(t - \tau) \delta^T(\tau - s) ds$$

$$\kappa(t, s) = \mu \sum_{k=1}^n c_k \chi(t - t_k) \kappa(t_k, s) + h \chi(t - s)$$

The latter equality defines a system of linear nonhomogeneous equations in $\kappa(t_k, s)$. The solution of the above system fully describes the PPGF of system (3.19). If the PPGF is defined and $\det[X_{l,m}] \neq 0$, convolution (3.16) describes the forced vibration. Let $g(t)$ be force excitation from (3.19). Similarly,

$$x(t) = \mu \sum_{k=1}^n c_k \chi(t - t_k) x(t_k) + h f(t)$$

$$f(t) = \int_0^T \chi(t - s) g(s) ds$$

Again, the latter equations produce a set of linear equations which may be solved using the Cramer's rule. The periodic solutions may exist even in the degenerated case of the zero principal determinant of the above system, $\det[X_{l,m}] = 0$. For the corresponding analysis, the well-known Kronecker-Capelli theorem [156] is applicable. The multiple periodic solutions will exist if the rank of the extended matrix of the algebraic equations coincides with that of the primary rank. In another case there are no periodic solutions at all. This particular case is of theoretical interest and it will not be considered further.

If the variable coefficient may be represented as a combination of Dirac δ -functions, the analysis of corresponding integral equations may be reduced to an analysis of a system of linear algebraic equations. This leads to a significant simplification of the analysis of such a parametric system.

(B) In Fig. 3.1a, a pendulum is subjected to the base-induced excitation which is given by the "triangle sine" function of time (see Fig. 3.1b). Let x denote the angular deflections of the pendulum from the vertical axis. Under the assumptions that this angle is small and that viscous damping is present, we obtain the equation of motion

$$\ddot{x} + \Omega^2 x + \mu \dot{x}(t)x + 2bx = 0 \quad (3.20)$$

where $\Omega^2 = g/l$, b is the damping factor and $\mu = \text{const}$ is a parameter defining the amplitude of the T -periodic "triangle sine" vibration $u(t)$ (see Fig. 3.1b). This

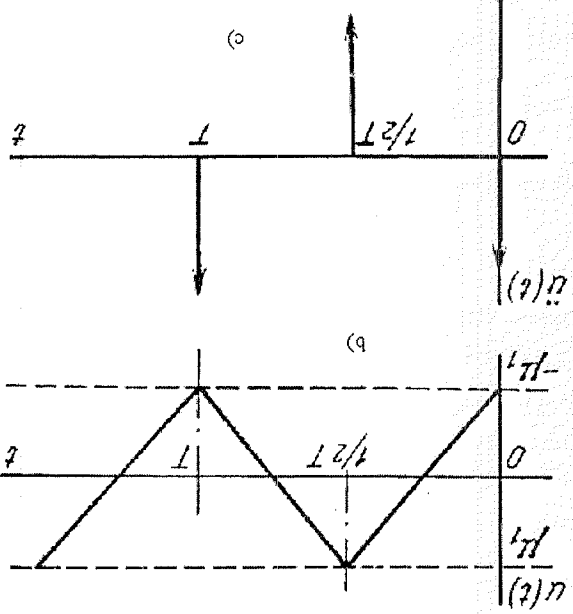
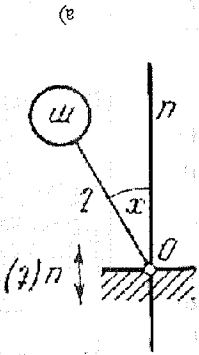


Fig. 3.1

It may be shown easily that

$$\mu u(t) = \mu \delta_{T/2}^T(t) = \frac{4\mu}{T} \sum_{k=0}^J \cos[(2k+1)\omega t]$$

As a second example, consider the model of the mechanism in Fig. 3.2a. The body of mass m is hinged by means of the rigid rod to the slider the mass of which is much greater than m . The above slider is involved in the motion with collisions against a stationary obstacle which take place at the times $t = kT$, $k \in \mathbb{Z}$. Hence, the motion of the slider consists of the parabolic segments $\mu_1 u(t) = -\frac{1}{2} \mu_1 (t^2 - Tt)$, $t \in [0, T]$ with further periodic extension (see Fig. 3.2b) in the time domain. The corresponding equation of motion may be represented easily in the form of (3.20), where arrows symbolise the δ -impulses.

vibration is defined as $\mu u(t) = \mu [t/2 - T/8]$ for $t \in [0, T/2]$ with further periodic symmetric extension $u(t + T/2) = -u(t)$, $\mu_1 = u(T/2) = T/8$. The "graph" of parametric excitation, which is defined by the translational force of inertia and proportional to $\ddot{u}(t)$, is represented in Fig. 3.1c.

$\mu_1 \dot{u}(t) = -\mu_1 + \mu_1 T \delta^T(t)$. It may be verified using generalised differentiation of the velocity $u(t)$, see Fig. 3.2c.

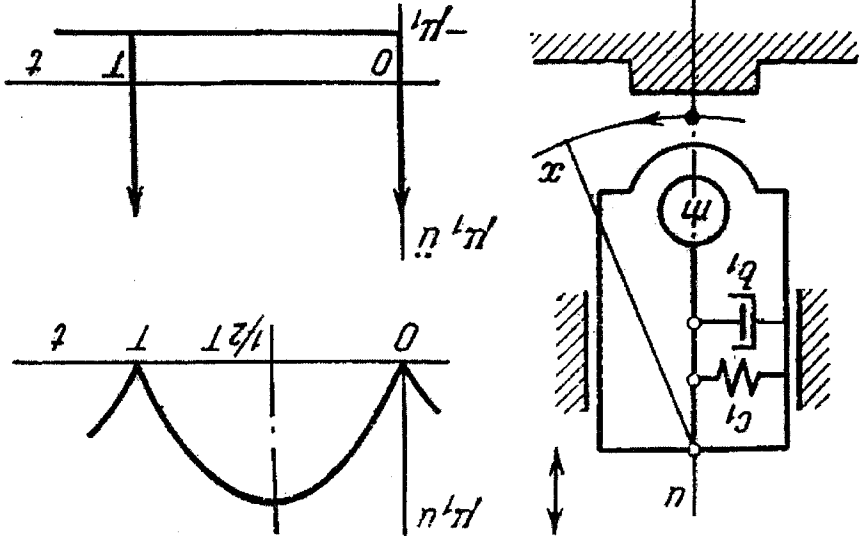


Fig. 3.2

For the analysis of the second problem, Eq. (3.20) was rearranged to take the form

$$\ddot{x} + [1 - \mu + \mu T \delta^T(t)] \dot{x} = 0, \quad \mu = \mu_1 T^{-1}$$

where, for simplicity, $\Omega = 1$, $b_1 = 0$. In agreement with the general theory, the appearance of the periodic regimes with periods T and $2T$ is possible. For the T -periodic vibration we have the integral equation

$$x(t) = -\mu T \int_T^0 \dot{x}(t-s) \delta^T(s) x(s) ds$$

which yields the relationship

$$x(0) = -\mu T \dot{x}(0) x(0) \quad (3.21)$$

An expression for the PGF $\chi(t)$ may be obtained from (2.10). It may be considered that the "natural frequency" of the steady-state response of the above system is $\sqrt{1-\mu}$. Therefore,

$$\chi(t) = \begin{cases} \chi_{10}(t), & \mu \in [0, 1] \\ \chi_{11}(t), & \mu > 1 \end{cases}$$

(compare with formula (2.14)). The conditions for the existence of the T -periodic solutions for $\mu \in [0, 1]$ may be obtained from (3.21) after cancelling $x(0)$

$$\chi_{10}(t) = \frac{2\sqrt{1-\mu} \cos[\sqrt{1-\mu}(t-T/2)]}{1 - \coth[\sqrt{1-\mu}(t-T/2)]} \sinh(\mu-1T/2)$$

$$\frac{\mu T}{\omega \sqrt{1-\mu}} = -\tan \frac{\omega}{\pi \sqrt{1-\mu}}, \quad T = 2\pi\omega^{-1}$$

If $\mu = 1$, then the PGF $\chi(t)$ is not defined since the nontrivial solutions do not exist. For $\mu > 1$ the latter equation yields

$$\frac{\mu T}{\pi \sqrt{1-\mu}} = \tanh \frac{\omega \sqrt{\mu-1}}{\mu \pi}$$

In agreement with the method of Sect. 3.4 (A), the definition of the stability conditions for the trivial solution to Eq. (3.20) requires knowledge of the PGF $\chi(\lambda; t)$ corresponding to the operator $L(p+\lambda) = [p+\lambda]^2 + 1 - \mu$. Using formula

(2.18) yields

$$\chi(\lambda; t) = \frac{1}{\exp[t(p_1 - \lambda)]} + \frac{2p_1}{1 - \exp[T(p_1 - \lambda)]} \frac{\exp[t(p_2 - \lambda)]}{\exp[T(p_2 - \lambda)]}$$

where $p_{1,2}$ are the poles of $L(p)$. After simple rearrangements we find the

characteristic equation

$$1 = -\mu T \chi(\lambda; 0)$$

which takes the form

$$\frac{\exp(-\lambda T) \sin(\sqrt{1-\mu} T)}{\sqrt{1-\mu} [1 + \exp(-2\lambda T) - 2 \exp(-\lambda T) \cos(\sqrt{1-\mu} T)]} = (\mu T)^{-1}$$

for $\mu \in [0, 1]$. The substitution $\exp(-\lambda T) = V$ yields the quadratic equation in V (system multiplier)

$$V^2 - \{2 \cos[\sqrt{1-\mu} T] - \mu T [1 - \mu]^{-1/2} \sin[\sqrt{1-\mu} T]\} V + 1 = 0$$

The above expression may be rewritten in the form $V^2 - R_0 V + 1 = 0$. It can be shown easily that the desired roots are $V_{1,2} = \frac{1}{2} R_0 \pm \left(\frac{1}{4} R_0^2 - 1 \right)^{1/2}$, $V_1 V_2 = 1$. It is also seen

that if $1/4 R_0^2 > 1$, the absolute value of one of the roots is greater than unity. Therefore, an evaluation of the above inequality causes *parametric resonance* to occur. Since $\mu > 0$, the condition of resonance for $0 < \mu < 1$ takes the form

$$(3.22) \quad \frac{\mu\pi}{\pi\sqrt{1-\mu}} > -\tan \frac{\omega}{\pi\sqrt{1-\mu}} > 0, \quad \frac{\omega}{\mu\pi} > \cot \frac{\omega}{\pi\sqrt{1-\mu}} > 0$$

The first condition corresponds to the boundary case $e^{-\lambda T} = 1$ of T -periodic regimes. The second condition $e^{-\lambda T} = -1$ corresponds to the regimes with period $2T$ (see below). Consider $\mu > 1$. Then after rearrangements, the characteristic equation $1 = -\mu T \chi(\lambda; 0)$ takes the form

$$\mu^2 - 2 \coth(\sqrt{\mu-1}T) - \mu T(\mu-1)^{-1/2} \sinh(\sqrt{\mu-1}T) \Big] \mu + 1 = 0$$

which yields the conditions of parametric resonance

$$(3.23) \quad \frac{\mu\pi}{\pi\sqrt{\mu-1}} > \tanh \frac{\omega}{\pi\sqrt{\mu-1}}, \quad \frac{\omega}{\mu\pi} > \coth \frac{\omega}{\pi\sqrt{\mu-1}}, \quad \mu > 1$$

The second condition again corresponds to the regimes with period $2T$.

To define the above regimes, the equation of the $2T$ -periodic vibration is to be used. Since the excitation has a period which is smaller by a factor of two, we can expect the vibration mode to be symmetric, $x(t+T) = -x(t)$. Hence, the integral equation of symmetric vibration with period $2T$ will be used (compare with (2.3))

$$x(t) = -\mu T \int_0^T \chi_*(t-s) \delta^T(s) x(s) ds$$

where the PGF $\chi_*(t)$, in accordance with (2.15) and (2.11), takes the form

$$\chi_*(t) = \begin{cases} \chi_{20}^*(t), & \mu \in [0, 1] \\ \chi_{21}^*(t), & \mu = 1 \\ \chi_{22}^*(t), & \mu > 1 \end{cases}$$

for $t \in [0, T]$, but not for $t \in [0, T/2]$.

$$\chi_{20}^*(t) = \frac{2\sqrt{1-\mu}}{1} \frac{\sin[\sqrt{1-\mu}(t-T/2)]}{\sin[\sqrt{1-\mu}(t-T/2)]}, \quad \chi_{21}^*(t) = \frac{2}{t} - \frac{4}{T},$$

$$\chi_{22}^*(t) = \frac{2\sqrt{1-\mu}}{1} \frac{\cosh(\sqrt{\mu-1}T/2)}{\sinh[\sqrt{\mu-1}(t-T/2)]}$$

The conditions for the existence of the $2T$ -periodic regimes follow from the above for $\mu \in [0, 1]$, $\mu > 1$, $\mu = 1$,

$$\frac{\mu\pi}{\pi\sqrt{1-\mu}} = \cot \frac{\omega}{\pi\sqrt{1-\mu}}, \quad \frac{\omega}{\mu\pi} = \coth \frac{\omega}{\pi\sqrt{1-\mu}}, \quad \omega = \pi$$

In the case $\mu = 1$, the resonance condition is $\omega < \pi$ (compare with [168]). The zones of parametric instability of the mechanism in Fig. 3.2a are, therefore, defined by formulae (3.22) and (3.23) along with the latter inequality. For $\mu > 1$ the above system may be considered as a system with negative "eigenfrequency". This corresponds to the case of the "inverted" pendulum. The obtained results correspond to the fact that an "inverted" pendulum with an oscillating hinge may be stabilised [99]. Let us consider now the model in Fig. 3.1a which is described by Eq. (3.20) with $\mu \delta^{T/2}(t) = \mu \delta^{T/2}(t)$ and $\Omega = 1$, $b = 0$. For the T -periodic solutions for $\mu \in [-\infty, \infty]$ we have

$$x(t) = \mu \chi(t)x(0) - \mu \chi(t-T/2)x(T/2)$$

Successive substitution $t = 0, T/2$ yields the homogeneous system of algebraic equations

$$\begin{aligned} x(0)[1 - \mu \chi(0)] + \mu \chi(T/2)x(T/2) &= 0 \\ x(0)[- \mu \chi(T/2)] + x(T/2)[1 + \mu \chi(0)] &= 0 \end{aligned}$$

with the principal determinant $1 - \mu^2 \chi^2(0) + \mu^2 \chi^2(T/2) \neq 0$ because

$$\chi(t) = \frac{1}{2} \cos(t - T/2) [\sin(T/2)]^{-1} \text{ and } \chi^2(T/2) \geq \chi^2(0)$$

Therefore, there are no T -periodic regimes. For $2T$ -periodic symmetric regimes, simple rearrangements yield the extreme condition: $\mu = \pm 2 \cot(\pi/\omega)$. The characteristic equation may be constructed in accordance with the above procedure. The reader may verify easily that the condition of parametric resonance is given by the inequality $\mu^2 > 4 \cot^2(\pi/\omega)$ (compare with references [121, 167, 168]). In Fig. 3.3a the corresponding zones of instability are hatched.

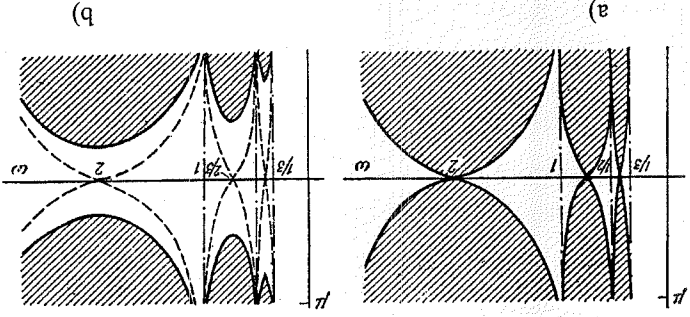


Fig. 3.3

The case of the "inverted pendulum" may be considered similarly. It may be demonstrated that its upper position of equilibrium is stable if the condition $4 < \mu^2 < \coth^2(\pi/\omega)$ holds. Analogous calculations in the presence of damping ($b \neq 0$) demonstrate the narrowing of the zones of parametric instability (see Fig. 3.3b). The corresponding tedious rearrangements are not represented here. At $b \neq 0$ the PGF takes a form similar to that in (2.12) and (2.13).

The present analysis may be supplemented by a study of the system behaviour at linear resonance where the PGF $\chi(t)$ is not defined (see Sect. 2.2).

We calculate the PPGF $\kappa(t, s; \mu)$. The model in Fig. 3.2b will be considered. By definition, a PPGF is a response of a dynamic system to a periodic sequence of δ -functions. Hence,

$$\kappa(t, s) = -\mu T \int_0^T \chi(t - \tau) \delta^T(\tau) \kappa(\tau, s) d\tau + \int_0^t \chi(t - \tau) \delta^T(\tau - s) d\tau$$

where the PGF $\chi(t)$ corresponds to the operator $L(p) = [p^2 + 1 - \mu]^{-1}$. Using the δ -

functions features yields

$$\kappa(t, s) = -\mu T \chi(t) \kappa(0, s) + \chi(t - s)$$

The equation for the definition of $\kappa(0, s)$ may be obtained if we set $t = 0$ in the above expression. As a result of simple manipulations, we find

$$\kappa(t, s) = -\frac{\mu T \chi(t) \chi(s)}{1 + \mu T \chi(0)} + \chi(t - s) \quad (3.24)$$

Similarly, the PPGF for the $2T$ -periodic regimes or for the model in Fig. 3.1a may be obtained.

Obviously, Eq. (3.24) may be obtained using formulae (3.12) and (3.17) of Sect. 3.3. The calculation of the values $D^*(t, s; \mu)$ and $D(\mu)$ from (3.17) using (3.12) and the accepted notation yields

$$D^*(t, s; \mu) = \chi(t - s) [1 + \mu T \chi(0)] - \mu T \chi(t) \chi(s), \quad D(\mu) = 1 + \mu T \chi(0)$$

because $B_n = 0$ and $D_n(t, s) \equiv 0$ for $n > 1$. Formula (3.24) may be obtained using the above reasoning. If the system is subjected to the external T -periodic force $g(t)$, Eq. (3.16) yields the representation for the forced vibration. Such a motion develops if, for example, the excitation $u(t)$ in Figs. 3.1a and 3.2a forms a certain angle with the vertical axis. Such problems in the nonlinear approach are analysed in detail below. The explicit results, obtained in the analysis of such a class of parametric system, may be used for the construction of approximate solutions for much more general systems. Su and Chjen obtained some particular results, see reference [167].

Chapter 2 Strongly Nonlinear Single-Degree-of-Freedom Systems

§4 Conservative Systems

4.1. Classification of Nonlinear Systems

Here we consider only nonlinear systems. First, the SDOF system will be studied. Let the equation of motion of an absolutely rigid body of unit mass with coordinate x take the form

$$\ddot{x} + 2bx + \Omega^2 x + \Phi(x) = g(x, \dot{x}, t) \quad (4.1)$$

where $b > 0$ and Ω are the parameters of the linear part of system: b is the damping factor, Ω is the angular natural frequency, $b < \Omega$. Force, which is defined by the function g , is a nonlinear function of its arguments. It is impossible to obtain an exact form for the solution of Eq. (4.1) for arbitrary functions g .

Consider an auxiliary (reference) system which allows for a simple exact solution (such a system, of course, must be close to system (4.1) in a certain sense). The approximate solution of Eq. (4.1) then may be obtained by successive improvement of the "reference" solution. This idea belongs to Poincaré. All founders of approximate methods of nonlinear mechanics used it in different forms.

The most popular reference system is a linear system. Let Eq. (4.1) be written as

$$\ddot{x} + 2bx + \Omega^2 x + \Phi(x) = \varepsilon g(x, \dot{x}, t) \quad (4.2)$$

where ε is a *small parameter*, i.e. a number which characterises the closeness of the system to a linear system. It is assumed that, when the parameter ε is small enough, we can neglect the values with order lower than ε^1 .

Systems (4.2) are said to be *weakly nonlinear* or *quasi-linear* systems. The theory of such systems has been developed well recently³. Nevertheless, there exist systems which do not allow, in principle, representations of type (4.2). These are the systems where the quasi-linear approach is contrary to the

¹ Remainder of notation: $\alpha = O(\varepsilon)$ means that the order of α equals that of ε ;

² In quasi-linear systems, it is often assumed that $b = O(\varepsilon)$.

³ The references may be found in [186].

physical sense. If in the equation of motion at least one nonlinear term is not factored by a small parameter, then such a system is said to be *essentially nonlinear system*. A representative class of essentially nonlinear systems is formed by the *quasi-conservative systems* with the nonsmall potential forces $\Phi(x)$. The corresponding equations of motion take the form

$$(4.3) \quad \ddot{x} + \Omega^2 x + \Phi(x) = \text{eg}(x, \dot{x}, t)$$

If the equation of motion of essentially nonlinear system contains at least one nonlinear term factored by the large parameter $\lambda = O(\varepsilon^{-\alpha})$, $\alpha > 0$ then the system is said to be strongly nonlinear. The object of our future study will be the strongly nonlinear quasi-conservative systems. For such systems we have $\Phi(x) = \lambda\Phi_0(x)$ in

Eq. (4.3). Other representative classes of strongly nonlinear systems were studied in references [5, 98, 115, 182] etc¹.

The study of strongly nonlinear systems requires the application of special methods. This is evident from the following reasoning. Let it be known a priori that the periodic solution $x(t)$ of Eq. (4.1) is represented by the Fourier series $x_k e_k(t)$ and

$x_k \sim O(k^{-2})$. Then, the function $x(t)$ is continuous, but its derivative may possess

discontinuities at $t = lT$, where T is the period of vibration. It is evident that calculation of the arbitrary number of quasi-linear approximations gives $x(t)$ in the

form of the finite sum

$$(4.4) \quad x(t) = \sum_{k=-N}^N x_k e_k(t)$$

which defines a smooth function. Therefore, the method of quasi-linear approximations in principle can not reveal the effect of the discontinuity of a derivative of $x(t)$. The above example is related to the study of the very important class of applications based on *vibro-impact systems* (Sect. 4.3).

4.2. Equations of Conservative Systems

We will briefly reiterate the fundamental results of the theory of SDOF conservative systems. The equation of motion takes the form

$$(4.5) \quad \ddot{x} + \Phi_0(x) = 0$$

where $\Phi_0(x)$ is the force defined by the potential $\Pi_0(x)$ as $\Phi_0(x) = \frac{d}{dx} \Pi_0(x)$. For

¹ See Foreword.

² For the SDOF system every positional force allows the existence of a potential.

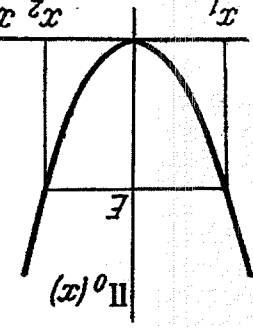


Fig. 4.1

minimum and monotonously increasing "walls" (see Fig. 4.1). Here we are concerned with oscillatory processes only. Therefore, let us assume that the potential $\Pi_0(x)$ takes the specific form of a potential well and possesses a single

motion is expressed in terms of special functions [160]. Here E and the initial time instant t_0 are arbitrary constants. Typically, the desired calculation of the above quadrature by inverting the function $t(x)$. Here the total

where t_0 is arbitrary initial time instant. The desired solution $x(t)$ may be obtained by

using the method of separation of variables,

$$H(x, \dot{x}) = \frac{\dot{x}^2}{2} + \Pi_0(x) = E = \text{const}$$

system (4.5) the total energy, which is the first integral,

Then, for each fixed level of energy $E \in [0, \infty]$, the period of the oscillatory motion may be expressed as

$$(4.7) \quad T(E) = \int_{x_2(E)}^{x_1(E)} \frac{\sqrt{2} dx}{\sqrt{E - \Pi_0(x)}}$$

where the equation $\Pi(x_1, x_2) = E$ defines the halt points which correspond to zeros of

kinetic energy.

The periodic motion $x(t)$ may be resolved in terms of the Fourier series

$$x(t) = \sum_{k=-\infty}^{\infty} x_k(E) e^{ik\varphi(E,t)}$$

where $\varphi(E, t) = \omega(E)t + t_0$ is the phase and $\omega(E) = 2\pi T^{-1}(E)$ is the lowest angular frequency of vibration¹. In the above representation the coefficients $x_k(E)$ are real numbers: $x_{-k}(E) = x_k^*(E)$. Hence $x(t)$ may be resolved into the Fourier series in cosines of multiple arcs [68].

Let $E \in [0, \infty]$. In agreement with expression (4.7) the potential $\Pi_0(x)$ defines the interval of frequencies $\omega(E) \in [\omega_1, \omega_2]$. Frequencies from this interval are said to be the *eigenfrequencies* of the conservative system.

If $\omega(E)$ is energy-independent, the interval of the eigenfrequencies collapses. Such a conservative system is said to be *isochronous*. At every level of initial energy of excitation the successive vibration will be of the same fixed frequency (period). If in some particular subinterval of eigenfrequencies an increase in frequency results in an increase (decrease) in energy, such systems are said to be *strongly (softly) anisochronous* in this subinterval.

Let $A(E)$ be a monotonously increasing function of energy. It might be, for example, half of the peak-to-peak value of the vibration (*amplitude*) or some other measure of motion. Since the function A is uniquely related to the total energy of a system, it may be thought of as an integral of motion. The graphical representation of the dependence $A(\omega)$ is said to be a *backbone curve* of system (4.5) for the integral of motion A .

Suppose that $\Phi_0(x) = \Omega^2 x + \Phi(x)$. Substitution of this force into (4.5) yields,

$$\ddot{x} + \Omega^2 x + \Phi(x) = 0$$

or, in the operator notation,

$$x = -\frac{\Phi(x)}{p^2 + \Omega^2}, \quad p \equiv \frac{d}{dt}$$

The force $\Phi(x)$ may be considered formally as external. In accordance with Eq. (2.2), by making use of the periodic Green function (PGF) of the linear system for the T -periodic regimes, we find

$$x(t) = - \int_{-T}^0 \chi(t-s) \Phi[x(s)] ds \quad (4.8)$$

From formula (2.14), for the time interval $0 \leq t < T$,

¹ Further, if it does not result in misunderstanding, the term "frequency" will be used instead of the term "lowest frequency of vibration".

$$\chi(t) = \frac{1}{2\Omega} \frac{\cos[\Omega(t-T/2)]}{\sin(\Omega T/2)}$$

Relationship (4.8) is the *Hammerstein nonlinear integral equation* in the unknown motion $x(t)$. It can be resolved uniquely for the periods $T(E)$ given by Eq. (4.7). Relationship (4.6) defines the solution.

The above equation is said to be a *free vibration integral equation* of a conservative system. If $\Phi(-x) = -\Phi(x)$, then the Fourier series representing $x(t)$ will contain odd harmonics only and the function $x(t)$ will be symmetric, $x(t+T/2) = -x(t)$. The corresponding free vibration integral equation of (see (2.3)) may be written using the symmetric PGF $\chi^*(t)$

$$x(t) = - \int_{-T/2}^0 \chi^*(t-s) \Phi[x(s)] ds \quad (4.9)$$

where, in accordance with formula (2.15) for the time interval $0 \leq t < T/2$,

$$\chi^*(t) = \frac{1}{2\Omega} \frac{\sin[\Omega(t-T/4)]}{\cos(\Omega T/4)}$$

Detailed information concerning conservative systems may be found in references [6,55,99,103] etc.

4.3. Vibro-impact Systems

Consider a linear conservative system given by the equation of motion

$$\ddot{x} + \Omega^2 x = 0$$

which describes harmonic vibration of a body of unit mass attached to a spring the stiffness of which is Ω^2 .

The phase portraits of such a system are ellipses given by the equations

$$H(x, \dot{x}) = \frac{1}{2}(\dot{x}^2 + \Omega^2 x^2) = E = \text{const}$$

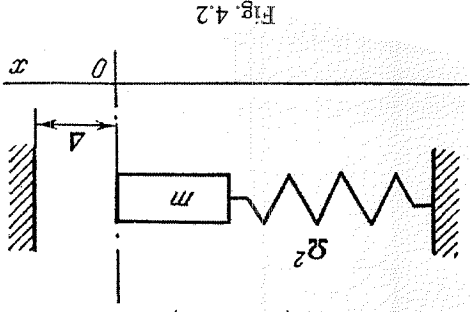


Fig. 4.2

obstacle for $\Delta > 0$, impactless motion is possible. Note that in this case the amplitude of the vibro-impact motion takes a constant value; the half-swing of the vibration equals Δ . Therefore, the amplitude can not be treated as a single-valued function of the energy integral.

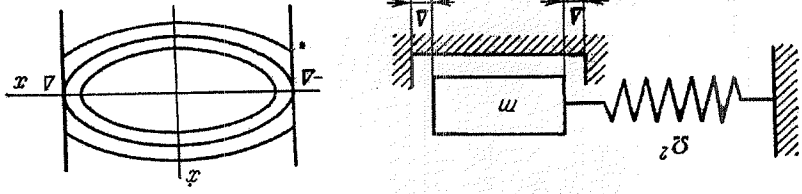


Fig. 4.4

4.4. Singular Force of Impact

The condition of impact (4.10) may be introduced into the equation of motion as an equivalent force

$$m\ddot{x} + \Omega^2 x + \Phi_0 = 0$$

where the force Φ_0 symbolises the force of impact and m is the mass of the colliding body. It is clear that the above force may be expressed in terms of a generalised singular function. The momentary impact is localised both in time and space domains.

Therefore, the desired representation of the function Φ_0 may be given in terms of generalised functions of time or of spatial variables. Consider first the time-domain representation. Condition (4.10) governs the variation of the impulse of the force Φ_0 (impact impulse) in the close vicinity of the instant of collision t_a

$$J = m\dot{x}_- - m\dot{x}_+ = 2m\dot{x}_-, \quad \dot{x}_- > 0$$

$$\text{where } \dot{x}_\pm \equiv \dot{x}(t_a \pm 0).$$

The resultant force is localised at $t = t_a$. This may be expressed in terms of the Dirac δ -function by

$$\Phi_0|_{t=t_a} = J\delta(t - t_a) \quad (4.15)$$

$$\text{where } \int_{t_a-0}^{t_a+0} \Phi_0 dt = J.$$

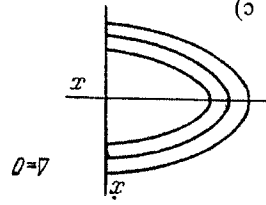
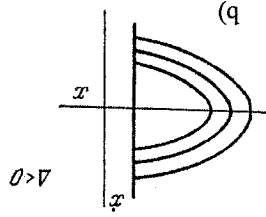
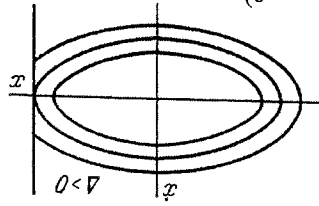


Fig. 4.3

(4.10)
$$\dot{x}(t_a - 0) = -\dot{x}(t_a + 0)$$
 In accordance with the sign of Δ , the system is capable of demonstrating different phase portraits.

A single stationary obstacle is "installed" at the point $x = \Delta$ (see Fig. 4.2). We further assume that a momentary elastic impact [54] takes place when the coordinate x achieves the Δ level. The conditions of such an impact, which takes place at $x = \Delta$ at the time instant t_a , are as

(1) In the case $\Delta > 0$ (clearance) the ellipses corresponding to a linear system are "trimmed" by the vertical line $x = \Delta$ and the true trajectories are their left-handed portions (see Fig. 4.3a). As the level of initial excitation is not enough to transcend the level $x = \Delta$, linear vibration with the frequency Ω takes place. In the presence of impacts, the frequency of vibration is $\omega > \Omega$; the increase in the energy level results in the increase in the frequency of vibration up to the value 2Ω . Therefore

$$\Omega < \omega < 2\Omega, \quad \Delta > 0 \quad (4.11)$$

and the vibro-impact system with clearance is strongly anisochronous. (2) Fig. 4.3b shows the phase trajectories for the case $\Delta < 0$ (preload). By analogy,

$$2\Omega < \omega < \infty, \quad \Delta < 0 \quad (4.12)$$

and the system is softly anisochronous. Vibration in this case may develop if the level of the initial energy is enough to overcome the force of the initial preload $\Omega^2|\Delta|$.

(3) In the case $\Delta = 0$, we obtain the ellipses which are trimmed exactly in the middle (see Fig. 4.3b). Thus, the angular frequency is 2Ω for all levels of initial energy,

$$\omega = 2\Omega, \quad \Delta = 0 \quad (4.13)$$

The vibro-impact system with zero clearance is isochronous. Imagine now that two symmetric obstacles are installed, as Fig. 4.4 shows. Two symmetric lines trim two symmetric parts from the original ellipses (see Fig. 4.5).

In this case

$$\Omega < \omega < \infty \quad (4.14)$$

and the energy level grows as the frequency increases. Therefore, a system with symmetric obstacles is strongly nonisochronous. Here, as in the case of the single-side

It is evident from the previous discussion that collisions take place periodically at the time instances $t_a = t_0 + \alpha T$, $\alpha \in \mathbb{Z}$, and T is the period of impacts which is calculated using the equation $T = 2\pi\omega^{-1}$ and equations (4.11)–(4.13). For $-\infty < t < \infty$ we find the T -periodic extension for expression (4.15)

$$\Phi_0 = J\delta^T(t - t_0) \quad (4.16)$$

where, in accordance with Sect. 2.1

$$\delta^T(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT) = T^{-1} \sum_{k=-\infty}^{\infty} e^{ik\omega t}, \quad T = 2\pi\omega^{-1}$$

Sometimes it is worth examining two phases of impact, namely loading and restitution.

In the loading phase, a momentary deformation of the obstacle takes place and the colliding body transfers the loading impulse $J_- = mx_-$ to the obstacle. In the

restitution phase the obstacle returns the accumulated energy to the body and, thus, imposes the impulse $J_+ = -mx_+$. Therefore, the total impulse will be $J = J_- + J_+$

and force (4.15) may be expressed as

$$\Phi_0|_{t_a} = J\delta^-(t - t_a) + J_+\delta^+(t - t_a)$$

where the functions $\delta^\pm(t)$ are normalised to unity.

The equivalence of this notation and expression (4.15) may be verified easily using

Sect. 1.2,3 and the properties of the left-hand and right-hand δ -functions.

Now consider the spatial presentation of the singular force of impact Φ_0 . Condition

(4.10) may be analysed using the energy concept.

We assume that impact takes place when the coordinate x achieves the level of Δ .

The stationary and absolutely rigid obstacle may be thought of as a half-space which is

capable of momentary accumulation of potential energy. When $x = \Delta - 0$ this energy

is

$$\mathcal{K}|_{x=\Delta-0} = 0$$

At the same time, if the velocity of the body which approaches the obstacle is x_- , then

$$\mathcal{K}|_{x=\Delta+0} = \frac{1}{2}mx_-^2$$

The energy increment, therefore

$$\delta\mathcal{K} = \mathcal{K}|_{x=\Delta-0} - \mathcal{K}|_{x=\Delta+0} = -\frac{1}{2}mx_-^2$$

The force of the impact in the coordinate representation is localised at $x = \Delta + 0$ and

may be expressed as

¹ Here $\mathbb{Z}$ denotes the set of integers.

$\Phi_0 \equiv \Phi_0(x, x) = -\Phi(x, x)$ where in accordance with the expression for $\delta\mathcal{K}$,

$$\Phi(x, x) = -\frac{1}{2}mx_-^2\delta^+(x - \Delta) \quad (4.17)$$

and $\delta^+(x)$ is the right-hand δ -function (see Sect. 1.2 and 1.3), which is normalised to unity. Hence, the work done by the force (4.17) in the vicinity of the point $x = \Delta$ equals $\delta\mathcal{K}$ exactly, and the initial equation of motion may be written as

$$m\ddot{x} + \Omega^2 x + \Phi_0(x, x) = 0 \quad (4.18)$$

where the condition of impact is used already. In the time-domain representation we have, therefore,

$$m\ddot{x} + \Omega^2 x + \Phi_0[x(t), x(t)] = 0$$

where the force $\Phi_0[x(t), x(t)]$ is given by expression (4.16). The evident geometrical conditions

$$x(t) \leq \Delta, \quad x_- > 0 \quad (4.19)$$

are to be added to the above equations. The first of the geometrical conditions forbids the disposition of the body behind the obstacle, the second requires the preimpact velocity to be positive. If the second of the above conditions does not hold, the first also can not be satisfied, but not vice versa.

The solution of Eq. (4.18) will be the T -periodic function $x(t)$, which, when substituted into the above equation, converts it into an identity (in the sense of the theory of generalised functions).

$$m\ddot{x}(t) + \Omega^2 x(t) + J\delta^T(t - t_0) = 0$$

where t_0 is an undefined variable, and for all $\alpha = 0, \pm 1, \dots$

$$x(t_0 + \alpha T) = \Delta, \quad J = 2m\dot{x}_-(t_0 + \alpha T)$$

As this takes place, conditions (4.19) hold and the periods of vibration are defined by the sign of Δ in the frequency intervals given by (4.11)–(4.13).

For the symmetric system (described in Sect. 4.3) the following may be established easily.

The condition of impact (4.10) holds twice within the period of vibration $T = 2\pi\omega^{-1}$, where the frequency ω is given by interval (4.14). In the coordinate representation [12] we find

$$\Phi_0^*(x, x) = \frac{1}{2}mx_-^2\delta^\pm(|x| - \Delta)\operatorname{sgn} x \quad (4.20)$$

where the symbol $\delta^\pm(|x| - \Delta)\operatorname{sgn} x$, $x \neq 0$ denotes the generalised function

$$\delta^\pm(|x| - \Delta)\operatorname{sgn} x = \begin{cases} \delta^+(x - \Delta), & x > 0 \\ -\delta^-(x + \Delta), & x < 0 \end{cases}$$

and the δ -functions $\delta_{\pm}(x)$ are normalised to unity.

In the time-based representation

$$\Phi_0^*[x(t), x(t)] = J\delta_{T/2}(t - t_0) \quad (4.21)$$

where $\delta_{T/2}(t)$ is the symmetric T -periodic δ -function (see Sect. 2.1)

$$\delta_{T/2}(t) = -\delta_{T/2}(t + T/2)$$

$$\delta_{T/2}(t) = \sum_{k=-\infty}^{\infty} \left[\delta(t - kT) - \delta\left(t - kT - \frac{T}{2}\right) \right] = \frac{T}{2} \sum_{k=-\infty}^{\infty} e^{(2k+1)i\omega t}$$

and the impulse of impact $J = 2mx_-(t_0 + \alpha T)$, $0 \leq t_0 < \frac{1}{2}T$, $\alpha = 0, \pm 1, \dots$

$$x(t_0 + \alpha T) = \Delta, \quad x\left(t_0 + \frac{T}{2} + \alpha T\right) = -\Delta$$

Conditions (4.19) are transformed to

$$|x(t)| \leq \Delta, \quad x(t_0 - 0) > 0 \quad (4.22)$$

The arbitrary constant t_0 defines the sequence of the collisions. If

$\alpha T \leq t_0 < \frac{1}{2}T + \alpha T$, then the right-hand side obstacle is involved in the first impact. If

$\frac{T}{2} + \alpha T \leq t_0 < (\alpha + 1)T$, then the left-hand side obstacle is involved in the first impact.

Eq. (4.18) retains the form in accordance with the substitution $\Phi_0(x, \dot{x}) \rightarrow \Phi_0^*(x, \dot{x})$.

Note that for the simplification of notation in this chapter we consider $m = 1$.

4.5. Motions of Vibro-impact Systems

The solutions of Eq. (4.18) with asymmetric (Φ_0) and symmetric (Φ_0^*) impact

forces may be obtained easily now.

In the first case we use Eq. (4.8). With provision for formula (4.16) we find

$$x(t) = -\int_T^0 \chi(t-s) J \delta_T(s-t_0) ds = -J \chi(t-t_0) \quad (4.23)$$

where the condition of impact $x(t_0) = \Delta$ yields

$$J = -\frac{\chi(0)}{\Delta}, \quad \Delta \neq 0 \quad (4.24)$$

Due to the arbitrary choice of the initial time t_0 , and for the simplest description of the function $x(t)$, we can consider $t_0 = 0$. Thus, the initial time coincides with the moment of impact. Therefore, in accordance with Eq. (2.14), (4.23) and (4.24) for the time interval $0 \leq t < T$ we find

$$x(t) = -\frac{J}{2\Omega} \frac{\sin(\Omega T/2)}{\cos[\Omega(t-T/2)]}, \quad J = -2\Omega\Delta \tan \frac{\Omega T}{2} \quad (4.25)$$

where the second condition of impact $J = -2\dot{x}_-$ holds automatically (it can be verified by direct substitution). This also follows from the arbitrariness of the constant t_0 .

The obtained solution must be continued periodically.

Solution (4.25) might be written formally for all T subject to the condition $x(t) < \infty$. In this case the geometrical conditions of impact $J \geq 0$, $x(t) \leq \Delta$ yield the frequency intervals $\Omega < \omega < 2\Omega$ for $\Delta > 0$ and $\omega > 2\Omega$ for $\Delta < 0$, where $(\omega = 2\pi T^{-1})$ (compare with (4.11) and (4.12)).

The second expression from Eq. (4.25) indicates that $J \rightarrow \infty$ as $\omega \rightarrow 2\Omega$. This corresponds to the case of infinitely large energy.

In spite of the fact that when $\omega = \Omega$ the PGF $\chi(t)$ is not defined, solution (4.25) may be rearranged formally. The numerator and denominator might be divided by $-2\Omega \sin(\Omega T/2)$, thus producing

$$x(t) = \Delta \frac{\cos[\Omega(t-T/2)]}{\cos(\Omega T/2)} \quad (4.26)$$

and the quantity

$$A = \frac{1}{2}(\Delta - \min x) = \frac{1}{2}\Delta[1 - \sec(\Omega T/2)] \quad (4.27)$$

defines the amplitude of vibration. When $T = 2\pi\Omega^{-1}$, the solution of (4.25) degenerates into the solution of the corresponding linear system $x(t) = \Delta \cos \Omega t$.

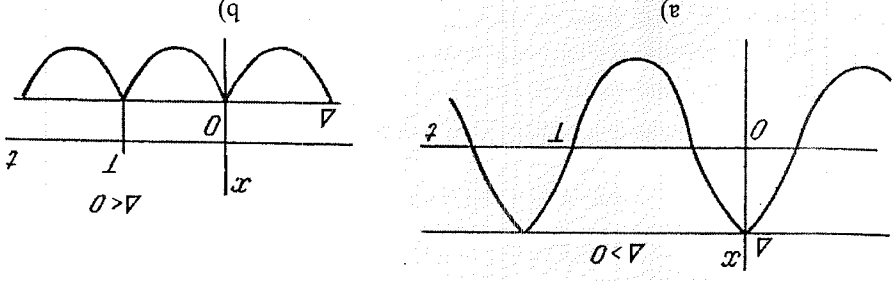


Fig. 4.6

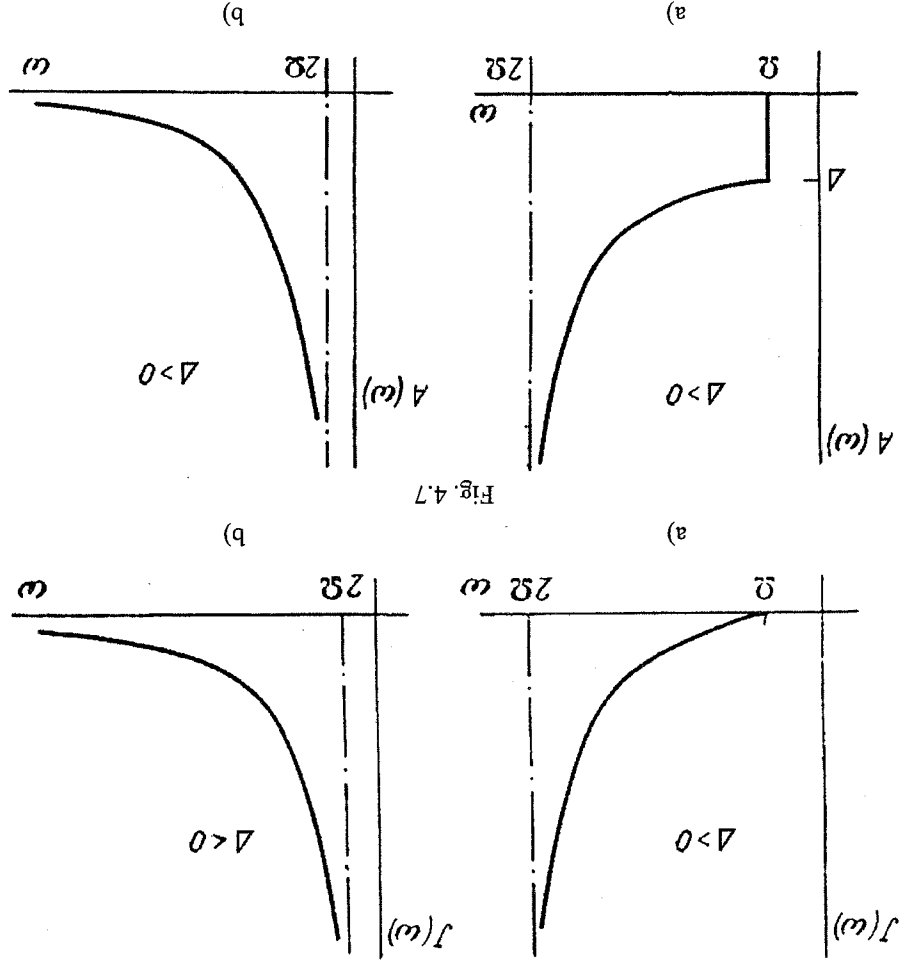


Fig. 4.7

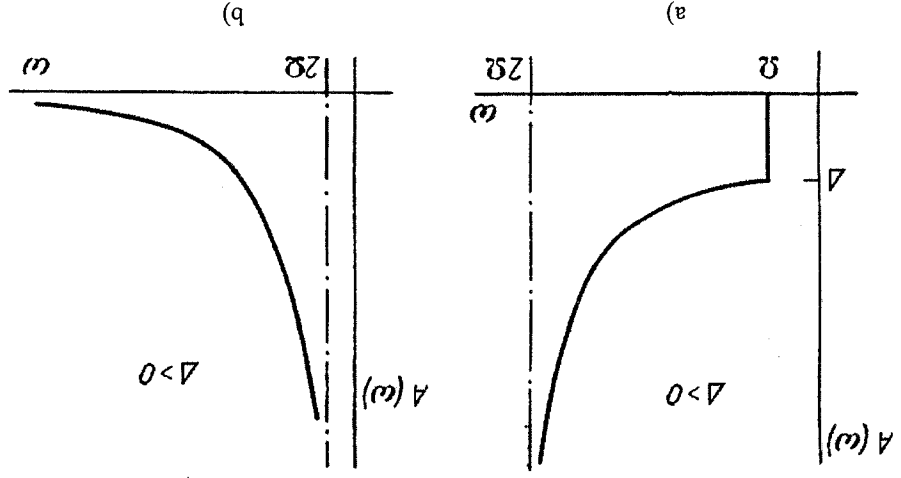


Fig. 4.8

This regime of vibration is said to be a *grazing regime*. For $\Delta < 0$ the grazing regime is impossible. Solution (4.26) is usually obtained as a result of an application of the stitching method [17,75].

Figure 4.6a,b portrays the examples of the time histories $x(t)$ for $\Delta > 0$ and $\Delta < 0$. In Fig. 4.7a,b and Fig. 4.8a,b the dependencies $J(\omega)$ and $A(\omega)$ are represented, respectively.

The vertical portion of the curve in Fig. 4.8a represents the dynamic behaviour of the

linear (impactless) system. The above figures define the backbone curves for the integrals of motion J and A . Similar graphical representation may be obtained also for the integral of energy E .

In the isochronous case $\Delta = 0$, $\omega = 2\Omega$ (see (4.13)) and, therefore, $\omega(J) = \text{const}$. The solution $x(t)$, for $0 \leq t < \pi/\Omega$, takes the form

$$(4.28) \quad x(t) = -\frac{2\Omega}{J} \sin \Omega t$$

where J is the undefined constant which is frequency independent.

Since the PGF $\chi(t)$ may be written explicitly in the form of (2.14) for the time interval $0 \leq t < T$ only (see Sect. 2.7), an introduction of the second undefined constant t_0 into the solution will complicate, to some extent, the corresponding notation.

In Fig. 4.9a typical time history for the case $\Delta = 0$ is shown. Fig. 4.9b portrays the corresponding backbone curve.

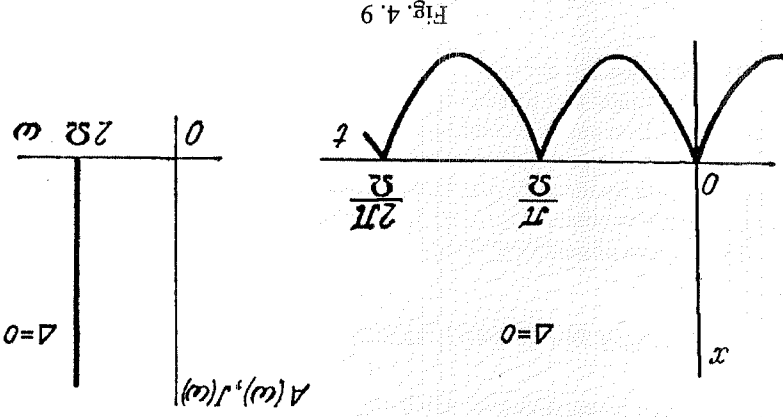


Fig. 4.9

For a symmetric system we can use (4.9). Taking into consideration (4.21) we find

$$(4.29) \quad x(t) = -\int_{T/2}^0 \chi_*(t-s) J \delta^{T/2}(s-t_0) ds = -J \chi_*(t-t_0)$$

where the condition of impact $x(t_0) = \Delta$ is applied again. The assumption $t_0 = 0$ for the time interval $0 \leq t < T/2$ yields

$$(4.30) \quad x(t) = -\frac{J}{2\Omega} \sin[\Omega(t-T/4)], \quad J = 2\Delta\Omega \cot \frac{\Omega T}{4}$$

where the restrictions $J > 0$ and $|x(t)| \leq \Delta$ produce the allowable frequency range

$\omega > \Omega$ (see (4.14)).

The substitution of the first of the equations of (4.30) into the second yields

$$x(t) = -\frac{\Delta \sin[\Omega(t - T/4)]}{\sin(\Omega T/4)} \quad (4.31)$$

Compare this with references [17,75].

When $\omega = \Omega$ we obtain the grazing mode of vibration $x(t) = \Delta \cos \Omega t$. From (4.30) it follows that as $\omega \rightarrow \infty$ ($T \rightarrow 0$), the impulse $J(\omega) \rightarrow \infty$. Thus, infinite energy may be supplied to the system at infinite frequency only.

Note, that, in this case, the amplitude of vibration $A(\omega)$ can not be considered as an integral of motion. For all frequencies $A(\omega) = \Delta = \text{const}$.

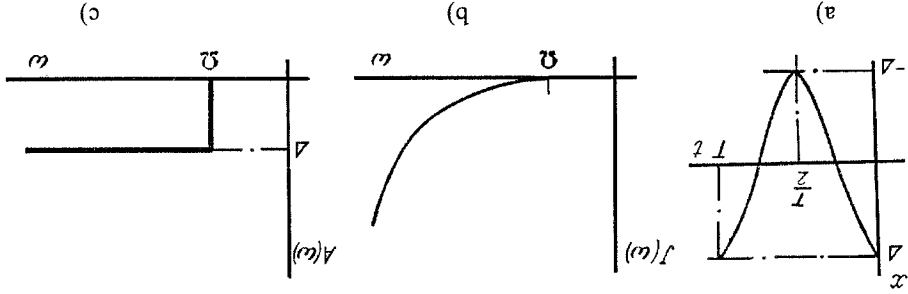


Fig. 4.10

In Fig. 4.10a-c, the time history of motion and the dependencies $J(\omega)$ and $A(\omega)$ are respectively shown for the symmetric vibro-impact system. Solution (4.30) may be obtained using the general equation (4.8). That will require more complicated calculations. The period of vibration contains two opposing impacts ("positive" and "negative") against obstacles.

We will discuss briefly the solutions which are not feasible in the "vibro-impact sense", i.e. solutions of (4.25) and (4.30) which are formally defined at some period T but do not correspond to geometrical conditions (4.19).

Such solutions may be of different types. For example, if the condition $x(t) \leq \Delta$ does not hold, the solutions with *additional crossovers* may be obtained (see Fig. 4.11). Such solutions may be thought of for a system where the obstacle appears from time to time and, therefore, sometimes there is no impact when $x = \Delta$. The failure to meet the condition $J \geq 0$ means that the impacting body approaches the obstacle with negative velocity, i.e. before the impact it is situated behind the obstacle and the condition $x(t) \leq \Delta$ does not hold.

4.6. Strongly Nonlinear Systems

For many types of potential forces which are represented in the equation of motion $\ddot{x} + \Phi(x) = 0$

it appears to be extremely difficult to obtain a general form of the solution. The inversion of quadrature (4.6) leads to the appearance of special complex functions. In the above equation, let the force $\Phi(x)$ take the form

$$\Phi(x) = \Phi_1(x) + \lambda \Phi_2(x) + \lambda^2 \Phi_3(x) + \dots + \lambda^n \Phi_n(x)$$

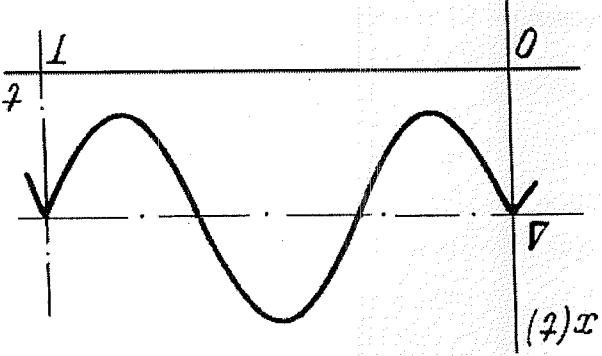
where λ is a *large* parameter, and the number of terms in this representation is finite. Systems with such a nonlinearity are known as *strongly nonlinear systems* (see Sect. 4.1). For a better physical interpretation we introduce the small parameter $\varepsilon = \lambda^{-1/2}$.

Transition to the fast time $\tau_n = \varepsilon^{-n} t$ [190] yields

$$\frac{dx(t)}{dt} = \frac{dx(\tau_n)}{d\tau_n} \frac{d\tau_n}{dt} = \frac{1}{\varepsilon^n} \frac{dx}{d\tau_n}, \quad \frac{d^2 x}{dt^2} = \frac{1}{\varepsilon^{2n}} \frac{d^2 x}{d\tau_n^2}$$

and the equation of motion takes the form

Fig. 4.11



Such solutions must be eliminated from consideration. In a number of cases it is difficult to segregate the feasible solutions even using numerical simulations. In practical calculations only the condition $J \geq 0$ may be verified easily. The theory of vibro-impact systems attracts considerable interest. This may be explained not only by the wide practical application of vibro-impact systems. The theoretical solutions obtained for such systems may be used in the analysis of systems with different types of nonlinear forces.

$$\frac{d^2x}{dt^2} + \varepsilon^2 \mu^n \Phi_1(x) + \lambda \varepsilon^{2n} \Phi_2(x) + \dots + \lambda^n \varepsilon^{2n} \Phi_n(x) = 0$$

Introduction of the small parameter $\mu = \varepsilon^2$ yields

$$\frac{d^2x}{dt^2} + \mu^n \Phi_1(x) + \mu^{n-1} \Phi_2(x) + \dots + \mu \Phi_{n-1}(x) + \Phi_n(x) = 0$$

The above equation may be solved using the asymptotic method (see §5) or another approximate method considering smaller terms¹.

Nevertheless, special classification of strongly nonlinear conservative systems is senseless if all the functions $\Phi_k(x)$ from the equation of motion are of order unity and defined on $-\infty < x < \infty$. Using a simple time transformation, the strong nonlinearity may be "killed" and originally multi-scaled-in-time processes which correspond to the different nonlinear terms $\lambda^k \Phi_k(x)$ may be uniscaled. Therefore, the simultaneous

improvement of the solution is possible at all x .

It will not take place, however, if $\Phi_k(x) \equiv 0$ on particular disjoint intervals $a_k \leq x \leq b_k$ for some k . In this case it might be necessary to find the desired solutions separately in each interval of x and then to "stitch" the obtained "portions" of the solution. As a result, multi-scaled-in-time physical processes may also be diversified

in space. Study of such nonlinear systems requires special attention. Analysis of vibro-impact systems with provision for the process of collision [17,131], the specific systems of automatic and cycle control, the positioners and resonance machines is associated with the necessity for the consideration of special types of strongly nonlinear forces of threshold type. Such forces are nonzero after the coordinate x exceeds a given threshold level [2,5,7,20,40,78,80,88,89,178,186] etc. The nonlinear functions with similar properties are used in the analysis of electromechanical devices [102].

Here, one special case of conservative systems with strong *threshold* nonlinearities will be analysed. Possible generalisations may be done easily.

4.7. Strongly Nonlinear Systems of Threshold Type

Consider the class $\{\Phi\}_\Delta$ of threshold functions [94] as shown in Fig. 4.12.

¹ The above equation may be rewritten in the quasi-linear form. In this particular case it does not produce any advantages.

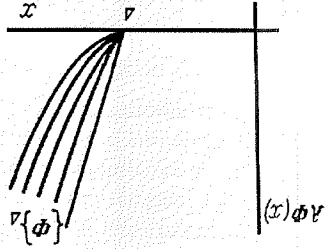


Fig. 4.12

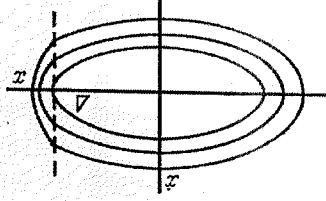


Fig. 4.14

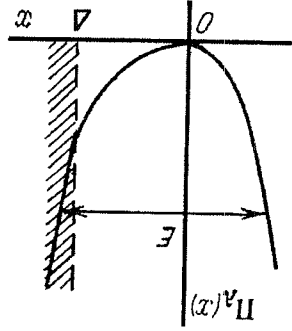


Fig. 4.13

The functions $\Phi(x) = \psi(x - \Delta)\eta(x - \Delta)$ belong to the class of threshold functions. Here $\eta(x)$ is a the Heaviside function and $\psi(x)$ is a monotonously increasing smooth function with $\psi(0) = 0$. We will consider further the most important convex functions. It will be shown that the class $\{\Phi\}_\Delta$ may be significantly expanded.

Consider the equation of motion

$$\ddot{x} + \Omega^2 x + \lambda \Phi(x) = 0 \quad (4.32)$$

where λ is a large parameter and $\Phi(x) \in \{\Phi\}_\Delta$.

The potential $\Pi_\lambda(x) = \frac{1}{2}\Omega^2 x^2 + \lambda U(x)$, $\Phi(x) = U'(x)$ is represented as a potential well with a single minimum and monotonously increasing "walls", as schematically shown in Fig. 4.13. The hatched zone corresponds to the abrupt increase of the potential. A disposition of the coordinate x inside the zone $x > \Delta$ is said to be an *interaction*. The sketch of the phase portrait of system (4.32) is represented in Fig. 4.14.

The solution of (4.32) may be obtained in terms of a Fourier series based on the trigonometric system $\{e_k(t)\} = \{\exp(ik\omega_\lambda t)\}$, $k = 0, \pm 1, \dots$; $\omega_\lambda = 2\pi T_\lambda^{-1}$, where

$$T_\lambda(E) = \int_{x_2(E)}^{x_1(E)} \frac{\sqrt{2} dx}{\sqrt{E - \Pi_\lambda(x)}} \quad (4.33)$$

and E denotes the total energy of the system, $\Pi_\lambda[x_{1,2}(E)] = E$.

If $x(t)$ is the solution of (4.32), then $x = x[t - t_0, A_2(E)]$, where t_0 and $A_2(E)$ are

undefined constants and $A_2(E)$ is a single-valued function of the total energy. For the definition of x the value of t_0 is not important, therefore we can put $t_0 = 0$ and consider that the interaction starts at $t = 0$, $x = \Delta$. Denote the velocity by

$x(0) = V_{02} = V_{02}(E)$, where, by virtue of the persistence of the problem, V_{02} is a

mutually single-valued function of total energy E . Therefore, the quantity

$J_2(E) = 2V_{02}(E)$ is the integral of motion. It will be convenient to use it in our further

analysis.

Let $x_-(t) = x(t) \leq \Delta$ be the motion before interaction and $x_+(t) > \Delta$ be the

motion after interaction. Then $x_-(0) = x_+(0) \leq \Delta$, and by virtue of the autonomy of

conservative system there exists an instant t_2 such that $x_-(t_2) = x_+(t_2) = \Delta$.

The time interval $0 \leq t_2 < T$ is the time of interaction. For $t \in [0, t_2]$ we have

$x(t) > \Delta$.

Eq. (4.32) is written separately for time intervals $0 \leq t \leq t_2$, $t_2 \leq t \leq T$ and further, by

periodicity,

$$(4.34) \quad \ddot{x} + \Omega^2 x = 0, \quad \dot{x} + \Omega^2 x + \lambda \Phi(x) = 0$$

In accordance with this (4.33) may be transformed to

$$(4.35) \quad T_2(E) = \int_{\Delta}^{x_1(E)} \frac{\sqrt{2} dx}{\sqrt{E - \frac{1}{2} \Omega^2 x^2}} + \int_{x_2(E)}^{\Delta} \frac{\sqrt{2} dx}{\sqrt{E - \Pi_2(x)}}$$

The first integral here will be denoted as T_∞ (the sense of the subscript symbol will be revealed later). The second integral equals t_2 , i.e. $T_2(E) = T_\infty(E) + t_2$. For large λ

$x_2(E) \approx \Delta$, and therefore $t_2 \rightarrow \infty$ as $\lambda \rightarrow \infty$. The above interaction may be

considered as "fast".

To obtain the general solution $x(t)$, the solutions of (4.34) must be stitched. The

function $x(t)$ may be obtained using integral equation (4.8) [94]. Let the function

$x_+(t)$, which is the solution to the second equation of (4.34) at the initial conditions

$$x_+(0) = \Delta, \quad \dot{x}_+(0) = \frac{2}{\lambda} J_2, \quad \text{be known i.e. } x_+ = x_+(t, J_2). \quad \text{Equation (4.8)}$$

$$x(t) = -\lambda \int_{t_2}^0 x(t-s) \Phi[x_+(s, J_2)] ds$$

might be solved for periods (4.33). Thus we obtain

4.8. Singularisation

For large λ , any nonlinearities from the class $\{\Phi\}_\Delta$ may be accurately substituted with a singular nonlinearity (4.17). Such a substitution (singularisation) is widely applied if it turns out to be possible to ignore the duration of the interaction and to consider it as a momentary event. The above substitution ignores the details of the interaction and reflects only its effect. It allows for simple analytical delivery of the important characteristics of motion. Such a transition is widely used, not only for the study of vibro-impact systems. The solutions obtained in a singular approximation usually adequately describe the dynamic behaviour of the systems containing strongly

4.4).

substituted by a singular nonlinearity describing the force of the impact (see Sect. interaction. As a result, the threshold-type nonlinearities from the class $\{\Phi\}_\Delta$ will be simplify this procedure, we can consider the momentary impact instead of the fast Relations (4.36) and (4.37) "switch" the solutions of the two equations in (4.34). To which may be treated as an equation in J_2 .

(4.37)

$$\Delta = -T_2^{-1} L_k F^{-k}(\lambda, J_2)$$

conditions yields

equivalent conditions $x(0) = \Delta$ and $x(t_2) = \Delta$ may be used. The first of these

In representation (4.36), the integral of motion J_2 is undefined. For its definition two

$$F^{-k}(\lambda, J_2) = \int_{t_2(J_2)}^{\infty} \lambda \Phi[x_+(t, J_2)] e^{-ik\omega_2 t} dt = \int_{t_2(J_2)}^{\infty} \lambda \Phi[x_+(t, J_2)] e^{-ik\omega_2 t} dt$$

where

$$(4.36) \quad x(t) = -T_2^{-1} L_k F^{-k}(\lambda, J_2) e_k(t)$$

desired solution

Substitution of the series representing the PGF into the above integral yields the

where $L_k = (\Omega^2 - k^2 \omega_2^2)^{-1}$ and summation is performed over all integers k .

$$x(t) = T_2^{-1} L_k e_k(t) = \frac{\cos \left[\Omega \left(t - \frac{T_2}{2} \right) \right]}{2\Omega \sin \left(\frac{\Omega T_2}{2} \right)}, \quad t \in [0, T_2]$$

where the PGF $x(t)$ by virtue of (2.1) and (2.14), is written

$$x(t) = -\lambda \int_{t_2(J_2)}^0 x(t-s) \Phi[x_+(s, J_2)] ds$$

nonlinear elements of threshold type. Unlike the method of direct linearisation (replacing a nonlinear system with a linear system), singularisation replaces a strongly nonlinear system with a system with much stronger nonlinearity. As a matter of fact, in all studies on the dynamics of vibro-impact systems with momentary impact, the strongly nonlinear forces of threshold type are considered in a singular approximation. Consider equation of motion (4.32) along with the equation of motion of the asymmetric vibro-impact system, which may be written using (4.16) as

$$\ddot{x} + \Omega^2 x = -J\delta_T(t), \quad x(t) \leq \Delta$$

where the impact coincides with the initial time, and the impulse of the impact in accordance with the condition $x(0) = \Delta$ is given by the expression

$$J = -\frac{\mathcal{X}(0)}{\Delta}$$

and the possible periods of vibration T are defined by formulae (4.11)–(4.13). Let the vibration with impacts develops at some particular level of the full energy $E \in [E_0, \infty]$. This will be for all systems from the class $\{\Phi\}_\Delta$ because the velocity at the beginning of the interaction is $V_0 \neq 0$. It may be revealed that as $\lambda \rightarrow \infty$, the solution (4.36) which satisfies (4.37) changes into the “vibro-impact” solution

$$x(t) = -JT^{-1}L_k e^{\lambda t}(t), \quad J = 2V(-0) = V_0, \quad x(0) = \Delta$$

As $\lambda \rightarrow \infty$ the right-hand stop point of the interacting body is $x_2(E) \rightarrow \Delta$. That is why $x(t) \leq \Delta$ for all t as $\lambda \rightarrow \infty$. An expression for the potential of the system takes the form

$$\Pi_\lambda(x) = \frac{1}{2}\Omega^2 x^2 + \lambda \int_x^\Delta \Phi(x) dx$$

and contains the integral with upper limit $x \geq \Delta$. As $\lambda \rightarrow \infty$, the integral $\int_x^\Delta \Phi(x) dx$ tends to zero. The entire term $\lambda \int_x^\Delta \Phi(x) dx$ in the latter expression represents the kinetic energy, which is accumulated by the obstacle. In accordance with the law of energy conservation, this term is constant and equals $\frac{V_0^2}{2}$ where V_0 is the velocity at the beginning of the interaction. Therefore, $\Pi_\infty(x) = \frac{\Omega^2 x^2}{2}$ when $x = \Delta - 0$, and

$\Pi_\infty(x) = \frac{\Omega^2 x^2 + V_0^2}{2}$ when $x = \Delta + 0$. It can be observed readily that the limiting potential possesses the step-wise increase which equals the energy of the interaction.

wall of height $\left(\frac{2}{V_0^2}\right)$.

In Fig. 4.15, the limiting potential is sketched as a potential well with a rectangular

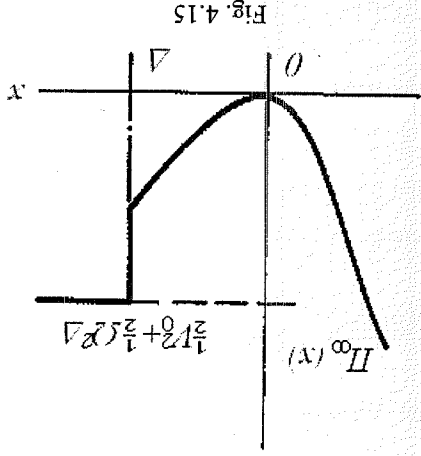


Fig. 4.15

It can be proven that for the interaction time given by the second integral from (4.35) $\lim_{\lambda \rightarrow \infty} t_\lambda = \lim_{\lambda \rightarrow \infty} \int_{x_2(E)}^\Delta \sqrt{2} dx \sqrt{E - \Pi_\lambda(x)} = 0$ and the impulse of interaction as $\lambda \rightarrow \infty$, is

$$J_\lambda = \int_{t_\lambda}^0 \left\{ \Omega^2 x_+^+(t) + \lambda \Phi[x_+^+(t)] \right\} dt \rightarrow J = 2V_0$$

From here it follows that the sequence $\{\lambda \Phi[x_+^+(t), J_\lambda(t)]\}$ is δ -shaped, and in the sense of generalised functions it tends to $J\delta(t)$, $0 \leq t < T$. Hence, $F_{-k}(\lambda, J_\lambda) \rightarrow J$. Representation (4.36) transforms into the vibro-impact representation and the condition of interaction $x(0) = x(t_\lambda) = \Delta$ transforms into the condition of impact $x(0) = \Delta$. The period of vibration T_∞ is given by the first integral from (4.36). The aforementioned reasoning for large λ allows for the replacement of the “preliminary” solutions with that obtained in the vibro-impact case. However, in a number of cases (even for large λ) it is important to keep the specifics of the initial nonlinearity. This can be done using an improved singularisation procedure.

4.9. Improved Singularity

To obtain more satisfactory solutions, the specifics of the initial nonlinearity may be introduced into the motion laws which are obtained using the singularisation procedure. Series (4.36) may be evaluated by considering

$$x(t) = -T^{-1} L_k F_{-k}(\lambda, J_\lambda) e_k(t)$$

where values λ and J_λ are treated as parameters, $F_{-k} \equiv F(-ik\omega)$ and

$$L_k = (\Omega^2 - k^2 \omega^2)^{-1}. \text{ By making use of the quantities } |F(-ik\omega)| \text{ and } \arg[F(-ik\omega)],$$

we find

$$F_{-k} = |F(-ik\omega)| e^{-i \arg[F(-ik\omega)]}$$

It may be shown readily that

$$\arg[F(-ik\omega)] = -\frac{1}{2} t_\lambda k \omega$$

To prove this we note that $\varphi_\lambda(t) = \lambda \Phi[x^+(t)]$ and

$$\arg \left[F(-ik\omega) e^{\frac{ik\omega t_\lambda}{2}} \right] = \arg \left\{ e^{\frac{ik\omega t_\lambda}{2}} \int_{t_\lambda}^0 \varphi_\lambda(t) e^{-ik\omega t} dt \right\} = 0 \quad (4.38)$$

since

$$\operatorname{Im} I = \int_{t_\lambda}^0 \sin \left[k\omega \left(\frac{1}{2} t_\lambda - t \right) \right] \varphi_\lambda(t) dt = - \int_{t_\lambda/2}^{-t_\lambda/2} \sin(k\omega u) \varphi_\lambda \left(u + \frac{1}{2} t_\lambda \right) du = 0$$

In fact, the quantity

$$I = \exp \left(\frac{ik\omega t_\lambda}{2} \right) \int_{t_\lambda}^0 \varphi_\lambda(t) \exp(-ik\omega t) dt$$

$$= \left[\cos \left(\frac{k\omega t_\lambda}{2} \right) + i \sin \left(\frac{k\omega t_\lambda}{2} \right) \right] \int_{t_\lambda}^0 (\cos k\omega t - i \sin k\omega t) \varphi_\lambda(t) dt$$

is real, since

$$\operatorname{Im} I = \int_{t_\lambda}^0 \sin \left[k\omega \left(\frac{1}{2} t_\lambda - t \right) \right] \varphi_\lambda(t) dt = - \int_{t_\lambda/2}^{-t_\lambda/2} \sin(k\omega u) \varphi_\lambda \left(u + \frac{1}{2} t_\lambda \right) du = 0$$

which is due to the fact that interaction is elastic and, therefore, the function

$\varphi_\lambda(u + \frac{1}{2} t_\lambda)$ is symmetrical with respect to the line $u = 0$. Hence, equality (4.38)

holds. For two complex numbers z_1 and z_2 ,

$$\arg z_1 z_2 = \arg z_1 + \arg z_2$$

and so from (4.38) we find

$$\arg F(-ik\omega) + \arg e^{\frac{ik\omega t_\lambda}{2}} = 0$$

and, finally,

$$\arg[F(-ik\omega)] = \frac{1}{2} k\omega t_\lambda$$

Q.E.D.

The introduction of the above formulae into the representation for $x(t)$ yields

$$x(t) = -T^{-1} L_k |F_{-k}| e_k \left(\frac{t}{2} \right) e_k(t)$$

Since $e_{-k}(\frac{1}{2} t_\lambda) e_k(t) = e_k(-\frac{1}{2} t_\lambda) e_k(t) = e_k(t - \frac{1}{2} t_\lambda)$, we find

$$x(t) = -T^{-1} L_k |F_{-k}| e_{-k}(t'), \quad t' = t - \frac{1}{2} t_\lambda$$

The above series may be summed on $0 \leq t' < T$ using the procedure as discussed in Sect. 2.4, where the formula

$$\chi(t) = \sum_{p_n} \operatorname{Res}_{(p_n)} \left\{ L(p) e^{pt} \right\} \frac{1 - e^{pT}}{1 - e^{pT}}$$

for the explicit representation of the PGF on $0 \leq t < T$ was obtained. Here $\operatorname{Res}_{L(p)}$ denotes the residual of the function $L(p)$ at the singular point p_n and the summation is provided for the poles of the function $L(p)$ only. Similarly, for $x(t')$ on $0 \leq t' < T$,

$$\chi(t') = - \sum_{p_n} \operatorname{Res}_{(p_n)} \left\{ L(p) F_0(-p; \lambda, J_\lambda) e^{pt'} \right\} \frac{1 - e^{pT}}{1 - e^{pT}} \quad (4.39)$$

where $L(p) = (p^2 + \Omega^2)^{-1}$ and the functions $F_0(-p; \lambda, J_\lambda)$ may be obtained from the function $|F(-ik\omega; \lambda, J_\lambda)| = |F_{-k}|$ using of the substitution $p = ik\omega$. In formula (4.39) the summation is over the poles of the function $L(p) F_0(-p)$ only. The first two poles $p_1 = i\Omega$, $p_2 = -i\Omega$ correspond to the operator $L(p)$. Thus, equation (4.39) may be

rewritten

$$\chi(t') = - \left\{ \sum_{\pm i\Omega} \operatorname{Res}_{(\pm i\Omega)} \left[\frac{L(p) F_0(-p; \lambda, J_\lambda) e^{pt'}}{1 - e^{pT}} \right] + \sum_{n>2} \operatorname{Res}_{(p_n)} \left[\frac{L(p) F_0(-p; \lambda, J_\lambda) e^{pt'}}{1 - e^{pT}} \right] \right\}$$

where p_n , $n > 2$, denote the poles $F_0(-p)$. Since the time interval t_λ which defines

$$F_0(-p; \lambda, J_\lambda) = \left| \int_{t_\lambda}^0 \varphi_\lambda(t') e^{pt'} dt' \right|$$

is small, the poles $F_0(-p)$ are situated far to the left of the imaginary axis, so all $\operatorname{Re} p_n$ at large λ and $n > 2$ are small quantities, and the second sum in the right-hand

part of the representation of $x(t')$ may be neglected. As a result of simple manipulations (see Sect. 2.4) we find

$$x(t') = -|F(i\Omega; \lambda, J_\lambda)| \left\{ \frac{1}{1 - e^{i\Omega t'}} - \frac{1}{2i\Omega} \frac{1 - e^{-i\Omega T}}{1 - e^{-i\Omega T}} + \dots \right\}$$

or, with the accuracy of the neglected terms

$$x(t') = -|F(i\Omega; \lambda, J_\lambda)| \frac{\sin(\Omega T/2)}{\cos[\Omega(t' - T/2)]}, \quad t - \frac{1}{2}t_\lambda = t' \in [0, T]$$

which is equivalent to the notation

$$x(t) = -|F(i\Omega; \lambda, J_\lambda)| \chi(t - \frac{1}{2}t_\lambda) \quad (4.40)$$

The function $\chi(t - \frac{1}{2}t_\lambda)$ may be given using formula (2.20)

$$\chi\left(t - \frac{1}{2}t_\lambda\right) = \begin{cases} \frac{\cos[\Omega(t - t_\lambda/2 - T/2)]}{\cos[\Omega(t - t_\lambda/2 + T/2)]}, & -T \leq t - \frac{1}{2}t_\lambda \leq 0 \\ \frac{\sin(\Omega T/2)}{\cos[\Omega(t - t_\lambda/2 - T/2)]}, & 0 \leq t - \frac{1}{2}t_\lambda < T \end{cases}$$

From (4.40) we find that in the framework of the approximation approach used, the solution possesses a derivative discontinuity at $t = \frac{1}{2}t_\lambda$ which corresponds to the momentary impact taking place within time $\frac{1}{2}t_\lambda$ after the coordinate x attains the level of the obstacle Δ . Therefore, the solution (4.36) may be correlated with the singular nonlinearity (4.17) with the time lag $\frac{1}{2}t_\lambda$ (see (4.16))

$$\Phi_0[x(t), x(t)] = J\delta_T^2(t - \frac{1}{2}t_\lambda)$$

The condition of the impact $x(0) = x(t_\lambda) = \Delta$ yields the transcendental equation for the impulse of the interaction force

$$-|F(i\Omega; \lambda, J_\lambda)| = \frac{\chi(\frac{1}{2}t_\lambda)}{\Delta}, \quad t_\lambda = t_\lambda(J_\lambda) \quad (4.41)$$

where $x(-t) = \chi(t)$, and t_λ depends on the total energy E , and therefore on the impulse J_λ .

The above transformations allow for improvement of the singular approximation for the solution of problems with threshold-type strongly nonlinear forces. The improved singular approximation was obtained under the assumption that the system with finite-time interactions is replaced with the vibro-impact system containing the "floating" absolutely rigid obstacle. The coordinate of the above obstacle $-|F(i\Omega; \lambda, J_\lambda)|\chi(0)$ depends on the value of the impact force because the equivalent impacts take place in the time instants at the full halt $t = \frac{1}{2}t_\lambda(J_\lambda) + kT$, $k = 0, \pm 1, \dots$, instead of the instants

$t = kT$. Equation (4.41) may be simplified using the assumption that for large λ (small t_λ)

$$|F(i\Omega; \lambda, J_\lambda)| = \left| \int_{t_\lambda}^0 \varphi_\lambda(t) e^{-i\Omega t} dt \right| \approx \left| \int_{t_\lambda}^0 \varphi_\lambda(t) dt \right|$$

It can be seen readily that the latter integral in the above expression represents the impulse $J_\Phi \approx J_\lambda$ of the force $\lambda\Phi[x^+(t)]$ over the time of interaction t_λ . Such a simplification yields, for formula (4.41),

$$J_\lambda = -\frac{\chi(\frac{1}{2}t_\lambda)}{\Delta}, \quad t_\lambda = t_\lambda(J_\lambda) \quad (4.42)$$

The relations (4.40) and (4.42) may be derived in an easier way. Solution (4.36) may be obtained through the integral equation

$$x(t) = -\lambda \int_{t_\lambda(J_\lambda)}^0 \chi(t-s)\Phi[x^+(s), J_\lambda] ds$$

where due to the smallness of t_λ this integral may be rearranged using the mean-value theorem. By definition $\Phi[x^+(s)] \geq 0$, therefore

$$x(t) = -\chi[t - \Theta(t)] \left[\int_{t_\lambda(J_\lambda)}^0 \lambda\Phi[x^+(s), J_\lambda] ds \right]$$

where $0 < \Theta(t) < t_\lambda(J_\lambda)$ and the integral in the above expression equals J_Φ which represents the impulse of the force $\Phi[x^+(t)]$ over the time t_λ , i.e.

$$x(t) = -J\Phi\chi[t - \Theta(t)]$$

The approximations $J_\Phi \approx J_\lambda$, $\Theta(t) \approx \frac{1}{2}t_\lambda(J_\lambda)$, $|F(i\Omega; \lambda, J_\lambda)| \approx J_\lambda$ yield (4.40). Since $\chi(0) = \chi(0; \omega)$ with $\omega = 2\pi T^{-1}$, Eq. (4.41) or Eq. (4.42) define the dependence of the impulse of the interaction on the frequency of vibration, or in other words, an approximate equation of the backbone curve for the integral of motion J_λ . The use of the definition for the PGF yields instead of (4.42),

$$J_\lambda = -2\Omega\Delta \frac{\cos\left[\frac{1}{2}\Omega(t_\lambda - 2\pi/\omega)\right]}{\sin(\pi\Omega/\omega)}, \quad t_\lambda = t_\lambda(J_\lambda)$$

The above approximate solutions for large enough λ approach the exact solutions for parameters of the initial problem. Hence, the regime defined in the framework of the improved singular approach reflects the specifics of the initial system. The problem of restoration of the potential, which is based on the time of motion, is solved in reference [99]. It is shown that under certain conditions the above problem possesses a

unique solution. Note, since the arbitrary constant t_0 ($x = x(t+t_0)$) is introduced into the solution we can not consider $t_2 = 0$, because the value of t_2 is a function of J_λ , which in turn is independent of t_0 .

In the following sections examples of singularisation for specific systems are considered.

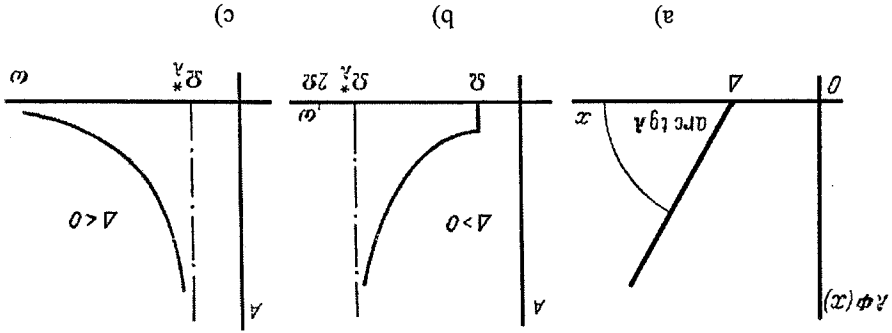


Fig. 4.16

4.10. Piecewise Linear Force of Threshold Type

In this example the force $\Phi(x) \in \{\Phi\}_\Delta$ is described using the piecewise linear

function $\lambda\Phi(x) = \lambda(x - \Delta)\eta(x - \Delta)$ (see Fig. 4.16a), where

$$\eta(x) = \begin{cases} 1, & x > 0 \\ 0, & x < 0 \end{cases}$$

is the Heaviside function and λ is large parameter. The equation of motion

$$x + \Omega^2 x + \lambda(x - \Delta)\eta(x - \Delta) = 0 \quad (4.43)$$

may be rewritten in the form of two linear equations

$$x + \Omega^2 x = 0, \quad x^- \leq \Delta; \quad x^+ + (\Omega^2 + \lambda)x^+ = \lambda\Delta, \quad x^+ \geq \Delta$$

and a general form of the solution may be obtained by stitching particular solutions.

However, it turns out to be too tedious.

Nevertheless, Eq. (4.43) along with different analogues, are well studied, see

references [1, 104, 196]. In Fig. 4.16b,c the schematic backbone curves of this system

are portrayed for the cases $\Delta > 0$ and $\Delta < 0$, respectively. The frequency Ω_λ^* which

corresponds to the case of infinite energy is given by the formula

$$\Omega_\lambda^* = 2\sqrt{\frac{\lambda + \Omega^2}{\lambda + \Omega^2 + \Omega}} \approx 2\Omega \left(1 - \frac{\sqrt{\lambda}}{\Omega}\right) \quad (4.44)$$

where $\lim_{\lambda \rightarrow \infty} \Omega_\lambda^* = 2\Omega$. It reflects the fact that for large λ the solution of (4.43) may be

approached adequately by the solution of the vibro-impact system (see Sect. 4.9). The above methods may be applied to Eq. (4.43). In order to obtain an improved singular approximation we have to define the time of interaction t_λ .

The motion for $x \geq \Delta$ is described by the following set

$$x^+ + (\Omega^2 + \lambda)x^+ = \lambda\Delta, \quad x^+(0) = \Delta, \quad x^+(0) = \frac{1}{2}J_\lambda$$

which yields

$$x^+(t) = \frac{J_\lambda}{2\sqrt{\Omega^2 + \lambda}} \sin(\sqrt{\Omega^2 + \lambda}t)$$

$$+ \frac{\Delta\Omega^2}{\lambda + \Omega^2} \cos(\sqrt{\Omega^2 + \lambda}t) + \frac{\lambda + \Omega^2}{\lambda\Delta}$$

The condition of full halt $x^+(\frac{1}{2}t_\lambda) = 0$ yields

$$t_\lambda = \frac{2\pi - 2\alpha_\lambda}{\Omega} = O\left(\frac{\sqrt{\lambda}}{1}\right)$$

where

$$\sin \alpha_\lambda = \frac{1}{2}J_\lambda A_\lambda^{-1}, \quad \cos \alpha_\lambda = -\frac{\lambda + \Omega^2}{\Delta\Omega^2} A_\lambda^{-1}$$

$$A_\lambda = \left[\frac{1}{4}J_\lambda^2 + \cos \alpha_\lambda + \frac{\lambda + \Omega^2}{\Delta\Omega^4} \right]^{1/2}$$

The calculation of $|F(1/2; \lambda, J_\lambda)|$ is required for a construction of the solution of

(4.40). In this case, using the representation for $x^+(t)$, it may be done easily, but the expression for impulse (see (4.41)) will be too awkward. For the analytical calculations it is more convenient to use formula (4.42), which yields

$$J_\lambda = -\frac{\chi(1/2t_\lambda)}{\Delta}, \quad t_\lambda = t_\lambda(J_\lambda)$$

where $t_\lambda(J_\lambda)$ was obtained above.

In the framework of study of resonant vibration associated with large impulses of the interaction force. Additional simplification may be considered. For the above system, the total energy takes the form

$$E = \frac{1}{8}J_\lambda^2 + \frac{1}{2}\Omega^2\Delta^2, \quad J_\lambda = \sqrt{8E - 4\Omega^2\Delta^2} \quad (4.45)$$

In the case of intensive interactions $E \approx \frac{1}{8}J_\lambda^2$, the expressions for t_λ may be simplified. An assumption that the quantities of orders Δ and Ω^2 may be neglected

in comparison with the quantities of orders $O(J_1^2)$ and $O(\lambda)$, respectively, yields simplifications in the formulae for t_2 and J_2 ,

$$(4.46) \quad t_2 \approx \frac{\sqrt{\lambda}}{\pi}, \quad J_2 = -\frac{\Delta}{\Delta} \frac{\chi(\frac{1}{2}t_2)}{\Delta} \approx -\frac{\chi(0) + \pi/(4\sqrt{\lambda})}{\Delta}$$

In our case $\chi(0) = (2\Omega)^{-1} \cot(\pi\Omega/\omega)$ and $\omega > \Omega$. The frequency ω_* corresponds to the case of infinite energy. At this frequency $\chi(\frac{1}{2}t_2) = 0$. With provision for (4.46) the above frequency may be obtained from the equation

$$(4.47) \quad \frac{\pi\Omega}{2} = \text{atan} \left(-\frac{2\sqrt{\lambda}}{\Omega\pi} \right)$$

The use of the linear terms in the expansion $\text{atan } x = \frac{\pi}{2} - x + \dots$ ($|x| < 1$) in (4.47) yields

$$(4.48) \quad \omega_* \approx 2\Omega(1 - \Omega/\sqrt{\lambda})$$

The value of the frequency ω_* may be estimated in a different manner. Take into account the period of vibration T_2 comprising the period of vibration of the vibro-impact system T_∞ and the time t_2 . In the case of infinite energy, $E = \infty$, the period of vibration is $T_\infty = \pi\Omega^{-1}$, and using the first equation of set (4.46) we find again

$$\omega_* = \frac{2\sqrt{\lambda}\Omega}{\sqrt{\lambda} + \Omega} \approx 2\Omega \left(1 - \frac{\sqrt{\lambda}}{\Omega} \right)$$

A comparison of (4.48) and (4.44) yields $\Omega_* \approx \omega_*$, i.e. the frequency of vibration in the case of infinite energy is defined practically exactly.

The solution

$$x(t) = -J_2 \chi \left(t - \frac{\pi}{2\sqrt{\lambda}} \right)$$

where J_2 is given by formula (4.46), approximates the desired solution well in the case of large impulses, since $E \approx \frac{1}{8}J_2^2$. The frequency of free vibration is close to (4.44), and in the zone of interaction, $x \geq \Delta$, the term $\Omega^2 x$ is neglected.

For small energies, the above neglect is not valid. If Eq. (4.43) is replaced by the set

$$(4.49) \quad \ddot{x}^- + \Omega^2 x^- = 0, \quad \dot{x}^- \leq 0; \quad \ddot{x}^+ + \lambda x^+ = \lambda\Delta, \quad x^+ > 0$$

It was considered here that $\dot{\chi}(0) = \frac{1}{2}$.

then instead of the restoring force $\Omega^2 x + \lambda(x - \Delta)\eta(x - \Delta)$, the discontinuous force $\Omega^2 x + [\lambda(x - \Delta) - \Omega^2 x]\eta(x - \Delta)$ (see Fig. 4.17a) is considered.

Such a force corresponds to the problem of vibration of a unit mass body which is suspended by a spring with a stiffness of Ω^2 under the condition that during the interaction time $t_2 = \pi(\sqrt{\lambda})^{-1}$ this spring is "switched off" and replaced with a very stiff spring λ . The backbone curve of such a system in the case $\Delta > 0$ is portrayed in Fig. 4.17b. This graph indicates that at low energies (amplitudes) spurious conclusions may be drawn. Within a small vicinity of the frequency $\omega = \Omega$ the system is softly

anisochronous if $\omega < \Omega$. With an increase in the energy level, the curve in Fig. 4.17b progressively better approximates the original one. The false "boundary" information at $x = \Delta + 0$ becomes less important at higher levels of energy.

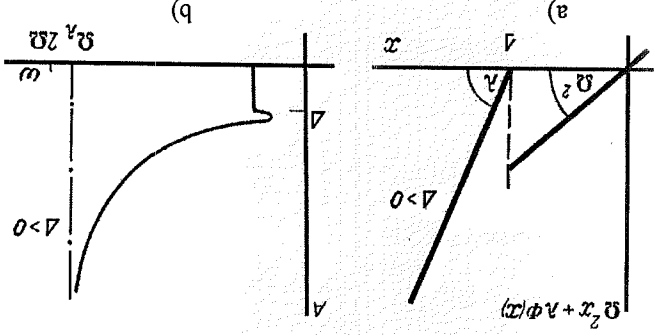


Fig. 4.17

System (4.49) may be easily solved in an explicit form [94]. For the case $\Delta \neq 0$ we find

$$x^+(t) = \Delta + \frac{1}{2}J_2 \chi(\sqrt{\lambda})^{-1} \sin \sqrt{\lambda} t, \quad t \in [0, t_2], \quad t_2 = \pi(\sqrt{\lambda})^{-1}$$

Then $\lambda \Phi[x^+(t)] = \frac{1}{2}J_2 \sqrt{\lambda} \sin \sqrt{\lambda} t, \quad t \in [0, t_2]$. Calculation of the quantity of $F_{-k}(\lambda, J_2)$, from (4.36) yields

$$F_{-k}(\lambda, J_2) = \frac{1}{2}J_2 \lambda J_2 (\lambda - k^2 \omega^2)^{-1} [\exp(-ik\omega t_2) + 1]$$

For $\Delta \neq 0$ this series may be rewritten in the form

$$(4.50) \quad x(t) = -J_2 T^{-1} L_k(\lambda) e_k(t), \quad L_k(\lambda) = \frac{1}{\lambda} [\exp(-ik\omega t_2) + 1] \frac{2(\Omega^2 - k^2 \omega^2)(\lambda - k^2 \omega^2)}{2(\Omega^2 - k^2 \omega^2)(\lambda - k^2 \omega^2)}$$

4.11. Threshold-type Force Defined by the Power Function

Now consider the equation of motion

$$\ddot{x} + \Omega^2 x + \lambda(x - \Delta)^\alpha \eta(x - \Delta) = 0, \quad \alpha \geq 1$$

An improved singular approximation to the exact solution may be defined using inversion of the quadrature

$$t - t_0 = \frac{1}{\lambda} \int \left[E - \frac{1}{2} \Omega^2 x^2 - \frac{\alpha+1}{\lambda} (x - \Delta)^{\alpha+1} \eta(x - \Delta) \right]^{-1/2} dx$$

The initial equation of motion may be expressed as

$$\ddot{x}^- + \Omega^2 x^- = 0, \quad x^- \leq \Delta; \quad \ddot{x}^+ + \Omega^2 x^+ + \lambda(x^+ - \Delta)^\alpha = 0, \quad x^+ \geq \Delta$$

In the case of large impulses, $E \approx \frac{8}{J_\lambda^2}$, the second equation may be replaced by

$$x^+ + \lambda(x^+ - \Delta)^\alpha = 0$$

The first root of the equation $8\lambda(x - \Delta)^{\alpha+1} = (\alpha+1)J_\lambda^2$ defines the coordinate of the

full arrest $x_2(J_\lambda)$,

$$x_2(J_\lambda) = \Delta + \left[\frac{(\alpha+1)J_\lambda^2}{8\lambda} \right]^{1/(\alpha+1)}$$

The time of interaction (see [99]) is

$$t_\lambda = \int_{x_2(J_\lambda)}^\Delta \frac{\sqrt{2} dx}{\sqrt{\frac{1}{8} J_\lambda^2 - c_\lambda^2 (x - \Delta)^{\alpha+1}}}$$

where $c_\lambda \equiv \lambda(\alpha+1)^{-1}$. Due to the representation for $x_2(J_\lambda)$ we find

$$t_\lambda = \int_{B(J_\lambda)}^0 \frac{\sqrt{2} dx}{\sqrt{\frac{1}{8} J_\lambda^2 - c_\lambda^2 x^{\alpha+1}}}$$

where $B(J_\lambda) = \left(\frac{1}{8} c_\lambda^{-1} J_\lambda^2 \right)^{1/(\alpha+1)}$. The substitution $y^{\alpha+1} = 8c_\lambda J_\lambda^{-2} x^{\alpha+1}$ and rearrangements yield

For the solution of the contact problems it is usually considered that $\alpha = (2n+1)(2n)^{-1}$, where n is a natural number. In this case $1 \leq \alpha \leq \frac{3}{2}$. The value $\alpha = \frac{3}{2}$ corresponds to the hypothesis of Hertz [17,54].

where $J_\lambda = -\frac{\Delta T}{\sum_k^k L_k(\lambda)}$ in accordance with the condition $x(0) = \Delta$. It can be expressed in a finite form using the procedure of Sect. 2.4 because it defines the PGF corresponding to the operator

$$L_\lambda(p) = \frac{\lambda [\exp(-pt_\lambda) + 1]}{\lambda \exp(-pt_\lambda)} + \frac{2(p^2 + \Omega^2)(p^2 + \lambda)}{\lambda} + \frac{2(p^2 + \Omega^2)(p^2 + \lambda)}{\lambda}$$

Application of formula (2.20) yields

$$x(t) = \begin{cases} -J_\lambda \left[x_\lambda^1(t) + x_\lambda^3(t) \right], & t \in [0, t_\lambda] \\ -J_\lambda \left[x_\lambda^2(t) + x_\lambda^3(t) \right], & t \in [t_\lambda, T] \end{cases} \quad (4.51)$$

where the following functions are used

$$x_{1,2}^{(\lambda)}(t) = \frac{\lambda}{\cos \Omega(t - t_\lambda \pm \frac{1}{2}T)} \left[\frac{\cos \Omega(t - t_\lambda \pm \frac{1}{2}T)}{\cos \Omega(t - \frac{1}{2}T)} - \frac{\sqrt{\lambda} \sin \frac{1}{2} \Omega T}{\cos \sqrt{\lambda}(t - t_\lambda \pm \frac{1}{2}T)} \right]$$

$$x_3^{(\lambda)}(t) = \frac{\lambda}{\cos \Omega(t - \frac{1}{2}T)} \left[\frac{\cos \Omega(t - \frac{1}{2}T)}{\cos \Omega(t - \frac{1}{2}T)} - \frac{\sqrt{\lambda} \sin \frac{1}{2} \Omega T}{\cos \sqrt{\lambda}(t - \frac{1}{2}T)} \right]$$

The value of the impulse is obtained in the form

$$J_\lambda = -\Delta \left\{ \frac{\lambda}{\cos \Omega(t_\lambda - \frac{1}{2}T)} \left[\frac{\Omega}{\sin \frac{1}{2} \Omega T} + \cot \frac{\Omega T}{2} \right] - \frac{1}{\cos \sqrt{\lambda}(t_\lambda - \frac{1}{2}T)} \left[\frac{\sqrt{\lambda} T}{\sin \frac{1}{2} \sqrt{\lambda} T} + \cot \frac{\sqrt{\lambda} T}{2} \right] \right\}^{-1}, \quad t_\lambda = \pi(\sqrt{\lambda})^{-1}$$

Series (4.50) is valid for all $\omega > \Omega$ (when $\lambda = k\omega$ the singularity vanishes) except at the resonant frequency (4.48).

The above solution is based on the equality $t_\lambda = \pi\sqrt{\lambda}$, and therefore is valid only for high levels of energy.

This may be established using the finiteness of the function $\Phi[x^+(t)]$, see [62].

$$t_\lambda = \frac{(\frac{1}{8}J_\lambda^2)^{1/(a+1)-1/2}}{\int_1^{\sqrt{2}} \sqrt{2} dy} \frac{c_{1/(a+1)}^{1/(a+1)}}{\sqrt{1-y^{a+1}}}$$

The integral in the above expression may be calculated in a standard way using B - and therefore, the Γ -functions of Euler. As a result of the calculations taking into account the definition of quantity c_λ we find

$$t_\lambda = D_\lambda J_\lambda^\gamma, \quad \gamma = (1-\alpha)/(1+\alpha)$$

$$D_\lambda = \frac{\sqrt{2\pi}(\frac{1}{8})^{1/2} \Gamma(\alpha+1)^{-1}}{\left[\lambda(\alpha+1)^{-1} \Gamma(\alpha+1) \right]^{1/(a+1)} (\alpha+1) \Gamma(\alpha+3)/(2(\alpha+1))} \quad (4.52)$$

Here $D_\lambda = O(\lambda^{-1/(a+1)})$, and smaller terms are neglected. Taking into account (4.42) we find

$$J_\lambda = -\frac{\Delta}{\Delta} \frac{\chi(\frac{1}{2}t_\lambda) + \frac{1}{4}D_\lambda J_\lambda^\gamma}{\Delta} \approx -\frac{\chi(0) + \frac{1}{4}D_\lambda J_\lambda^\gamma}{\Delta} \quad (4.53)$$

$$\frac{1}{4}D_\lambda J_\lambda^{\gamma+1} + \chi(0)J_\lambda + \Delta = 0, \quad \gamma+1 = 2(\alpha+1)^{-1} \quad (4.54)$$

Since $\alpha \geq 1$, in the above equation $\gamma+1 \leq 1$. It may be solved graphically or numerically, and in explicit form in a number of particular cases. The period of vibration with large impulses of interaction is obviously defined as the sum $T_\infty + t_\lambda$, where T_∞ is the period of free vibration of the vibro-impact system and t_λ is the time of interaction. The general form of the improved singular approximation in the case of large impulses is given by the function $x(t) = -J_\lambda \chi[t - t_\lambda(J_\lambda) + t_0]$, where the impulse J_λ and the time $t_\lambda(J_\lambda)$ are defined in (4.54) and (4.52), t_0 is an arbitrary constant. For practical calculations it may be more convenient to use the impulse of the impact taken from the limiting solutions. Other types of threshold nonlinearities may be studied similarly.

The B -function is defined as follows [97].

$$B(a, b) = \int_0^1 y^{a-1} (1-y)^{b-1} dy, \quad a, b > 0. \text{ Or } B(a, b) = \Gamma(a)\Gamma(b)/\Gamma(a+b), \text{ where the}$$

Γ -function is given by $\Gamma(x) = \int_0^\infty y^{x-1} e^{-y} dy$, $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$. For integer n , $\Gamma(n) = n!$

4.12. Symmetric Threshold-Type Forces

The conclusions of Sections 4.7–4.11 may be applied in full to the study of systems with symmetric threshold-type forces which are described by the equations of motion

$$\ddot{x} + \Omega^2 x + \lambda \Phi^*(x) = 0 \quad (4.55)$$

where the forces $\lambda \Phi^*(x)$ are given by symmetric threshold-type functions as sketched in Fig. 4.18.

In order to obtain analogues of the main formulae of Sections 4.7–4.11 it is enough to replace the PGF $\chi(t)$ with the symmetric PGF $\chi^*(t)$ (see (4.9)). Corresponding replacements yield relationships similar to those obtained above.

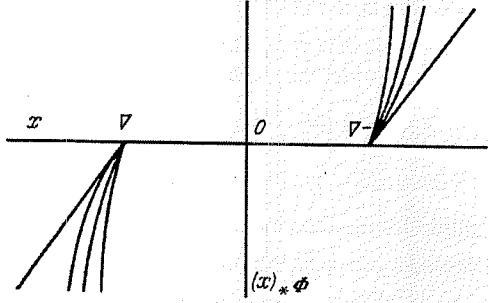


Fig. 4.18

As $\lambda \rightarrow \infty$ the solution of the equation of motion transforms into the "vibro-impact" solution (4.30). The improved singular approximations are represented in the form

$$x(t) = -J_\lambda \chi^*(t - \frac{1}{2}t_\lambda), \quad J_\lambda = -\Delta/\chi^*(\frac{1}{2}t_\lambda), \quad t_\lambda = t_\lambda(J_\lambda)$$

etc.

The examples in Sect. 4.10 and 4.11 may be recalculated similarly.

§5 Forced Vibration

5.1. Problem Statement

In this chapter we study the forced vibration of strongly nonlinear threshold-type systems. Such vibration is described by the equations

$$\ddot{x} + \Omega^2 x + \lambda \Phi(x) = \varepsilon [g_0(x, x) + P(t)] \quad (5.1)$$

where $\Phi(x) \in \{\Phi\}_V$. The function $g_0(x, x)$ reflects the additional small forces acting in the system (dissipative forces, for example); $P(t)$ is the T -periodic external forcing, $P(t+T) = P(t)$; λ and ε are large and small parameters, respectively. It is considered also that $\lambda = O(\varepsilon^{-\alpha})$, where α is a positive number.

Our goal is the search for the periodic vibration which develops in system (5.1). Along with the problem of periodic solutions, the Cauchy problem for (5.1) with the initial conditions

$$x(+0) = x_0, \quad \dot{x}(+0) = v_0 \quad (5.2)$$

is to be considered.

The solution of the Cauchy problem describes *transient processes*. Each time we assume that functions from the equation of motion provide the existence and the uniqueness of the solution for problems (5.1) and (5.2). The related mathematical problems are discussed in references [84-86, 135, 145].

The mechanical model in Fig. 5.1 comprises the body of the unit mass suspended by the resilient element, the elasticity and damping of which are given by the functions $\lambda\Phi(x)$ and $g_1(x, x)$, under the action of the external periodic excitation $\varepsilon P(t)$. Such a system corresponds to the above equation. As the coordinate x of the body crosses over the threshold level $x = \Delta$, the interaction with the obstacle, which is represented as a parallel combination of the purely elastic and dissipative elements $\lambda\Phi(x)$ and $g_2(x, x)$, takes place. In this case the damping force takes the form

$$g_0(x, x) = g_1(x, x) + g_2(x, x)$$

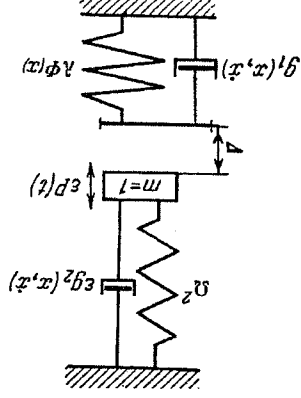


Fig. 5.1

5.2. Change of Variables

Denote the right-hand part of Eq. (5.1) by $g(x, x, t)$

(5.3)

$$\ddot{x} + \Omega^2 x + \lambda \Phi(x) = g(x, x, t)$$

Such an equation also defines the system with *parametric excitation*. The *standard form* of the above equation is $\dot{z} = \varepsilon Z(z, t)$, where z is the vector of phase variables and Z is the vector-function. The components of the vector z are slow time variables. Thus, it is reasonable to consider the system with the *averaged right-hand part*

$$\dot{z}_0 = \varepsilon Z_0(z_0), \text{ where } Z_0(z_0) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T Z(z_0, t) dt \text{ is the time mean value of the function } Z(z, t).$$

Such a transformation is an essential part of the *averaging principle*. The equation for the function $z_0(t)$ does not contain time in an explicit form and may be solved more easily than the original. We may expect that for $t = O(\varepsilon^{-1})$ for all t , the difference between $z(t)$ and $z_0(t)$ will not be too large [117].

The variable on which the averaging is produced (in this particular case it is time t) may be a dependent variable. In this case, it may be more effective to use a new transformed *fast phase* as such a variable. To transform Eq. (5.3) into the z variables, consider the conservative generating system where $\varepsilon = 0$

Our presentation mainly follows the monograph [119]. More precise analysis is given in Appendix 2.

The same method may be applied if the external force g is a function of the "slow time" $t = \varepsilon t$, $g = g(x, x, t, t)$. In this case $z_0 = \varepsilon Z_0(z_0, t)$. Such problems are beyond the scope of the present monograph (see reference [117]).

The general form solution for (5.4) takes the form

$$\ddot{x} + \Omega^2 x + \lambda \Phi(x) = 0 \quad (5.4)$$

depending on two integrals of motion A and ϕ [68]. Here the integral A is uniquely related to the total energy of system (5.4) $A = A(E)$, where $E = \frac{1}{2}\dot{x}^2 + \Pi_\lambda(x)$ denotes the total energy of the system, and ϕ is defined by the quadrature (4.6),

$$\frac{dx}{\sqrt{2[E - \Pi_\lambda(x)]}} = t - \phi$$

The integrals A and ϕ define the motion of the system and by virtue of the energy conservation principle

$$\dot{A} = \dot{\phi} = 0$$

Since the right-hand part of Eq. (5.3) is small it is reasonable to suggest that in a perturbed motion the quantities A and ϕ vary slowly: $\dot{A} = O(\varepsilon)$, $\dot{\phi} = O(\varepsilon)$. Hence, the new set of variables A , ϕ or A , $\psi = \omega_0(A)(t - \phi)$ may be used as a set of phase variables. The representation for the general solution of a conservative system may be considered as a formula of *transformation of variables* [119]

$$x = X(A, \psi), \quad \dot{x} = \omega_0(A)X_\psi(A, \psi), \quad X(A, \psi + 2\pi) = X(A, \psi) \quad (5.5)$$

where the subscript symbol denotes differentiation. Transformation (5.5) is reversible. Therefore, since $A(t)$ and $\psi(t)$ for the perturbed system are defined, it is possible to define $x(t)$ and $\dot{x}(t)$, and vice versa.

Transformation (5.5) with provision for (5.4) yields

$$\omega_0^2 X_{\psi\psi} + \Omega^2 X + \lambda \Phi(X) \equiv 0 \quad (5.6)$$

These transformations are compatible if

$$X_A A + X_\psi \psi = \omega_0 X_\psi$$

Substitution of (5.5) into (5.3) yields

$$(X_\psi \omega_0' + \omega_0 X_{\psi A} A + \omega_0 X_{\psi\psi} \psi) = -\Omega^2 X - \lambda \Phi(X) + \varepsilon g(X, \omega_0 X_\psi, t)$$

$$\omega_0' = \frac{d}{dt} \omega_0(A)$$

The above two relationships represent the system of linear equations in A and ψ with the principal determinant

$$d_0 = - \begin{vmatrix} X_{\psi A} X_\psi - X_{\psi\psi} X_A \\ \omega_0 - X_\psi^2 \omega_0' \end{vmatrix}$$

With provision for identity (5.6) the solution of the above system will be

$$A = -\frac{d_0}{\varepsilon} g(X, \omega_0 X_\psi, t) X_\psi, \quad \psi = \omega_0(A) A + \frac{d_0}{\varepsilon} g(X, \omega_0 X_\psi, t) X_A \quad (5.7)$$

where the latter equation is added to make system (5.7) autonomous.

The obtained equations, by virtue of the reversibility of transformation (5.5) are equivalent to the initial equation of motion, but can be analysed using the averaging principle. The above reasoning is very general and may be applied to a wide class of quasi-conservative systems. The specifics of strongly nonlinear systems are revealed below, where transformation (5.5) is constructed on the basis of the solutions obtained in the previous paragraph.

It is worth noting that the determinant d_0 is ψ -independent. This may be verified using the reasoning of reference [119].

Consider the expression for the total energy of the generative system

$$E(A, \psi) = \frac{1}{2} X_\psi^2 \omega_0^2 + \int_X [\Omega^2 X + \lambda \Phi(X)] dX$$

and find the derivatives E_A and E_ψ

$$E_A = \frac{d}{dA} \left[\frac{1}{2} X_\psi^2 \omega_0^2 + \int_X [\Omega^2 X + \lambda \Phi(X)] dX \right]$$

$$= \omega_0' \omega_0 X_\psi^2 + \omega_0^2 X_\psi X_{\psi A} + \frac{d}{dA} \int_X [\Omega^2 X + \lambda \Phi(X)] X_A dX$$

where

$$\frac{\partial E}{\partial A} dX = \frac{d}{dA} X_A dX$$

Using identity (5.6)

$$E_A = \omega_0' \left\{ X_{\psi A} X_\psi - X_{\psi\psi} X_A \right\} + X_\psi^2 \omega_0' = -\omega_0' d_0 \quad (5.8)$$

Similarly,

$$E_\psi = \omega_0^2 X_{\psi\psi} X_\psi + \frac{d}{d\psi} \int_X [\Omega^2 X + \lambda \Phi(X)] X_\psi dX$$

$$= X_\psi^2 \left[\omega_0^2 X_{\psi\psi} + \Omega^2 X + \lambda \Phi(X) \right] = 0$$

Hence, E and E_A are both ψ -independent, and (5.8) shows that d_0 is also ψ -independent, Q.E.D.

Relationship (5.8) reveals the physical sense of the parameter $d_0 = -\omega_0'(A)$ and demonstrates that Eq. (5.7) may be simplified significantly if the energy E is used as the variable A .

System (5.7) represents the particular case of so-called systems with *two fast phases*. The methods of analysis of such systems were developed by Bogolubov and Zubarev, Volosov and Morgunov, Mitropolsky, NN Moiseev, etc [117, 119, 189]. Note that equations like (5.7) may be obtained without knowledge of the general form

of the solution of the generating system. Following Volosov [190], if the first integrals of the system are known, it becomes possible to construct corresponding equations which are better suited for the direct application of the averaging technique directly from the form of the above integrals.

The analysis of equations (5.7) indicates that there exist two types of variables. The variable A possesses the derivative which is proportional to the small parameter; this variable describes the slow evolution (envelope) of the process. The variables ψ and t are fast variables whose derivatives have order of unity. The above fast variables describe a small vibration which is superimposed on the slow evolution of a process. The presence of such a vibration causes the equations of motion to be nonintegrable. An application of the averaging method smoothes the fast vibration, and therefore, results in more simple equations for the envelope of the process. The influence of the averaged fast variables may be accounted for by means of so-called higher approximations. The calculation of the higher approximations is beyond the scope of this monograph. The general schemes of asymptotic methods are not discussed here (see recommended references).

As indicated above, in accordance with the averaging principle, the system $\dot{z} = \varepsilon Z(z, t)$, which contains the time t as a fast variable, is to be replaced with the averaged system $\dot{z}_0 = \varepsilon Z_0(z_0)$. The case of system (5.7) is more complicated. Here the fast variable is a two-component vector. Direct averaging of the fast variables may result in simultaneous averaging of the slow variable reflecting the process evolution. Such cases are said to be resonance cases. At resonance, the amount of slow variables increases as the amount of fast variables decreases.

5.3. Resonant Processes

The resonant processes which are close to the free vibration of the generating conservative system are of special importance in nonlinear mechanics. At resonance, the work done by the external source mainly compensates the energy losses due to the dissipation. As a matter of fact, the system at resonance shows dynamic behaviour which is typical to an undamped system in free vibration. The resonant vibration, in general, might be recognised through the build-up of relatively large amplitudes under a relatively small ($\propto \varepsilon$) level of external forcing and dissipation.

In Eq. (5.3) let $g(x, x, t)$ be a T -periodic function of time, $\omega = 2\pi T^{-1}$. Let $\omega_0(A)$ be the natural frequency of a generative system, a function of the integral A . Here we assume that the function $X[A, \omega_0(t - \varphi)]$ is a T_0 -periodic function of time, $T_0 = 2\pi\omega_0^{-1}$, and the function $\omega_0(A)$ is continuously differentiable at all A . The smallness of the detuning

$$\varepsilon\theta(A) = \omega - \omega_0(A) \quad (5.9)$$

indicates the presence of *fundamental resonance* in system (5.3). In accordance with this definition, we introduce the new variable φ into Eq. (5.7),

$$\omega\varphi = \omega t - \psi, \quad \omega\varphi = \omega - \psi \quad (5.10)$$

We are looking for the approximate solutions which differ slightly from the exact solutions by $O(\varepsilon)$ for all times $t \propto 1/\varepsilon$ or even $-\infty < t < \infty$.

Therefore, it is reasonable not to keep the quantities of order $O(\varepsilon^2)$ and higher in the equations of motion. The substitution of (5.9) and (5.10) into (5.7) with provision for (5.8)¹ yields

$$\begin{aligned} \dot{A} &= \frac{\varepsilon}{E} g \left\{ X[A, \omega(t - \varphi)], X[A, \omega(t - \varphi)], \omega t \right\} X[A, \omega(t - \varphi)] + \varepsilon\omega^{-1}\theta(A) \\ \dot{\varphi} &= \frac{\varepsilon}{E} g \left\{ X[A, \omega(t - \varphi)], X[A, \omega(t - \varphi)], \omega t \right\} X[A, \omega(t - \varphi)] + \varepsilon\omega^{-1}\theta(A) \end{aligned} \quad (5.11)$$

The use of the notation $\varepsilon E_1(A, \varphi, t)$ and $\varepsilon E_2(A, \varphi, t)$ for the right-hand parts of these equations yields

$$\dot{A} = \varepsilon E_1(A, \varphi, t), \quad \dot{\varphi} = \varepsilon E_2(A, \varphi, t) \quad (5.12)$$

Since A and φ are slow functions of time, the equations of system (5.12) may be replaced by time-averaged equations,

$$\dot{A} = \varepsilon A_{10}(A, \varphi), \quad \dot{\varphi} = \varepsilon E_{20}(A, \varphi) \quad (5.13)$$

These equations are said to be the *abbreviated equations*, that is:

$$E_{j0}(A, \varphi) = \frac{1}{T} \int_0^T E_j(A, \varphi, t) dt, \quad j = 1, 2$$

It is known that slowly varying functions might be accurately replaced by their mean values, reflecting an evolution of the process (envelope). Averaging smoothes the fast vibration represented in a process. Equations (5.13) do not contain time in an explicit form and, therefore, are more simple than the original equations of motion.

$$\varepsilon\theta(A) = p\omega - q\omega_0(A) \quad (5.14)$$

where p and q are coprime numbers, indicates the presence of the *combination resonance*² $p:q$ in system (5.3). The above resonance is said to be *sub-periodic resonance* in the case where $p = 1$ (type of resonance is $1:q$) and *super-periodic (ultraharmonic) resonance* if $q = 1$ (type of resonance is $p:1$).

For a description of combination resonances the new slow variable³ is introduced

¹ From these expressions it is seen that at resonance we have two slow variables. In *multi-frequency* systems, such as linear parametric systems, in particular, the term *combination resonance* has another meaning.

² Such a transformation is not always sufficient for the description of combination

(5.15)

$$q\omega\phi = p\omega t - q\psi, \quad q\omega\phi = p\omega - q\psi$$

and system (5.7) is then transformed into

(5.16)

$$\begin{aligned} \dot{A} = & \frac{qE_A}{\varepsilon p} g \left\{ X \left[A, \omega \left(\frac{q}{p} t - \phi \right) \right], X \left[A, \omega \left(\frac{q}{p} t - \phi \right) \right], \omega t \right\} \\ & \times X \left[A, \omega \left(\frac{q}{p} t - \phi \right) \right] \\ \dot{\phi} = & \frac{qE_A}{\varepsilon p} g \left\{ X \left[A, \omega \left(\frac{q}{p} t - \phi \right) \right], X \left[A, \omega \left(\frac{q}{p} t - \phi \right) \right], \omega t \right\} \\ & \times X \left[A, \omega \left(\frac{q}{p} t - \phi \right) \right] \left[+ \varepsilon \frac{q}{\theta(A)} \right] \end{aligned}$$

The obtained system (within the accuracy of the notation) is of a structure which is similar to (5.12). Hence, the averaged system is given by expressions (5.13).

The above transformations are required in the analysis of resonant processes. The direct averaging of the right-hand part of (5.7) containing two fast variables yields, for

the first equation for A ,

$$\dot{A} = -\frac{1}{\varepsilon} \int_0^{2\pi} \int_0^T g(X, \omega_0 X^\psi, t) X^\psi d\psi dt \quad (5.17)$$

It turns out that under conditions (5.9) or (5.14) such a procedure of double averaging is senseless [119, 190]. Equation (5.17) is, therefore, feasible in nonresonant cases

only. The averaged equations in the above form are defined by the general solutions of the corresponding conservative problems. In the previous paragraph such exact and approximate solutions were obtained.

5.4. Averaging in Systems with Impact Interactions

Usually we assume that all forces acting upon a system are described using regular functions. For us, however, the most important case is when the potential force from (5.1) or (5.3) is given by formulae (4.17) or (4.20). Such *singular forces* (as shown in the previous section) give the possibility of obtaining the equiform approximate representations for the general solutions of the conservative threshold, strongly nonlinear systems, and even exact solutions in the limiting case $\lambda \rightarrow \infty$.

resonances. Very often we have to consider the term reflecting the vibrations with the frequency of the external force (see reference [98], for instance). As a result of such a procedure the slow variable ϕ will be averaged also. Such problems are not discussed in this monograph.

The material below mainly follows paper [14]. Zhuravlev [202, 204] suggested a different method of studying the systems with impacts based on the special transformation of the original variables and successive application of the averaging procedure.

Equations similar to (5.7) may be obtained in the case of singular forces. In order not to complicate the problem by accounting for the excessive formalities, we repeat the calculations of Sect. 5.2 as applied to the case of singular forces.

The case when (5.3) may be represented as

$$\ddot{x} + \Omega^2 x + \Phi_0(x, \dot{x}) = \varepsilon g(x, \dot{x}, t)$$

is considered first. Here the generalised force $\Phi_0(x, \dot{x})$ is given by expression (4.7). It describes an asymmetric interaction. The solution of this equation when $\varepsilon = 0$ (see (4.25)) takes the form

$$x(t) = -J\chi[\omega_0(J)(t - t_0), \omega_0(J)]$$

$$\chi(t) = \frac{\omega_0}{\omega_0^2} + \frac{\pi}{\omega_0} \sum_{k=1}^{\infty} \frac{\Omega^2 - k^2 \omega_0^2}{\cos k\omega_0 t}, \quad J(\omega_0) = -\frac{\chi(0, \omega_0)}{\Delta}, \quad J \geq 0 \quad (5.18)$$

where the third relationship when $\Delta \neq 0$ defines the smooth function $\omega_0(J)$ but $\omega_0 = 2\Omega$ when $\Delta = 0$ by virtue of the system isochronism (see Sect. 4.3). The PGF $\chi(t)$ itself, along with its two first derivatives, are represented as

$$\begin{aligned} \chi(t, \omega_0) = \frac{2\Omega}{1} \frac{\cos[\Omega(t - \frac{1}{2}T_0)]}{\sin[\Omega(t - \frac{1}{2}T_0)]}, \quad \chi(t, \omega_0) = -\frac{2}{1} \frac{\sin(\frac{1}{2}\Omega T_0)}{\sin[\Omega(t - \frac{1}{2}T_0)]} \\ \chi(t, \omega_0) = -\frac{2}{\Omega} \frac{\cos[\Omega(t - \frac{1}{2}T_0)]}{\sin(\frac{1}{2}\Omega T_0)} + \delta^{T_0}(t) \end{aligned} \quad (5.19)$$

where $\delta^{T_0}(t)$ is the T_0 -periodic δ -function (see Sect. 2.1). In accordance with the notation of Sect. 1.2 and 2.1, the third expression from (5.19) may be rewritten in the form $\chi(t, \omega_0) = \{\chi\} + \delta^{T_0}(t)$, where $\{\chi\}$ is the classical part of the generalised function.

The impulse of the impact $J = -2\Delta \tan(\Omega T_0/2)$ is related to the total energy E of the system,

$$E = \frac{1}{2}J^2 + \frac{1}{2}\Omega^2\Delta^2, \quad J \geq 0 \quad (5.20)$$

Transformation (5.5) now takes the form

$$\dot{x} = -J\chi[\psi, \omega_0(J)]\chi^\psi, \quad \dot{x} = -J\omega_0(J)\chi^\psi[\psi, \omega_0(J)]$$

$$\chi[\psi, \omega_0(J)] = \omega_0^{-1} \left[\frac{1}{2\pi\Omega_0^2} + \frac{\pi}{\sum_{k=1}^{\infty} \Omega_0^2 - k^2} \cos k\psi \right]$$

$$\Omega_0 = \Omega[\omega_0(J)]^{-1}$$

$$\chi[\psi, \omega_0(J)] = \frac{1}{\Omega} \frac{\cos[\Omega_0(\psi - \pi)]}{\sin(\pi\Omega_0)} \quad \text{for } 0 \leq \psi < 2\pi \quad (5.21)$$

It can be seen that new variables possess the physical sense of the impulse and phase. Transformation (5.21) is nonsmooth. At $\psi = 2l\pi$, $l \in \mathbf{Z}$, the value of χ_ψ possesses jump discontinuities. Hence, impacts in the new variables take place when $\psi = 2l\pi$ and the relationship $J = -\Delta\chi^{-1}[0, \omega_0(J)]$ holds. The above transformation is reversible since the variables (J, ψ) describe the system uniquely.

Now we can apply the results of Sect. 5.2. Identity (5.6) yields the new identity

$$\begin{aligned} & (-J\chi)_J^J - J\chi_\psi^\psi = -J\omega_0\chi^\psi \\ & \left[-J\chi^\psi\omega_0 + \omega_0(-J\chi^\psi)_J \right] \left[\chi^\psi_\psi \right] + \delta^{2\pi}(\psi) \left[\psi \right] \\ & = \Omega^2 J\chi - \Phi_0(-J\chi, -J\omega_0\chi^\psi) + \varepsilon g(-J\chi, -J\omega_0\chi^\psi, t) \end{aligned} \quad (5.22)$$

We took into account that the PGF $\chi[\psi, \omega_0(J)]$ depends on J implicitly. Hence, $\chi_J = \chi\omega_0(J)$, the superscript prime denotes differentiation. The principal

determinant of system (5.22) will not contain a $\delta^{2\pi}$ -function because

$$d_0 = - \left[(-J\chi)_J^\psi (-J\chi)_J^\psi - (-J\chi)_J^\psi (-J\chi)_J^\psi \right] - J^2 \chi_\psi^\psi \omega_0$$

and the second addend will be represented

$$(-J\chi)_J^\psi (-J\chi)_J^\psi = -J\omega_0 \left\{ \chi^\psi_\psi \right\} + \delta^{2\pi}(\psi) \left[(-J\chi)_J^\psi \right]$$

In accordance with the features of δ -functions

$$\delta(\psi) \left[-J\chi(\psi, \omega_0) \right]_J = \left[-J\chi(0, \omega_0) \right]_J \delta(\psi) = 0$$

because, by virtue of the third relationship from (5.18),

$$-J\chi(0, \omega_0) = \Delta \equiv \text{const}$$

As a matter of fact, using (5.8) and (5.20), $d_0 = -1/4 J\omega_0^{-1}(J)$, because now $A = J$. As a result of the above manipulations, system (5.7) in the new variables (J, ψ, t) takes the form

formulae

$$\begin{aligned} \omega_0(J) &= \frac{\pi^2}{\pi^2 - \text{atan}[J/(2\Omega\Delta)]}, \quad \Delta > 0, \quad \Omega < \omega_0 < 2\Omega \\ \omega_0(J) &= -\frac{\pi^2}{\pi^2 + \text{arctg}[J/(2\Omega\Delta)]}, \quad \Delta < 0, \quad 2\Omega < \omega_0 < \infty \end{aligned} \quad (5.24)$$

For resonance cases

$$p\omega - q\omega_0(J) = \varepsilon\theta(J)$$

the system takes the standard form (see (5.16))

$$\begin{aligned} J &= -4\varepsilon pq^{-1} g \left\{ -J\chi \left[\omega(pq^{-1}t - \phi), \omega \right], -J\chi \left[\omega(pq^{-1}t - \phi), \omega \right], \omega \right\} \\ &\times \chi \left[\omega(pq^{-1}t - \phi), \omega \right] \\ \phi &= -4p\varepsilon (Jq)^{-1} g \left\{ -J\chi \left[\omega(pq^{-1}t - \phi), \omega \right], -J\chi \left[\omega(pq^{-1}t - \phi), \omega \right], \omega \right\} \\ &\times \chi \left[\omega(pq^{-1}t - \phi), \omega \right] + \varepsilon\theta(J)(q\omega)^{-1} \end{aligned} \quad (5.25)$$

yields

$$J = \varepsilon E_{10}(J, \phi), \quad \phi = \varepsilon E_{20}(J, \phi)$$

In the previous paragraphs (sections 4.10 and 4.11) such approximate representations were obtained for the vibration of threshold strongly nonlinear conservative systems

(5.26)

$$x(t) = -J_\lambda \chi \left[t - 1/2 t_\lambda(J_\lambda) + t_0, \omega_0 \right]$$

where J_λ is the impulse and $t_\lambda(J_\lambda)$ is the time of interaction. In accordance with the

above we find that

$$J_\lambda = -\frac{\chi[1/2 t_\lambda(J_\lambda), \omega_0]}{\Delta}$$

The approximate representations similar to (5.26) were effectively used assuming intensive interaction whilst $E \approx 1/8 J_\lambda^2$. Hence, the above approximations are valid for

the analysis of resonant vibration with a frequency which corresponds to infinite energy in the conservative system $\omega \approx \omega_*$. For large λ this frequency is close to the value of 2Ω .

Using (5.26), (5.11) and (5.16), equations in a standard form may be obtained for the analysis of forced resonant vibration of strongly nonlinear threshold systems. The relevant calculations are similar to performed above.

The averaged equations for symmetric systems may be obtained similarly. For this purpose it is necessary to replace the PGF $\chi(t)$ with the symmetric PGF in the corresponding expressions,

$$\chi^*(t) = \frac{2\Omega}{1} \frac{\cos(\Omega T/4)}{\sin[\Omega(t - T/4)]}, \quad 0 \leq t < \frac{2}{T}$$

The equation $J = -\Delta/\chi^*(0, \omega_0)$ defines the natural frequency as a function of the impulse. It can be calculated using the formula

$$\omega_0(J) = \frac{\pi^2}{2 \tan[J/(2\Delta\Omega)]}, \quad 1 < \omega_0 < \infty \quad (5.27)$$

5.5. Steady-state Vibration and Stability

Replacement of the equations of motion $\dot{A} = \varepsilon E_1(A, \varphi, t)$, $\dot{\varphi} = \varepsilon E_2(A, \varphi, t)$ by the

averaged equations $\dot{A} = \varepsilon E_{10}(A, \varphi)$, $\dot{\varphi} = \varepsilon E_{20}(A, \varphi)$ allows us to obtain the first

approximation to the envelopes A, φ of the corresponding processes. In the case of periodic vibration, the above variables may be constant; $A = \text{const}$, $\varphi = \text{const}$, i.e.

$\dot{A} = 0$, $\dot{\varphi} = 0$. Therefore, the system of transcendental equations

$$\varepsilon E_{10}(A, \varphi) = 0, \quad \varepsilon E_{20}(A, \varphi) = 0$$

defines *steady-state* periodic processes.

Let A^0, φ^0 be a solution to the above system. It defines the desired steady-state process. Numerous simplifications are used below to define that solution. *Equations in variations* are required for analysis of the *stability* of steady-state solutions.

Let A^0, φ^0 define the steady-state regime. A and φ are current values, and $\xi = A - A^0$ and $v = \varphi - \varphi^0$ are small deviations. A *linearisation* of the averaged equations in the close vicinity of the steady-state point A^0, φ^0 yields the equations in variations

(5.28)

$$\begin{aligned} \xi &= \varepsilon [E_{10A}(A^0, \varphi^0) \xi + E_{10\varphi}(A^0, \varphi^0) v] \\ v &= \varepsilon [E_{20A}(A^0, \varphi^0) \xi + E_{20\varphi}(A^0, \varphi^0) v] \end{aligned}$$

The characteristic equation of this linear problem takes the form

$$\det \begin{bmatrix} \lambda - \varepsilon E_{10A} & \varepsilon E_{10\varphi} \\ \varepsilon E_{20A} & \lambda - \varepsilon E_{20\varphi} \end{bmatrix} = 0$$

The solution A^0, φ^0 will be asymptotically stable if small perturbations decay with time. The necessary and sufficient condition for such stability is that the real parts of the eigenvalues $\lambda_{1,2}$ are negative,

$$\text{Re } \lambda_{1,2} < 0$$

The conditions of asymptotic stability, therefore, take the form

$$E_{10A} + E_{20\varphi} < 0, \quad E_{10A}E_{20\varphi} - E_{10\varphi}E_{20A} > 0 \quad (5.29)$$

Let conditions (5.29) hold and assume that deviations from the steady-state conditions are small. The solutions $\xi(t), v(t)$ to system (5.28), therefore, describe the process of development of vibration with the parameters J^0, φ^0 . The *logarithmic decrement* $\gamma \approx \varepsilon |E_{10A} + E_{20\varphi}| T$ defines the rate of the above development.

In the framework of the present monograph most attention will be paid to the study of periodic processes. Thus, we do not consider the general case of the study of transient processes here. The above systems, nevertheless, are applicable for such an analysis.

5.6. Vibro-impact Systems Under Harmonic Excitation

The study of the forced vibration of strongly nonlinear systems starts from an analysis of the dynamics of the *impact oscillator* (see Fig. 5.2a-c) under external sine forcing $P(t) = \varepsilon P_1 \cos \omega t$ [151]. This problem was studied in detail in monographs [17, 75] etc. It is important that this problem allows the exact solution to be obtained in the explicit form, which may be used for the assessment of the efficiency of the approximate methods. Let the unit mass vibrate under the harmonic force $P(t)$. Energy dissipation in the system is represented by a viscous damping and $\Phi_0(x, x)$ denotes the force of the *elastic* impact against asymmetric (Fig. 5.2a,b) and symmetric (Fig. 5.2c) obstacles. The equation of motion takes the form

$$\ddot{x} + \Omega^2 x + \Phi_0(x, x) = \varepsilon [P_1 \cos \omega t - 2bx] \quad (5.30)$$

The structure of the impact force was discussed in Sect. 4.3. It can be eliminated from (5.30) using the conditions of impact:

$$\dot{x}_+ = -\dot{x}_-, \quad x_- > 0 \quad (5.31)$$

where $\dot{x}_\pm = \dot{x}(t_a \pm 0)$ and t_a is the instant of the impact which is defined by the equation $x(t_a) = \Delta$.

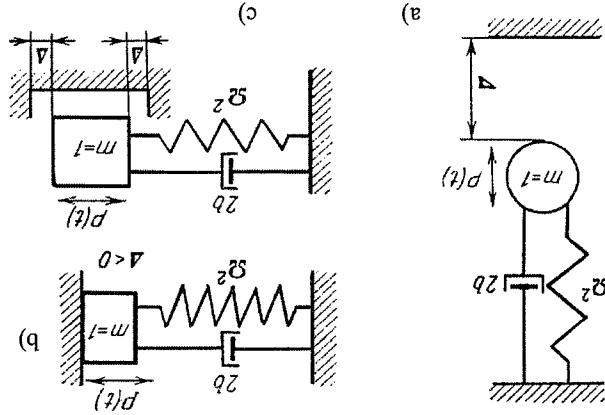


Fig. 5.2

The additional energy dissipation at momentary impact is usually accounted for by using the *Newton's hypothesis* considering the momentary loss of energy in accordance with the relationship

$$\dot{x}_+ = -R\dot{x}_-, \quad x_- > 0 \quad (5.32)$$

where $0 < R \leq 1$ is the *restitution ratio*. By definition, the value of R depends on the mechanical features of the colliding bodies only. It defines the energy dissipation at the impact (E_1) in accordance with the Carnot's theorem

$$E_1 = J^2 \frac{2m(1+R)}{1-R} \quad (5.33)$$

where $J = m(\dot{x}_- - \dot{x}_+) = m(1+R)\dot{x}_-$ is the variation of the momentum during the impact (*impact impulse*) and m is the mass of the colliding body (for simplicity of the calculations $m = 1$).

The physical sense, advantages and disadvantages of Newton's hypothesis [54, 131] are not discussed here. Similarly, we are not going to discuss the hypothesis of the nonmomentary impacts leading, in the limiting case $\lambda \rightarrow \infty$, to relationship (5.32). Some considerations on this problem may be found in reference [15].

Let $(1-R) = \varepsilon$ hold, hence the impact is quasi-elastic. In reference [14] it is shown that condition (5.32) may be accounted for in the equation of motion using singular forces

$$\Phi_1(x, x) = \frac{x_-^2(1+R)}{4} \delta_+(x-\Delta) + \frac{x_+^2(1-R^2)}{4} \delta_+(x-\Delta) \operatorname{sgn} \dot{x}$$

and

$$\Phi_1(x, x) = \frac{x_-^2(1+R)}{4} \delta_\pm(|x|-\Delta) \operatorname{sgn} \dot{x} + \frac{x_+^2(1-R^2)}{4} \delta_\pm(|x|-\Delta) \operatorname{sgn} \dot{x}$$

for asymmetric and symmetric interaction, respectively (compare with (4.17) and (4.20)). The above might be re-expressed as

$$\Phi_1(x, x) = \Phi_0(x, x) + \varepsilon \Phi_2(x, x), \quad \varepsilon = 1 - R$$

where Φ_0 which is given by (4.17) or (4.20) corresponds to the case of absolutely elastic impact, and Φ_2 is the dissipative component $\propto \varepsilon$ in the sense of generalised functions. Such decomposition was carried out in reference [12] and it will not be used here.

Substitute condition (5.32) with $R < 1$ into Eqs. (5.23) and (5.25). For the elastic impact, which takes place in time instants $\psi = 2k\pi$, $k = 0, \pm 1, \dots$, the impulse J is continuous and condition (5.31) holds.

Now calculate the variation $\delta J = J|_{\psi=2k\pi+0} - J|_{\psi=2k\pi-0}$ if $R < 1$ in (5.32).

By virtue of the transformation of the variables $x = -J\omega_0(J)\chi_\psi[\psi, \omega_0(J)]$, and,

therefore,

$$\dot{x}_- = -J\omega_0(J)\chi_\psi[\psi = 2k\pi - 0]$$

$$\dot{x}_+ = -J\omega_0(J)\chi_\psi[\psi = 2k\pi + 0]$$

By the definition of the PGF (see (5.21))

$$\chi_\psi[\psi = 2k\pi - 0] = -\chi_\psi[\psi = 2k\pi + 0] = -1/2\omega_0$$

hence

$$\dot{x}_- = \frac{1}{2}J|_{\psi=2k\pi-0}, \quad \dot{x}_+ = \frac{1}{2}J|_{\psi=2k\pi+0}$$

or

$$\delta J = -2(\dot{x}_+ + \dot{x}_-) = -2\dot{x}_-(1-R) = -\varepsilon J|_{\psi=2k\pi-0}$$

This finite jump may be introduced into the right-hand side of the first equation of (5.23) using unilateral δ -functions (see Sect. 1.2). Let $J_\pm = J|_{\psi=2k\pi \pm 0}$. The integration of the above jump yields

$$\delta J = \int_{J_0}^J dJ = \int_{J_0}^J J(\psi) [\delta^+(\psi) - \delta^-(\psi)] d\psi$$

Using this representation for the jump magnitude δJ , we find finally

$$\delta J = -\varepsilon r \int_{\psi+2k\pi}^{\psi+2k\pi} J(\psi) \delta^-(\psi) d\psi = -\varepsilon r J \Big|_{\psi=2k\pi-0}^{\psi=2k\pi}$$

The T -periodic time continuation of the function δ^-

$$\delta_T^-(t) = \sum_{k=-\infty}^{\infty} \delta^-(t - kT), \quad \delta_{2\pi}^-(\psi) = \sum_{k=-\infty}^{\infty} \delta^-(\psi - 2k\pi)$$

yields the new term¹ for the right-hand side of equation (5.23), namely

$$J = -\varepsilon \left[rJ\omega_0(J) \delta_{2\pi}^-(\psi) + 4\omega_0(J) g(-J, -J\omega_0\chi, \psi) \right] \quad (5.34)$$

The corresponding term is added also to the first equation of (5.25). In spite of the presence of a jump of the function J , the solution $x = -J\chi(\psi)$ remains continuous by virtue of the first equation of (5.22). The above continuity is due to the dependence of the PGF $\chi(\psi)$ on the function $\omega_0(J)$.

Now it is easy to obtain the transformed equations of type (5.25) for the equation of motion (5.30) when $R \leq 1$ and $\Delta \neq 0$. In the case of fundamental resonance $p = q = 1$, for example

$$J = -\varepsilon \left\{ rJ\delta_T^-(t - \phi) + [8bJ\chi(t - \phi) + 4r_1 \cos \omega t] \chi(t - \phi) \right. \\ \left. - 4\varepsilon J^{-1} [r_1 \cos \omega t + 2bJ\chi(t - \phi)] \chi(t - \phi) + \varepsilon \omega^{-1} \theta(J) \right\}$$

Here, for simplicity, the asymmetric problem is considered; $\theta(J)$ is calculated in accordance with (5.24). For the analysis of the symmetric case it is necessary to substitute $\chi \rightarrow \chi^*$ with successive use of (5.27).

With provision for (5.19), the averaged equations take the form

$$J = -\varepsilon \left\{ \frac{rJ}{bJ} + \frac{T}{\sin T\Omega} \left(1 - \frac{\sin T\Omega}{2\omega^2 R_1 \sin \omega\phi} \right) + \frac{\pi(\Omega^2 - \omega^2)}{2\omega^2 R_1 \cos \omega\phi} \theta(J) \right\} \\ \phi = -\varepsilon \left[\frac{\pi J(\Omega^2 - \omega^2)}{2\omega^2 R_1 \cos \omega\phi} - \frac{\omega}{\theta(J)} \right] \quad (5.35)$$

¹ Similar calculations may be carried out using the function $\delta^+(t)$ or even using the "regular" δ -function.

The above system may be rewritten as

$$J = \varepsilon E_{10}(J, \phi), \quad \phi = \varepsilon E_{20}(J, \phi)$$

with the evident notation. For the definition of the parameters of a stationary regime, the following transcendental equations were obtained from the above

$$E_{10}(J, \phi) = 0, \quad E_{20}(J, \phi) = 0$$

The following reasoning [119] simplifies the solution process.

Our analysis is carried out in the small ε -vicinity of the values $J^0 = J^0(\omega)$. Hence, to a declared accuracy, it may be considered that the detuning is $\varepsilon\theta(J) = -(J - J^0)\omega_0(J^0)$, and in the other terms of (5.35) we let $J = J^0 + \varepsilon \dots$. Then,

to $O(\varepsilon^2)$, we find

$$J = -\varepsilon \left[\frac{rJ^0}{bJ^0} + \frac{T}{\sin T\Omega} \left(1 - \frac{\sin T\Omega}{2\omega^2 R_1 \sin \omega\phi} \right) + \frac{\pi(\Omega^2 - \omega^2)}{2\omega^2 R_1 \cos \omega\phi} \omega_0(J^0) \right] \\ \phi = -\varepsilon \left[\frac{\pi J^0(\Omega^2 - \omega^2)}{2\omega^2 R_1 \cos \omega\phi} - \frac{\omega}{J^0 \omega_0(J^0)} \right]$$

where $J^0 = -2\Omega\Delta \tan(\pi\Omega/\omega)$. The determining system of transcendental equations is then simplified and takes the form

$$E_{10}(J^0, \phi) = 0, \quad E_{20}(J^0, \phi) = 0$$

From the first equation it follows that

$$\sin \omega\phi = \frac{\pi(\omega^2 - \Omega^2)}{rJ^0} \left[\frac{2\omega^2 R_1}{bJ^0} + \frac{T}{\sin T\Omega} \left(1 - \frac{\sin T\Omega}{2\omega^2 R_1} \right) \right] \quad (5.36)$$

hence the existence condition of the resonance process ($|\sin \omega\phi| \leq 1$) takes the form

$$\left| \pi(\omega^2 - \Omega^2) \left[\frac{rJ^0}{bJ^0} + \frac{T}{\sin T\Omega} \left(1 - \frac{\sin T\Omega}{2\omega^2 R_1} \right) \right] \right| \leq 1$$

Two different phases corresponding to two resonance regimes may also be obtained from the above relationship. Denoting the right-hand side of (5.36) by $W(J^0)$, we find

$$\phi_1 = \omega^{-1} \arcsin W(J^0), \quad \phi_2 = \omega^{-1} [\pi - \arcsin W(J^0)] \quad (5.37)$$

From the equation $E_{20}(J, \phi) = 0$,

$$J_{1,2} = J^0 - \frac{2\varepsilon R_1 \omega^2 \cos \omega\phi_{1,2}}{J^0 \pi(\Omega^2 - \omega^2) \omega_0(J^0)}, \quad J^0 = -2\Omega\Delta \tan \frac{\omega}{\pi\Omega} \quad (5.38)$$

where it is assumed that $\Delta \neq 0$. The first and the third relationships from (5.24) are used for the calculation of the function $\omega(J)$ from (5.38). For example, when $\Delta > 0$ from the first formula of (5.24) it follows that

$$\omega(J_0) = \frac{\cos^2(\pi\Delta/\omega)}{2\pi\Delta(1+\Omega\omega^{-1})^2}$$

In the case $\Delta = 0$, the generating system is isochronous. The corresponding analysis is significantly simplified. The *quasi-isochronous approach* $\Delta = \epsilon\Delta_1$, $\Delta_1 \sim 1$ will be frequently used below.

Formulae (5.37) and (5.38) define approximate expressions for the parameters of steady-state regimes in the case of fundamental resonance.

The stability of the obtained solutions may be analysed using expressions (5.29). With provision for the notation used, differentiation of the right-hand sides of (5.35) yields

$$E_{10J} = -\left[\frac{r}{b} + \frac{\sin T\Omega}{1 - \frac{\sin^2(\frac{1}{2}T\Omega)}{\sin^2(\frac{1}{2}T\Omega)}}\right], E_{10\phi} = \frac{2\omega^3 P_1 \cos \omega\phi}{\pi(\omega^2 - \Omega^2)}$$

$$E_{20J} = -\kappa_1 \Delta + \frac{2\omega P_1 \cos \omega\phi}{\pi J^2 (\Omega^2 - \omega^2)}, E_{20\phi} = \frac{\pi J (\Omega^2 - \omega^2)}{2\omega^2 P_1 \sin \omega\phi}$$

where

$$\kappa_1 = 2\pi\Omega^2 \left\{ \omega \left[a \tan J(2\Omega\Delta) \right]^{-1} \left[4\Omega^2 \Delta^2 + J^2 \right]^{-1} \right\} > 0, \Delta < 0$$

$$\kappa_1 = 2\pi\Omega^2 \left\{ \omega \left[\pi - \operatorname{atan} J(2\Omega\Delta) \right]^{-1} \left[4\Omega^2 \Delta^2 + J^2 \right]^{-1} \right\} > 0, \Delta > 0$$

Using the above calculations, the stability conditions (5.29) now take the form

$$E_{10J} + E_{20\phi} < 0, E_{10J} E_{20\phi} - E_{10\phi} E_{20J} > 0$$

In the frequency band of free vibration of the generative system $\omega > \Omega$, we have $J > 0$, and the values of the phase ϕ are selected in order to produce $\sin \omega\phi > 0$. Since $r, b \neq 0$, the first condition always holds. The first addend in the second inequality is positive and the second will be definitely positive if $\Delta \cos \omega\phi > 0$.

Therefore, the stability condition is reduced to an easily verified inequality. So, for $\Delta > 0$ in the case of a strongly anisochronous system, the phase ϕ_1 corresponds to the stable regime. In the case of the softly anisochronous system $\Delta < 0$, the stable process is defined by the phase ϕ_2 .

In accordance with the results of Sect. 5.5, the logarithmic decrement $\gamma \approx \epsilon |E_{10J} + E_{20\phi}| T$ characterises the decay rate of transient processes at the build-up of steady-state periodic vibration.

5.7. Exact Laws of Motion of Vibro-impact Systems

represented below.

The problem mentioned in the previous section may be solved exactly using the stitching method [17, 75, 133, 141]. For our purposes, however, the method of integral equations is more suitable. This method, as applied to systems of automatic control, was analysed in detail in references [149, 150].

The advantages of the method of integral equations are revealed with an increase in the system order [see Chap. 3]. The above method does not show any computational advantages as compared to the stitching method in application to an analysis of a SDOF system. Here we follow the results obtained in reference [17].

Consider a more general (as compared with (5.30)) equation of motion of the vibro-impact system

$$\ddot{x} + \Omega^2 x + 2bx + \Phi_1(x, \dot{x}) = P(t) \quad (5.39)$$

where $P(t+T) = P(t)$ and no assumptions about the smallness of the quantities involved were made. The expressions for the impact force $\Phi_1(x, \dot{x})$ in the symmetric and asymmetric cases were obtained in the previous section. The asymmetric case is studied here in detail.

We obtain, for example, the T -periodic solutions to system (5.39). Speaking generally, vibration with ν impacts within one period are possible, and a priori the special case of $\nu = \infty$ can not be excluded from consideration.

In accordance with (2.2) and similarly to the procedure for obtaining the T -periodic solution to Eq. (4.8), the integral equation which describes the forced periodic vibration may be written in the form

$$x(t) = \int_0^T \chi(t-s) P(s) ds - \int_T^t \chi(t-s) \Phi_1[x(s), \dot{x}(s)] ds \quad (5.40)$$

where the PGF $\chi(t)$ is given by (2.12).

The first integral here equals $f(t)$ which is the solution to the problem of forced vibrations in a linear system when $\Phi_1 \equiv 0$.

For the force of impact, on the period $[0, T]$ the following representation may be used

$$\Phi_1[x(t), \dot{x}(t)] = \sum_{k=1}^{\nu} J_k \delta(t - \phi_k), J_k = \dot{x}_-(\phi_k)(1+R)$$

where J_k is the variation of the impulse (impact impulse) in the time instance ϕ_k

where $\phi_k \in [0, T]$, so $x(\phi_k) = \Delta$, $k = 1, 2, \dots, \nu$. This formula was obtained in a similar way to that for (4.15). At all moments of impact $t = \phi_k$ the body of unit mass is subjected to the action of the force $-J_k \delta(t - \phi_k)$ and by virtue of Newton's hypothesis (5.32) $x_+(\phi_k) = -R\dot{x}_-(\phi_k)$. The impulse of impact, therefore, is

$$J_k = x_-(\phi_k) - x_+(\phi_k) = x_-(\phi_k)(1 + R)$$

In accordance with the features of the considered model $x_-(\phi_k) \geq 0$, $J_k \geq 0$. The forces of impact on the infinite time interval $-\infty < t < \infty$ are expressed using T -periodic functions $\delta^T(t - \phi_k)$.

Equation (5.40) was obtained under an assumption that the T -periodic regime does exist. The substitution of the expression for the impact force into (5.40) yields the 2ν -parametric representation

$$x(t) = f(t) - \sum_{k=1}^{\nu} J_k x(t - \phi_k) \quad (5.41)$$

The conditions of existence for the above solutions follow from the 2ν conditions of consistency

$$J_j = x_-(\phi_j)(1 + R), \quad x_-(\phi_j) = \Delta, \quad j = 1, \dots, \nu$$

which produces the system of transcendental equations in the unknowns J_k, ϕ_k , $j = 1, 2, \dots, \nu$,

$$\Delta = f(\phi_j) - \sum_{k=1}^{\nu} J_k x(\phi_j - \phi_k), \quad j = 1, 2, \dots, \nu, \quad \nu = 1, 2, \dots$$

$$J_j = \left[f(\phi_j) - \sum_{k=1}^{\nu} J_k x_-(\phi_j - \phi_k) \right] (1 + R) \quad (5.42)$$

If system (5.42) can be solved, the T -periodic regime exists and is defined by means of representation (5.41). The corresponding solutions must satisfy the additional conditions $J_k \geq 0$, $\phi_k \in [0, T]$ and representation (5.41) for $x(t) \leq \Delta$. If $\nu = \infty$, then it is required that $J_k \rightarrow 0$ for $k \rightarrow \infty$ in order that series (5.41) converges.

Regimes with large ν are called *chatter*. This reflects the appearance of a relatively large number of impacts within the period of vibration. The possibility exists of describing the cases with $\nu = \infty$ ([42, 43, 122] and §6). Chatter does not appear in resonance regimes due to the fact that at resonance the free motion of the system and the external force are concordant. The appearance of chatter reflects mismatch in a process. This point of view agrees completely with the concept of resonance. Resonance processes are similar in appearance to free vibration with single and double impacts within the period of vibration in asymmetric and symmetric cases, respectively.

Let $P(t + T/2) = -P(t)$, and ν alternative collisions take place with each obstacle. For the processes $x(t + T/2) = -x(t)$ we have

$$x(t) = f(t) - \sum_{k=1}^{\nu} J_k x^*(t - \phi_k), \quad f(t) = \int_0^{T/2} x^*(t - s) P(s) ds \quad (5.43)$$

where the symmetric PGF $x^*(t)$ is given by (2.13) and the conditions of consistency are similar to those discussed above. The reader can analyse this case independently. The calculation algorithm is reduced, therefore, to the solution of the system of transcendental equations (5.42) with given ν , the construction of the representation (5.41) (in case the above system may be resolved) and verification of the condition $x(t) \leq \Delta$. The order of system (5.42) and the structure of formula (5.41) are still the same independent of the system's order (see §9).

Consider the case when $\nu = 1$ and the external force is represented as a sine function of time [17, 198]. This case is important in applications. From (5.41),

$$x(t) = f(t) - J x(t - \phi), \quad x(\phi) = \Delta, \quad J = x_-(\phi)(1 + R) \quad (5.44)$$

The values J and ϕ may be calculated for the given $P(t) = P_1 \cos \omega t$.

The moment of impact may be taken to coincide with the initial time in (5.44), hence

$$x(t) = f_1 \cos(\omega t + \phi) - J x(t)$$

where the phase ϕ is unknown and the magnitude f_1 is the solution of the corresponding linear problem,

$$f_1 = \frac{\sqrt{(\Omega^2 - \omega^2)^2 + 4b^2 \omega^2}}{P_1}$$

The conditions of consistency for $x(t)$ yield

$$f_1 \cos \phi = \Delta + J x(0), \quad f_1 \sin \phi = -\frac{J}{\omega} \left[\dot{x}_-(0) + \frac{1}{1 + R} \right]$$

Finally,

$$J_{1,2} = \frac{-\Delta x(0) \pm \sqrt{f_1^2 x^2(0) - \left[\dot{x}_-(0) + (1 + R)^{-1} \right]^2 (\Delta^2 - f_1^2) / \omega^2}}{-\Delta x(0) + \left[\dot{x}_-(0) + (1 + R)^{-1} \right]^2 / \omega^2} \quad (5.45)$$

$$\sin \phi = -\frac{J_{1,2}}{\omega f_1} \left[\dot{x}_-(0) + \frac{1}{1 + R} \right], \quad \cos \phi = \frac{f_1}{\Delta + J_{1,2} x(0)}$$

and, therefore, the desired solution is completely defined.

The detailed analysis of the exact solutions with respect to the stability and whether the geometrical conditions are satisfied is carried out in monographs [17, 75] and is not reproduced here. For the analogues of (5.45) in the symmetric case, the formal

$$\chi(0) \rightarrow \chi^*(0), \chi_-(0) \rightarrow \chi_-^*(0)$$

is required.

Figure 5.3 represents the dependencies $J(\omega)$ for the asymmetric system in the resonant domains. Solid and dashed lines correspond to stable and unstable processes, respectively; dot-and-dashed lines represent the backbone curves. For simplicity, it was considered that $b = 0$; the presence of small viscous damping results in the minor distortion of the corresponding curves decreasing the level of attainable amplitudes. For the illustrations, the following cases are considered: $\Delta > 0$ (see Fig. 5.3a), $\Delta < 0$ (see Fig. 5.3b) and $\Delta = 0$ (see Fig. 5.3c). The existence conditions of two processes when $\Delta \neq 0$ were obtained in cited monographs [17,75].

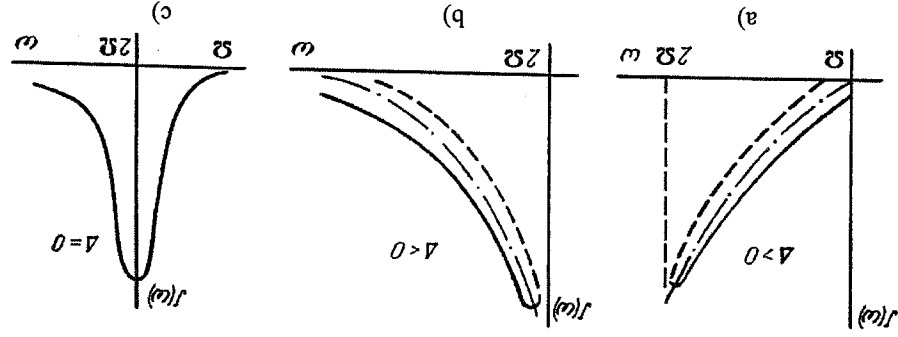


Fig. 5.3

The vibration magnitude depends on frequency $A(\omega)$ in a similar way to the dependence $J(\omega)$. After the backward jump (the transition from the vibro-impact regime to the linear regime) when $\Delta > 0$ harmonic vibration are developed in the system. When $\Delta < 0$ the vibration die out after a jump. Figure 5.4 represents the dependencies $J(\omega)$ and $A(\omega)$ which are obtained for the symmetric system. After the backward jump, a harmonic vibration is developed in the system, see Fig. 5.4b. Here ω_1 denotes the frequency of a backward jump to the linear branch of the amplitude-frequency characteristics of the system.

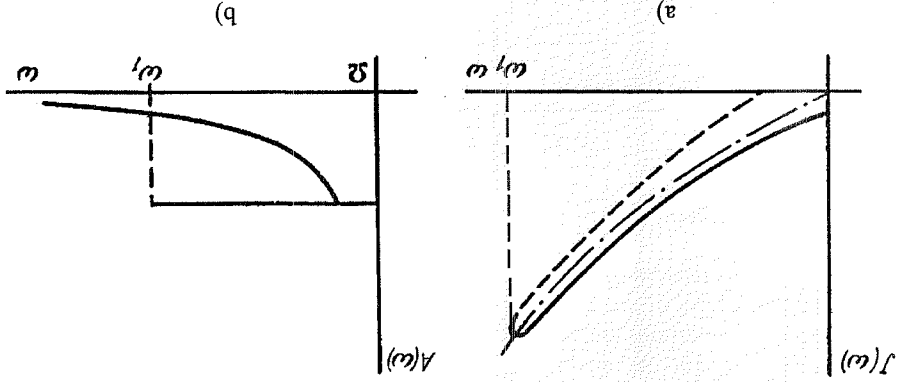


Fig. 5.4

The dynamic phenomena of frequency and amplitude pulling are analysed in detail in reference [17]. The above phenomena will be considered here for more complicated cases. Now we compare the exact solutions obtained in this section with the corresponding approximate solutions obtained using the averaging method in Sect. 5.6 [93]. For simplicity, let $b = 0$ (in this case $\chi_-(0) = -1/2$ and the corresponding expressions may be significantly simplified). It will be shown that the system of transcendental equations, defining the stationary values J and φ which are obtained using the above methods, coincide to within the accuracy of the small terms. Write the transcendental equations of Sect. 5.6 for the definition of the parameters of the steady-state process as

$$\varepsilon r J^0 + \frac{4}{\varepsilon \omega R_1^2 \sin \omega \varphi} = 0, \quad \varepsilon r = 1 - R$$

$$2 \varepsilon \omega R_1^2 \sin \omega \varphi + \frac{J^0 - J^0(\omega_1)}{\omega_1(J^0(\omega_1))} + \frac{J^0 \pi (\Omega^2 - \omega^2)}{2 \varepsilon \omega R_1^2 \sin \omega \varphi} = 0, \quad J^0 = -2 \Omega \Delta \tan \frac{\pi \Omega}{\omega}$$

and the equations following from (5.44)

$$J \left[\frac{2(1+R)}{1-R} + \frac{\omega R_1^2 \sin \omega \varphi}{\Omega^2 - \omega^2} \right] + \frac{\pi \Omega^2 \tan \frac{\omega}{\Omega} \left(\frac{R \cos \omega \varphi}{\Omega^2 - \omega^2} - \Delta \right)}{2 \varepsilon \omega R_1^2 \sin \omega \varphi} = 0, \quad J = 2 \Omega \tan \frac{\pi \Omega}{\omega}$$

where, for unification, the phase φ is multiplied by the frequency ω .

As in the latter equations, R_1 and $1 - R$ are of order ε , it may be readily seen that in the frequency domains of vibro-impact resonance the corresponding equations coincide within the accuracy of the second-order small values. In this particular case, since the solutions are close, formula (5.37) yields a high level of accuracy. The additional example of the comparison of the exact and averaged solutions (obtained

using the method of nonsmooth transformations) may be found in reference [202]. Similarly to the method of averaging, the method of integral equations gives the possibility of considering the combination resonance $p:q$. For this purpose, we will consider that the period T_1 of the vibration $x(t)$ in the exact integral equation (5.40) is related to the period of the external force T as $\frac{p}{q}T_1 = T$. The corresponding PGF

$\chi(t)$ and the basic relationships are based on the above expression. For example, while considering the subperiodic vibration $1:q$, the period of the PGF $\chi(t)$ is qT . The upper limit of the integral in (5.40) is also qT .

5.8. Resonance Vibration of a System with Piecewise Restoring Force

Consider the system in [89] (see also [106]) with the equation of motion in the form

$$\ddot{x} + \Omega^2 x + \lambda(x - \Delta)\eta(x - \Delta) = \varepsilon [P_1 \cos \omega t + \mu g_1(x, \dot{x}) + g_2(x, \dot{x})] \quad (5.46)$$

The orders of the parameters ε and λ are related as $\varepsilon = O(\lambda^{1/2})$. The function $\mu g_1(x, \dot{x})$ describes the impact damping and the function $g_2(x, \dot{x})$ describes the instant damping associated with the whole motion of the system. Meanwhile, the parameter μ is not specified. The mechanical model corresponding to this problem is depicted in Fig. 5.1.

We will study the resonance vibration arising in the considered system. The corresponding conservative problem with $\varepsilon = 0$ is studied in Sect. 4.10. The improved singular approximation to the motion of a vibro-impact system

$$x_0 = -J_\lambda \chi(t - \tau_\lambda + t_0), \quad 2\tau_\lambda = t_\lambda$$

will be used as a generative solution, where $t_\lambda = \pi\sqrt{\lambda}$ is the interaction time. Expression (4.46) may be used to calculate the impulse of interaction as a function of frequency,

$$J_\lambda(\omega_0) = -\frac{\chi(0, \omega_0) + \pi/(4\sqrt{\lambda})}{\Delta}$$

In the new "impulse-phase" variables for the case of fundamental resonance we obtain

We note here that the considered representation is only valid for high levels of energy (interaction impulses) corresponding to the resonant cases.

$$\begin{aligned} J &= -4\varepsilon \left\{ P_1 \cos \omega t + \mu g_1[-J\chi(t - \tau_\lambda - \varphi), -J\chi(t - \tau_\lambda - \varphi)] + g_2(-J\chi, -J\dot{\chi}) \right\} \\ &\quad \times \chi(t - \tau_\lambda - \varphi) \\ \phi &= -4\varepsilon J^{-1} \left\{ P_1 \cos \omega t + \mu g_1[-J\chi(t - \tau_\lambda - \varphi), -J\chi(t - \tau_\lambda - \varphi)] + g_2(-J\chi, -J\dot{\chi}) \right\} \\ &\quad \times \chi(t - \tau_\lambda - \varphi) + \varepsilon \omega^{-1} \theta(J) \end{aligned}$$

Here the PGF $\chi(t)$ is calculated at the frequency ω . The index λ in the notation J_λ is omitted for simplicity, the value $\theta(J)$ is calculated in accordance with the inverted function $J_\lambda(\omega_0)$, and the condition of fundamental resonance $\varepsilon\theta(J) = \omega - \omega_0(J)$, by virtue of Sect. 4.10 takes the form:

$$\text{Case } \Delta > 0: \quad \Omega < \omega < 2\Omega(1 - \Omega/\sqrt{\lambda}), \quad \omega_0(J) = \frac{\pi\Omega}{\pi\Omega - \text{atan}\left[\frac{1}{2}\pi\Omega\lambda^{1/2} + 2\Delta\Omega J^{-1}\right]}$$

$$\text{Case } \Delta < 0: \quad 2\Omega(1 - \Omega/\sqrt{\lambda}) < \omega < \infty, \quad \omega_0(J) = -\frac{\pi\Omega}{\pi\Omega\lambda^{1/2} + 2\Delta\Omega J^{-1} - \text{atan}\left[\frac{1}{2}\pi\Omega\lambda^{1/2} + 2\Delta\Omega J^{-1}\right]}$$

Case $\Delta = 0$:

$$\omega_0 = 2\Omega(1 - \Omega/\sqrt{\lambda}) = \text{const}$$

The case of the isochronous system (compare with (5.24)) is not considered below. In accordance with the type of the functions g_1 and g_2 , the right-hand sides of the above standard-form system may be averaged over the implicit time. Note that the force μg_1 acts within the small time interval $t_\lambda = O(\varepsilon)$. In order to account for the contribution of this force in the accepted approximation it is necessary to suppose

$$O(\mu) = O(1/\varepsilon)$$

Consider the case when the viscous force $g_2(x, \dot{x}) = -2b\dot{x}$ damps the system and the force of the interaction is the force of dry friction $\varepsilon\mu g_1(x, \dot{x}) = -\beta[\text{sgn } \dot{x}]\eta(x - \Delta)$, $\beta = O(1)$, where $\eta(x)$ is the Heaviside function.

Figure 5.5a represents the model of the system, and Fig. 5.5b shows the schematic plot of the bumper response F_0 . From Fig. 5.5b, the bumper response is represented by a hysteresis loop. Branches 1, 2 correspond to the loading phase ($\dot{x} > 0$); 3, 4 correspond to the restitution phase ($\dot{x} < 0$); 5 corresponds to the residual load.

we find
Since the PGF is continuous, $I_1 = O(1/\sqrt{\lambda}) = O(\varepsilon)$. In a similar manner to Sect. 5.6,

$$I_1 = -\frac{T}{2} \int_{\tau_1}^T \dot{\chi}(t) dt = -\frac{T}{2} [\chi(\tau_1) - \chi(0)], \quad \tau_1 = \frac{1}{2} \tau_2 = \frac{T}{\pi} \sqrt{\lambda}$$

Using the above and Fig. 5.6 we find $I_2 = 0$ and

rectangular impulse $\eta(t) - \eta(t - \tau_1)$.

where the function $\Lambda(T; \tau_1; \phi)$ represents the T -periodic time continuation of the

$$\eta[-J\chi(t - \tau_1 - \phi) - \Delta] = \Lambda(T; \tau_1; t - \tau_1 - \phi)$$

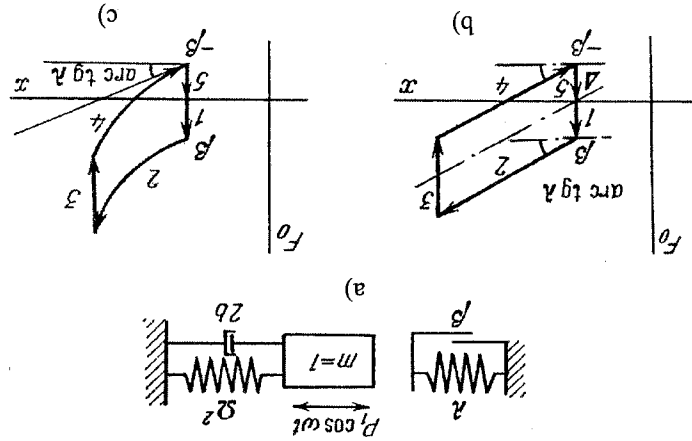
Additionally,

$$I_2 = \frac{1}{T} \int_0^T \operatorname{sgn}[-J\chi(t - \tau_1 - \phi)] [\chi(t - \tau_1 - \phi) - \eta[-J\chi(t - \tau_1 - \phi) - \Delta]] dt$$

$$I_1 = \frac{1}{T} \int_0^T \operatorname{sgn}[-J\chi(t - \tau_1 - \phi)] [\dot{\chi}(t - \tau_1 - \phi) - \eta[-J\chi(t - \tau_1 - \phi) - \Delta]] dt$$

accounted for in the following calculation
functions under the integral (see Fig. 5.6a,b) and the obvious condition $J \geq 0$ was
force of viscous damping are similar to those in Sect. 5.6. The periodicity of the
The terms of the averaged equations which correspond to the external force and to the

Fig. 5.5

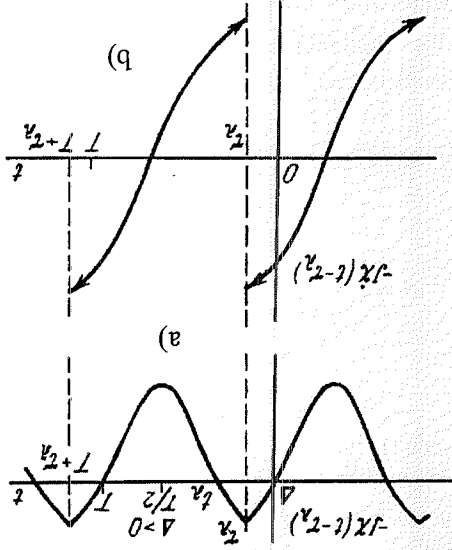


Under the condition $\Omega < \omega$ we always have $\chi(\tau_1) > \chi(0)$.

takes the form

The following reasoning is identical to that of Sect. 5.6. In particular, the stability condition of resonance regimes is $\Delta \cos \omega(\phi + \tau_1) > 0$ and the existence condition

Fig. 5.6



and the function $\omega_0(J)$ is determined above.

$$J_0 = J(\omega_0) \Big|_{\omega_0 = \omega} = -\frac{\chi(0, \omega) + \pi/(4\sqrt{\lambda})}{\Delta}$$

where

$$J = -\frac{8\beta}{T} [\chi(\tau_1) - \chi(0)] - \left[\frac{bJ_0}{\sin T\Omega} \left(1 - \frac{\sin T\Omega}{T\Omega} \right) + \frac{\pi \Omega^2 P_1 \sin \omega(\phi + \tau_1)}{2\omega^2 P_1 \sin \omega(\phi + \tau_1)} \right]$$

$$\phi = - \left[\frac{2\varepsilon \omega P_1 \cos \omega(\phi + \tau_1)}{J - J_0} + \frac{\omega}{J_0} \omega_0'(J_0) \right]$$

$$\left| \frac{\pi(\omega^2 - \Omega^2)}{2\omega^2 \varepsilon P_1} \left[\frac{T}{8\beta} [\chi(\tau_1) - \chi(0)] + \frac{\sin^2 \left(\frac{1}{2} \Omega T \right)}{e b J^0} \right] - \frac{\Omega T}{\sin \Omega T} \right| \leq 1$$

Expressions analogous to (5.37) and (5.38) are obtained similarly. The plots $J(\omega)$ are identical to those considered in the previous section. Note that for the improved description of the motion during the interaction we can use (after the parameters of the steady-state vibration are defined) the solutions to (5.41). The above solutions approximate the motion in the zone of interaction using the convex (see Fig. 5.7a), but not concave "impulses" (see Fig. 5.7b), as obtained in the case of singular approximation.

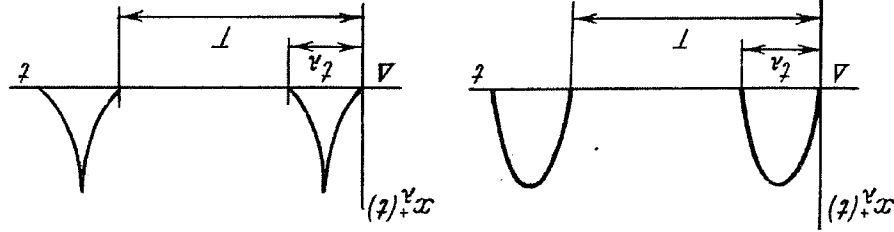


Fig. 5.7

The decrement of vibration $\gamma \approx \varepsilon [E_{10J} + E_{20\phi}] T$ represents the distortion of the above "impulses" in the transient process after the perturbation of the steady-state vibration. Finally, from the existence condition of a periodic resonant process, we obtain that in the case of the absence of the viscous friction ($b = 0$), $\chi(\tau_1) = 0$ and $J^0 = \infty$ at the frequency $\omega_* \approx 2\Omega(1 - \Omega/\sqrt{\lambda})$. Hence, the infinite impulses may appear even if the dry friction is represented in a buffer. With provision for $\chi(0) = (2\Omega)^{-1} \cot \pi\Omega/\omega$, the condition such that infinite amplitudes appear in the system, where the damping is represented in a buffer only, takes the form

$$\left| \frac{\omega_*^2 - \Omega^2}{\beta} \cot \frac{\pi\Omega}{\omega_*} \frac{\Omega\omega_* \varepsilon P_1}{\omega_*} \right| \leq 1$$

The expression under the modulus sign at the frequency ω_* is of order unity because the value of ω_* is close to 2Ω . Therefore, the value of $\cot \pi\Omega/\omega_*$ is small. The appearance of infinite amplitudes in systems with dry friction is well known [186]. A similar problem in a more complicated definition is analysed below in Sect. 9.11.

5.9. Piecewise Power Restoring Force

In equation of motion (5.46) let the strong nonlinearity be represented as $\lambda(x - \Delta)^\alpha \eta(x - \Delta)$ with $\alpha > 1$ and the functions $g_{1,2}$ have the same form as in the previous section. Then,

$$\ddot{x} + \Omega^2 x + \lambda(x - \Delta)^\alpha \eta(x - \Delta) = \varepsilon [P_1 \cos \omega t - 2bx] - \beta [\text{sgn } \dot{x}] \eta(x - \Delta)$$

where the parameters λ and ε are related by $\varepsilon = O(\lambda^{-1/(\alpha+1)})$. The hysteresis loop in Fig. 5.5c corresponds to the force of the buffer reaction F_0 . In accordance with Sect. 4.1 the general form solution of a generative conservative system with $\varepsilon = 0$ may be represented by the improved singular approximation,

$$x_0(t) = -J_\lambda \chi(t - \tau_\lambda - \phi), \quad \tau_\lambda = \tau_\lambda(J_\lambda)$$

Here the impulse of the interaction and the frequency of the vibration are related using the transcendental equations $\Delta = -J_\lambda \chi[\tau_\lambda(J_\lambda); \omega_0]$.

For high energy levels, the quantity $\tau_\lambda(J_\lambda) = 2\tau_\lambda(J_\lambda)$ may be expressed through the impulse using formula (4.52), $\tau_\lambda(J_\lambda) = D_\lambda J_\lambda^\gamma$, where $\gamma = (1 - \alpha)/(1 + \alpha)$ and the factor D_λ is

$$D_\lambda = \frac{\sqrt{2\pi} (1/8)^{1/2} \Gamma[1/(\alpha+1)]}{\left[\frac{\lambda(\alpha+1)}{(a+3)} \right] \Gamma \left[\frac{2(\alpha+1)}{2(\alpha+1)} \right]} = O(\lambda^{-1/(\alpha+1)}) = O(\varepsilon)$$

Since the expressions involved in the procedure of the averaging method are complex, the following simplifying approach may be suggested. Take the function $J_\lambda(\omega_0)$

from the limiting case $\lambda \rightarrow \infty$: $J(\omega_0) = -2\Delta\Omega \tan(\pi\Omega/\omega_0)$. Similarly, the functions $\omega_0(J)$ and $\omega_0(J)$ may be taken from the limiting conditions (see Sect. 5.3 and 5.6, in particular (5.24)). The interaction time is $t_2(J_\lambda) \approx t_2(J) = D_\lambda J^\gamma$ and, therefore, in the case of fundamental resonance $\varepsilon\theta(J) = \omega - \omega_0(J)$ the transformation of variables

$$x = -J_\lambda \chi[t - \tau_\lambda(J) - \phi], \quad \dot{x} = -J_\lambda \dot{\chi}[t - \tau_\lambda(J) - \phi]$$

yields the system in a standard form which is almost similar to that described in Sect. 5.8.

$$J = -4\varepsilon \{ P_1 \cos \omega t - 2b\chi[t - \tau_\lambda(J) - \phi] - \varepsilon^{-1} \beta \text{sgn} [-J\dot{\chi}[t - \tau_\lambda(J) - \phi]] \eta[-J\dot{\chi}[t - \tau_\lambda(J) - \phi]] \} \chi[t - \tau_\lambda(J) - \phi]$$

$$\begin{aligned} \phi = & -4\varepsilon \left\{ P_1 \cos \omega t - 2b\chi \left[t - \tau_\lambda(J) \right] - \phi \right\} \\ & - \varepsilon^{-1} \beta \operatorname{sgn} \left[-J\chi \left[t - \tau_\lambda(J) \right] - \phi \right] \left[\eta - J\chi \left[t - \tau_\lambda(J) \right] - \phi \right] \\ & \times \left\{ \chi \left[t - \tau_\lambda(J) \right] - \phi \right\} + J \frac{\partial \tau_\lambda}{\partial \omega} \left(J \right) \left\{ \chi \left[t - \tau_\lambda(J) \right] - \phi \right\} + \varepsilon \omega^{-1} \theta(J) \end{aligned}$$

For simplicity we denote $J = J_\lambda$ and $\tau_\lambda(J) = \frac{1}{2} \gamma D_\lambda J^{\gamma-1}$.

In the above equations the functions $\theta(J)$ and $J(\omega_0)$ are taken from the more simple limiting approximation $\lambda \rightarrow \infty$, but at the same time there exists the possibility of considering another hypothesis (different from Newton) of impact damping.

$$\begin{aligned} \frac{\partial \tau_\lambda}{\partial \omega} \left(J \right) \left[t - \tau_\lambda(J) \right] - \phi &= \left[\frac{\partial \tau_\lambda}{\partial \omega} \left(J \right) \left[t - \tau_\lambda(J) \right] - \phi \right] \left\{ T^{-1} L^k e_k \left[t - \tau_\lambda(J) \right] - \phi \right\} \\ &= -T^{-1} k \omega L^k e_k \left[t - \tau_\lambda(J) \right] - \phi = -\chi \left[t - \tau_\lambda(J) \right] - \phi \end{aligned}$$

The explicit time averaging in the standard-form system, the successive accounting for the results of the calculations carried out for the piecewise characteristics, and the usual simplification yields

$$\begin{aligned} J = & -\frac{8}{b} \left\{ \chi \left[\tau_\lambda(J) \right] - \chi(0) \right\} \\ & - \varepsilon \left\{ \frac{b J^0}{\sin T \Omega} \left(1 - \frac{T \Omega}{\sin T \Omega} \right) + \frac{\pi \Omega^2}{2 \omega^2 P_1 \sin \omega \left[\phi + \tau_\lambda(J) \right]} \right\} \\ & - \left\{ \frac{4 \varepsilon \omega P_1 \cos \omega \left[\phi + \tau_\lambda(J) \right]}{2 \pi J^0 (\Omega^2 - \omega^2)} \right\} \\ & - \frac{4}{b} \left\{ \chi \left[\tau_\lambda(J) \right] - \chi(0) \right\} \left\{ \gamma D_\lambda \left(J^0 \right)^{\gamma-1} + \frac{\omega}{J - J^0} \omega_0' \left(J^0 \right) \right\} \end{aligned}$$

where $J^0 = J(\omega_0 = \omega) = -2\Omega \Delta \tan(\pi \Omega / \omega)$, the function $\omega_0' \left(J^0 \right)$ may be obtained by differentiating (5.24), and $\tau_\lambda \left(J^0 \right) = 0.5 D_\lambda \left(J^0 \right)^\gamma$.

The following analysis is quite similar to that in Sect. 5.8. It will not be reproduced here. The main distinctions in the obtained relationships are discussed below. An additional term to the equation for the phase is governed by the friction coefficient and is of the second order of smallness, $\tau_\lambda = O(\varepsilon)$. Therefore, $\chi \left[\tau_\lambda(J) \right] - \chi(0) = O(\varepsilon)$ due to the continuity of the PGF, and $D_\lambda = O(\varepsilon)$. For

5.10. Principle of Energy Balance

Let the equation of motion takes the form

$$\ddot{x} + C(x) = \varepsilon g(x, \dot{x}, t)$$

Nonconservative forces here are defined by the T -periodic function g and conservative forces are defined by the function $C(x)$. Let $x(t) \equiv x(t + T)$ be a T_1 -periodic steady-state regime which is developed in such a system. Substitution of the above function x into the equation of motion, multiplication of both parts of the obtained equation by $\dot{x}(t)$ and successive integration over the period T_1 yields

$$(5.47) \quad \int_{T_1}^0 g \left[x(t), \dot{x}(t), t \right] \dot{x}(t) dt = 0$$

because the left-hand side identically produces zero,

$$\int_{T_1}^0 \dot{x}(t) \dot{x}(t) dt + \int_{T_1}^0 C(x(t)) \dot{x}(t) dt = \int_{x(T_1)}^{x(0)} \dot{x} dx + \int_{x(T_1)}^{x(0)} C(x) dx \equiv 0$$

i.e. on a periodic motion the work done by nonconservative forces equals zero. Equality (5.47) expresses the so-called principle of energy balance. In the case of forced vibration the force $g(x, \dot{x}, t)$ consists of two parts, namely $g_0(x, \dot{x})$ which is the damping force and $P(t)$ which is the excitation force. From (5.47) it follows, therefore, that

$$\int_{T_1}^0 g_0 \left[x(t), \dot{x}(t) \right] \dot{x}(t) dt + \int_{T_1}^0 P(t) \dot{x}(t) dt = 0$$

This equality determines a mechanism for sustaining continuous vibration: an external source balances the work done by a damping force (compare with [107]). The process $x(t)$ takes the form $x(\varphi, t)$, where φ is an unknown parameter. Thus, equality (5.47) transforms into the equation for the definition of all possible values of φ allowing the existence of periodic vibration.

Consider, as an example, the representation (5.44) describing the two-parametric class of solutions to the problem of the definition of the periodic regimes in vibro-impact system

$$x(t) = f(t + \varphi) - J\chi(t)$$

where the moment of impact coincides with the initial time $t = 0$. Two parameters are unknown here, namely φ and J . The equality $x(0) = \Delta$ is the geometric impact condition which takes place at the particular coordinate. The condition

$$J = \dot{x}_-(0)(1 + R) \text{ may be substituted by the equation of energy balance. It can be}$$

verified easily using the Carnot's theorem (5.33) for the unit mass, $m = 1$, that is:

$$E_1 = J^2 \frac{2(1+R)}{1-R}$$

The corresponding verification may be simplified significantly in the case of the absence of viscous friction because in this case $b = 0$ and $\chi_-(0) = -1/2$. The

integration of (5.47) after simple rearrangements yields (5.45).

The principle of energy balance is related to the averaging method. It produces a

simple recipe of analysis of resonant vibration of arbitrary order.

In the equation of motion let $\varepsilon = 0$. The conservative solution (which is supposed to be T_0 -periodic) possesses the structure $x = X[A, t - \varphi]$, where in the above-mentioned

nonisochronous case $A = A(\omega_0)$ (for simplicity, though it is not important, it is

considered that $A(\omega_0)$ is a mutually single-valued function). In accordance with the

averaging method, the approximate steady-state resonant solution to an equation of

motion takes the form $x = X[A^0, t - \varphi^0]$, where A^0, φ^0 are the parameters defined

from the equations $E_{10}(A, \varphi) = 0, E_{20}(A, \varphi) = 0$.

The parameters A^0, φ^0 may be defined in a different way. Let the frequency of the

excitation force ω fall into the frequency band of eigenfrequencies of the

corresponding conservative system. We will consider the case of fundamental

resonance only. Thus $A^0 \approx A(\omega)$, and we have the corresponding resonance solution

$$x = X[A(\omega); t - \varphi], \text{ where the period of the function } X \text{ is taken to be } T = 2\pi\omega^{-1}.$$

For the undefined phase of the process, from the condition of energy balance (5.47),

we find

$$E(\varphi) \equiv \int_0^T g(X, \dot{X}, t) \dot{X}(t - \varphi) dt = 0$$

It can be verified easily that the equations $E(\varphi) = 0$ and $E_{10}(A^0, \varphi) = 0$ are equivalent. The same relationships may be written in the case of more complicated resonances. This problem is solved below.

5.11. Conditions of Existence of Resonant Regimes Under Harmonic Excitation

Using this method we can not define different values of A^0 corresponding to different roots of the equation $E(\varphi) = 0$ and differing from one another by the value of order ε . However, in the majority of applications it is not required. The parameter $A^0 = A(\omega)$ accurately estimates the intensity of resonant processes (on the existence boundary the parameters of a resonant process may be accurately described using a backbone curve). The application of the energy balance method allows relatively easy definition of the existence conditions of such processes.

The mechanism for sustaining resonant vibration may be studied using the equation of energy balance.

Let $P(t) = P_1 \cos \omega t$, and a "conservative regime" can be represented in the form

$$x_0(t) = \sum_{k=0}^{\infty} X_k \cos k\omega_0 t$$

(see Sect. 4.2). We will construct the energy balance equation for the case when the frequency of excitation ω is a multiple of one of the eigenfrequencies of system $\omega_0 = \omega/q$, where q is an integer number. The equation of energy balance for the resonance solution $x(t) \approx x_0(t)$ of the period qT , $T = 2\pi\omega_0^{-1}$, takes the form

$$E_d = -P_1 \int_0^{qT} [\cos(\omega t + \varphi)] \left[-\sum_{k=1}^{\infty} X_k k\omega_0 \sin k\omega_0 t \right] dt \leq 0$$

where the undefined phase is related to the excitation and E_d denotes the work done by the dissipation force over the period qT . With provision for $\omega_0 = \omega/q$ we have

$$-E_d = -P_1 \int_0^{qT} \cos(\omega t + \varphi) X^q \omega \sin \omega t dt = P_1 X^q \pi q \sin \varphi$$

i.e. in the considered case the energy of excitation is "received" by the q th harmonics of the system's free motion. Such a resonance is said to be *subharmonic* (compare with Sect. 5.3) but it is more correct to call it a *subperiodic* resonance.

In particular, when $q = 1$ energy is "transmitted" by means of the first harmonics and the fundamental resonance takes place.

The phase of vibration controls the energy delivery, so

$$\sin \varphi = \frac{E_d}{P_1 X^q \pi q}$$

Therefore, the subperiodic resonance existence condition takes the form

$$\left| \frac{E_d}{E_d} X^q \right| \leq 1 \quad (5.48)$$

and on the boundary $\sin \varphi = 1$. This indicates the merging of two possible regimes.

For the definition of E_d the following quadrature is calculated

$$E_d = \int_0^{qT} E_0 \left[\sum_{k=0}^{\infty} X_k \cos k\omega_0 t - \sum_{k=1}^{\infty} k\omega_0 X_k \sin k\omega_0 t \right] dx_0$$

where $q\omega_0 = \omega$. When using (5.48) the relationships $X^q = X^q(A^q)$ and

$A^q = A(\omega/q)$ must be taken into the account.

It may be seen that an increase in q results in a decrease in the denominator in (5.48) because $\lim_{q \rightarrow \infty} X^q = 0$, and for the continuous solutions $x_0(t)$ the quantity X^q

decreases more quickly than $1/q^2$. However, in the strongly nonlinear systems the above decrease is slow enough. For the vibro-impact system, as an example,

$$x_0(t) = -J\chi(t) = -J \left[\frac{T_0}{2} \sum_{k=1}^{\infty} \Omega^2 - k^2 \omega_0^2 + \frac{T_0 \Omega^2}{1} \cos k\omega_0 t \right] \quad (5.49)$$

i.e. the asymptote of the Fourier components is minimal: $O(1/k^2)$.

The possibility of the appearance of subperiodic resonances is a common feature of all nonlinear systems, but in the presence of strong nonlinearities this feature manifests itself particularly explicitly.

The conditions of existence of resonant regimes of order $1:q$ may be revealed in the improved singular approximation using formula (5.48).

From formula (4.53), the frequencies corresponding to high energy levels may be assessed as

$$\omega_0 = 2\Omega + O(\lambda^{-1/(q+1)})$$

The improved singular approximation is required to increase the accuracy of the analysis of such processes. Let the frequency of excitation be $\omega_0 = \omega q^{-1}$, where

$q = 1, 2, \dots$. Then the approximate resonant solution takes the form

$$x_0(t) = -J\lambda\chi \left[t - \frac{1}{2}t_\lambda(J\lambda); \omega q^{-1} \right] \quad (5.50)$$

where the function $t_\lambda(J\lambda)$ is given by (4.52) and J_λ satisfies Eq. (4.54) with the frequency of PGF equals ωq^{-1} .

The condition (5.48) takes the form

$$\left| \frac{E_d}{E_d} X^q \right| \geq 1 \quad (5.51)$$

where expansion (5.49) is considered and E_d , E_{d_1} denote the work done by the

dissipative forces throughout the motion and in the contact zone. The reasoning of Sect. 4.12 may be used to obtain the conditions of existence for the systems with symmetric forces. It is worth noting that the approximate formulae obtained in §4 may be applied most effectively to the case of large energies corresponding to high frequencies (the interval of the eigenfrequencies of symmetric vibro-impact system is $1 < \omega_0 < \infty$).

The representation (5.50) in the symmetric case takes the form

$$x_0(t) = -J\lambda\chi \left[t - \frac{1}{2}t_\lambda(J\lambda); \omega(2q+1)^{-1} \right]$$

where the symmetric PGF may be represented using the Fourier series

$$\chi^*(t) = \frac{T_0}{4} \sum_{k=0}^{\infty} \frac{\cos[(2k+1)\omega_0 t]}{\Omega^2 - (2k+1)^2 \omega_0^2}$$

Hence the transmission of energy is possible due to odd harmonics and corresponding subperiodic resonances may be of odd order only. The equations for the determination of the impulse and the duration of the interaction are similar to (4.54) and (4.52).

Since the solution is symmetric, $\chi(t+T/2) = -\chi(t)$, it is feasible to calculate the balance of the work along the half-period of resonant motion $(2q+1)T/2$. In doing so, (5.51) changes into the inequality

$$\left| \frac{E_d}{E_d} X^q \right| \geq 1$$

From the above, the existence conditions of resonant vibration of arbitrary order for vibro-impact systems may be obtained immediately. Energy dissipation at contact is calculated here in accordance with Carnot's theorem (5.33). Note that the representation of the resonant solution in the form of conservative vibration is feasible in the case of small dissipation and excitation only. Since the duration of contact is short, it is reasonable to consider the magnitude of the dissipation force during the interaction to be of order unity. Now, consider the problem of the definition of the resonant processes under polyharmonic external excitation [93]. For a higher accuracy the method of averaging is used here, though, as revealed below, the general relationships may be obtained directly from the energy balance principle.

5.12. Bifurcation of Fundamental Resonant Regimes Under Polyharmonic Excitation

Let the equation of motion of a nonlinear system takes the form

$$\ddot{x} + C(x) = \varepsilon [g(x, \dot{x}) + P_n(t)] \quad (5.52)$$

where $g(x, \dot{x})$ defines the dissipative forces, and the external force is represented by the series

$$P_n(t) = \sum_{k=1}^n (\bar{Q}_k \cos k\omega t + D_k \sin k\omega t), \quad n > 1$$

The T -periodic regimes in system (5.52) will be studied.

Suppose that the general solution of a generative conservative system when $\varepsilon = 0$ will be in the form

$$x(t) = X[A, \omega_0(A)(t - \phi)] = \sum_{k=0}^{\infty} X_k(A) \cos[k\omega_0(A)(t - \phi)]$$

where, for definiteness, it is considered that $\omega_0(A)$ is a mutually single-valued

function. In the case of the fundamental resonance $\omega - \omega(A) = \varepsilon\theta(A)$, system (5.11)

after averaging and the usual simplifications $\varepsilon\theta(A) = -(A - A^0)\omega_0'(A^0)$,

$A = A^0 = A(\omega)$, (see Sect. 5.6) is represented in the form

$$\begin{aligned} \dot{A} &= \frac{\varepsilon}{T} \int_0^T \{g\} X(A^0, t - \phi) dt + P_n(t) \{X(A^0, t - \phi)\} \\ \dot{\phi} &= \frac{\varepsilon}{T} \int_0^T \{g\} X(A^0, t - \phi) dt + P_n(t) \{X(A^0, t - \phi)\} \end{aligned}$$

$$-(A - A^0)\omega_0'(A^0)$$

where the function X is taken at the frequency ω .

For the parameters of steady-state processes, where εE_{10} and εE_{20} denote the right-hand sides of the above equations, we obtain the system of transcendental equations

$$E_{10}(A^0, \phi) = 0, \quad E_{20}(A^0, \phi) = 0$$

The first equation representing the energy balance condition takes the form

$$E_{10}(A^0, \phi) \equiv E_{-1}(A^0) \{-\Lambda_1(A^0) + \frac{1}{2} \sum_{k=1}^n k\omega X_k(A^0) [\bar{Q}_k \sin k\omega\phi - D_k \cos k\omega\phi]\} = 0 \quad (5.53)$$

The second equation, respectively, is written as

$$E_{20}(A, \phi) \equiv E_{-1}(A^0) \left\{ \Lambda_2(A^0) + \frac{1}{2} \sum_{k=1}^n \frac{dX_k(A^0)}{dA} [\bar{Q}_k \sin k\omega\phi - D_k \cos k\omega\phi] \right\} = 0 \quad (5.54)$$

and serves as a definition of the undefined parameter A which is close to A^0 . The notation $\Lambda_{1,2}$ are obvious. For example,

$$-\Lambda_1(A^0) = \frac{1}{T} \int_0^T g[X(A^0, t - \phi)] X(A^0, t - \phi) dt$$

For a detailed analysis of Eq. (5.53) the above expression is rewritten in the form

$$\Lambda_1(A^0) = \frac{1}{2} \sum_{k=1}^n k\omega X_k P_k \sin(k\omega\phi + \beta_k) \quad (5.55)$$

where the coefficients P_k and β_k are calculated in the standard way.

Using well-known trigonometric formulae, the above equation may be reduced to an algebraic equation in one of the trigonometric functions. Note, that, since the coefficients D_k and $\bar{Q}_k$ are not subjected to conditions, they may be chosen so that the number of solutions in the system will be more than two. A detailed analysis may be carried out using general algebraic methods ([96]), but it is not reproduced here. Note, that an increase in the number n leads to the complication of the corresponding analysis.

Denote by $\alpha_n(\phi)$ the sum from Eq. (5.55). This function is known as a phase function.

The number of pairs of solutions to Eq. (5.55) (each solution represents a different resonance regime) equals the number of maxima of the phase function on the interval $0 \leq \phi < T$. The existence condition of at least one pair of solutions (for $\Lambda_1 \geq 0$) is of the form

$$\sup_{0 \leq \phi < T} \alpha_n(\phi) \geq \Lambda_1(A^0)$$

where the equality sign corresponds to the existence boundary, and the solution pair merges into one limiting solution.

Let the fundamental harmonic be a dominant harmonic in the spectrum of excitation, $P_1 \gg P_k$ for $k > 1$. Then the phase function behaves like a sine function possessing one maximum on the interval $0 \leq \phi < T$. In this case only one solution pair (or one on the boundary of existence) is possible in the system. Suppose now that in the

Only fundamental resonant processes are mentioned here.

spectrum of excitation the j th harmonic rose¹, so the number P_j became even greater than P_1 and, as a result, $P_1 X_1 \approx P_j X_j$. Then the phase function $\alpha_n(\phi)$ may be described approximately as a sum of two sine functions possessing two maxima if the value of the magnitude P_j^* exceeds some *critical value*. If the inequality

$$\max_{0 \leq \phi < T} \alpha_n(\phi) \geq \Lambda_1(A^0)$$

holds for the smaller maximum, then Eq. (5.55) possesses two pairs of solutions in the case of a rigorous inequality and one pair plus a limiting process in the case of equality.

If an additional harmonic rises, three maxima appear, etc.

The term "*bifurcation of solutions*" in (5.52) reflects the appearance of new regimes when parameters of the system attain particular critical values.

The possibility of bifurcation of solutions is an inherent feature of practically all nonlinear systems. In quasi-linear systems the magnitudes of the harmonics of the corresponding "conservative solution" P_k for $k > 1$ decrease very quickly with the increase in their number. Thus, the influence of the higher harmonics of excitation is less relevant.

For strongly nonlinear systems, and especially for vibro-impact systems (see Sect. 5.11), the above magnitudes X_k decrease most slowly. Similar to the case of subperiodic resonances, the phenomenon of bifurcation of solutions will be well manifested in vibro-impact systems.

The above may be formalised to a large extent, but it is more informative to analyse the bifurcation of solutions in a particular vibro-impact system. The physical meaning of additional resonance processes will be determined.

5.13. Bifurcation of Solutions in Vibro-impact System

Consider the system in Fig. 5.8.

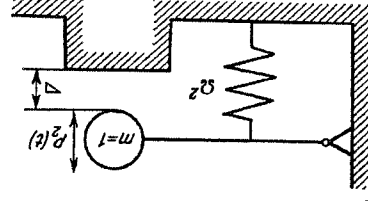


Fig. 5.8

¹ The term "rose" denotes such a feeble growth that it does cause a rise in the transient processes.

The external force is given by the T -periodic biharmonic function of time and damping is accounted for by means of the restitution ratio R such that $1 - R = \varepsilon$. The equation of motion takes the form

$$\ddot{x} + \Omega^2 x + \Phi_0(x, \dot{x}) = \varepsilon [\Phi_2(x, \dot{x}) + P_1 \cos \omega t + P_2 \cos 2\omega t] \quad (5.56)$$

where Φ_0 and Φ_2 are the elastic and dissipative components of the impact force.

A transition to "impulse-phase" variables in Eqs. (5.53) and (5.54) yields

$$\begin{aligned} \varepsilon E_{10}(J^0, \phi) &\equiv -4\varepsilon \left[\frac{r J^0}{\omega^2 P_2 \sin \omega \phi} + \frac{2\pi(\Omega^2 - \omega^2)}{\omega^2 P_2 \cos \omega \phi} + \frac{4T}{\pi(\Omega^2 - \omega^2)} \right] = 0 \\ \varepsilon E_{20}(J, \phi) &\equiv -4\varepsilon \left[\frac{\omega P_1 \cos \omega \phi}{\omega^2 P_2 \cos 2\omega \phi} + \frac{2\pi(\Omega^2 - \omega^2)}{\omega^2 P_2 \cos 2\omega \phi} + \frac{J^0}{\omega} \right] - \frac{\omega}{J - J^0} \omega'(J^0) = 0 \end{aligned} \quad (5.57)$$

Here Eqs. (5.25) with $p = q = 1$ were applied. Eqs. (5.35) were used for averaging of the impact damping, and the derivative of the function $\omega(J)$ (see (5.24)) was calculated above. Therefore, as usual, $J^0 = -2\Omega\Delta \tan(\pi\Delta/\omega)$.

In the considered case we always have $\omega > \Omega$. Hence, the phase function

$$\alpha_2(\phi) = \frac{\omega^2 P_1 \sin \omega \phi}{\omega^2 P_2 \sin 2\omega \phi} + \frac{2\pi(\omega^2 - \Omega^2)}{\pi(4\omega^2 - \Omega^2)}$$

possesses positive coefficients.

Under the condition $\sup[\alpha_2(\phi)] > (4T)^{-1} r J^0$, $0 \leq \phi < T$ there exists at least one pair of fundamental resonant solutions. The function $\alpha_2(\phi)$ may possess two maxima if four roots exist only within the interval $[0, T]$. That takes place if the condition of bifurcation

$$\frac{P_1(4\omega^2 - \Omega^2)}{4P_2(\omega^2 - \Omega^2)} \leq 1 \quad (5.58)$$

holds (at resonance $\omega > \Omega$ always). From (5.58), the above four roots may exist if $P_2 > P_1$. The order of quantity P_2^* may be assessed easily. Condition (5.58) for the resonance frequency $\omega = 2\Omega$ yields $5P_1 = 4P_2$. That corresponds to the existence boundary of additional solutions.

It is seen that, the dominance of the second harmonics is very small¹.

¹ Burd analysed the nonlinear system with the potential force $\sin x$ and demonstrated that the above relation takes the form $9P_1 \approx P_2$. That indicates a "lower probability"

The expressions representing the solution of the first equation (5.57) are very cumbersome. Therefore, the graphical solution was undertaken.

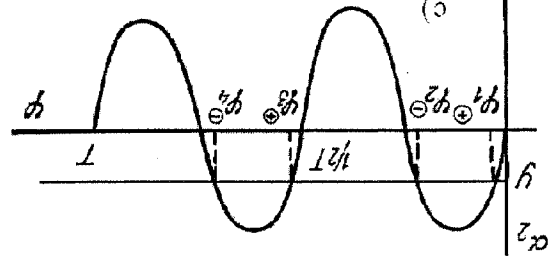
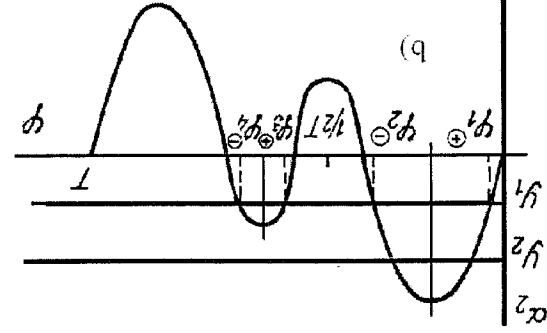
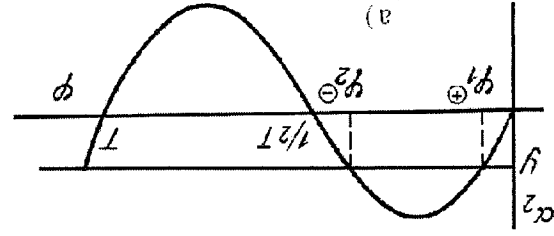


Fig. 5.9

In Fig. 5.9a the function $\alpha_2(\phi)$ is plotted with $P_1 \gg P_2$ (condition (5.58) is violated intentionally). Abscissas of the points of intersection of the straight lines $y = (4T)^{-1}r_0^0 = \text{const}$ with the curve $\alpha_2(\phi)$ represent the desired values $\phi = \phi_{1,2}$. Figure 5.9b corresponds to the case when condition (5.58) holds. Different possible

the bifurcation for systems with "weaker" nonlinearities.

variants of existence of resonant regimes are considered here. The straight line y_1 corresponds to the case of two pairs of solutions and y_2 to one pair.

Finally, Fig. 5.9c portrays the case $P_2 \gg P_1$. Two pairs of solutions may be observed here which are merged into one pair of subperiodic processes. In Eqs. (5.57) the small terms containing $\sin \omega \phi$ and $\cos \omega \phi$ may be neglected. The notation $\phi_0 = 2\phi$ may be introduced. The phase ϕ_0 will possess two values only and the first equation from (5.57) will describe the conditions for sustaining resonant vibration of period T due to the excitation with period $T/2$, i.e. subperiodic resonance of order 1:2. The above plots reflect the evolution of resonant regimes in system (5.56) depending upon the magnitude of the second harmonic of excitation force. While condition (5.58) holds, the system behaves similar to a system under pure sinusoidal excitation. Since the equality in (5.58) is satisfied in the conservative case ($r = 0$), new solutions to equation (5.57) may appear.

To demonstrate this, rewrite the above equation as

$$(5.59) \quad \alpha_2(\phi) = \sin \omega \phi \left[\frac{\omega^2 P_1}{2\pi(\omega^2 - \Omega^2)} + \frac{\pi(\omega^2 - \Omega^2)}{2\omega^2 P_2 \cos \omega \phi} \right] = 0$$

Then with a proviso for the case of equality in condition (5.58), this equation yields additional solutions,

$$\cos \omega \phi = -\frac{P_1(4\omega^2 - \Omega^2)}{4P_2(\omega^2 - \Omega^2)} = -1$$

i.e. $\phi_0 = \frac{\pi}{2}$. However, for the fundamental regimes here, $\sin \omega \phi = 0$ and the phase $\phi_0 = \frac{\pi}{2}$ correspond to one of the fundamental solutions. Thus, at the boundary of

existence the additional solutions merge with one of the fundamental solutions. The successive increase in the value P_2 results in the branching-off of the additional solutions. That may take place also in systems with nonzero damping (see Fig. 5.9b). The following growth of the magnitude P_2 in comparison with the magnitude P_1 results in the merging of additional solutions with the pair of subperiodic solutions.

An appearance of the above additional solutions reflects the mechanism of the process of continuous transition from the fundamental resonance to subperiodic resonant vibration with the increase in the magnitude ratio.

For the stability of the above solutions the definition of the phase function, criterion (5.29) and the results of Sect. 5.6, yield

$$E_{10r} = -T^{-1}r, \quad E_{10\phi} = \frac{2\omega^3}{\pi} \left[\frac{P_1 \cos \omega \phi}{\omega^2 - \Omega^2} + \frac{4P_2 \cos 2\omega \phi}{4\omega^2 - \Omega^2} \right] = 4\alpha_2'(\phi)$$

$$E_{20J} = -e_1 \Delta - \frac{J}{4} \left[\frac{\omega_2}{\alpha_2(\phi)} - \frac{3P_2 \cos 2\omega\phi}{2\pi(4\omega_2^2 - \Omega^2)} \right], E_{20\phi} = -4 \frac{J}{\alpha_2(\phi)}$$

where

$$\kappa_1 = 2\pi\Omega^2 \left\{ \omega \left[\tan J(2\Omega\Delta)^{-1} \right]^{-2} (4\Omega^2 \Delta^2 + J^2) \right\}^{-1} > 0$$

if $\Delta < 0$, and

$$\kappa_1 = 2\pi\Omega^2 \left\{ \omega \left[\pi - \arctan J(2\Omega\Delta)^{-1} \right]^{-2} (4\Omega^2 \Delta^2 + J^2) \right\}^{-1} > 0$$

if $\Delta > 0$.

Conditions (5.29) may be obtained from the above. The first condition is always satisfied if $r > 0$ ($R < 1$). Simple calculations show that the second condition is satisfied in the majority of cases (for example, if one of the harmonics dominates) if

$$\Delta \alpha_2^2(\phi) > 0$$

i.e. phases corresponding to the rising branches of the function $\alpha_2(\phi)$ define the asymptotically stable process when $\Delta > 0$. The phases from the falling branches of the phase function define the asymptotically stable process when $\Delta < 0$.

Therefore, additional solutions may arise from the unstable solution, which corresponds to the phase $\phi^0 = \frac{2}{T}$ when $\Delta > 0$. The above may be illustrated with the use of *branching diagrams* in Fig. 5.10a (case $\Delta > 0$) and Fig. 5.10b (case $\Delta < 0$).

The arrows escort us successively from the situation in Fig. 5.9a to the situation in Fig. 5.9c. The phases are numbered in accordance with the order of Fig. 5.9b. The positive and negative signs are used to indicate the stable and unstable regimes, respectively.

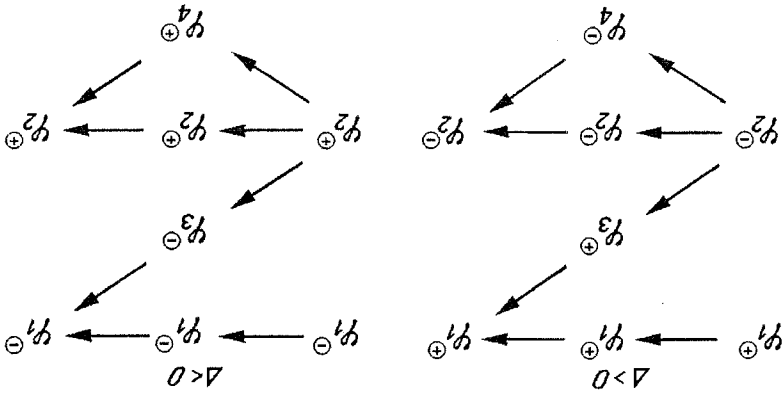


Fig. 5.10

The relationships defining the exact solution of the considered problem follow. Representation (5.44) yields

$$x(t) = \frac{eP_1 \cos \omega t}{eP_2 \cos 2\omega t} + \frac{\Omega^2 - \omega^2}{\Omega^2 - 4\omega^2} - J\chi(t - \phi)$$

where the impulse J and the phase ϕ follow the consistency conditions $x(\phi) = \Delta$, $J = (1 + R)x_-$, $x(t) \leq \Delta$, $J \geq 0$, $0 \leq \phi < T$. For the unknown values of J and ϕ we have the system of equations

$$\left[\frac{J}{1 - R} \right] = \varepsilon \left[\frac{\omega P_1 \sin \omega\phi}{2\omega P_2 \sin 2\omega\phi} + \frac{4\omega^2 - \Omega^2}{2\omega^2 - \Omega^2} \right] = \varepsilon \left[\frac{eP_1 \cos \omega\phi}{eP_2 \cos 2\omega\phi} + \frac{\Omega^2 - \omega^2}{\Omega^2 - 4\omega^2} - \Delta \right] \tan \frac{\omega}{\pi \Omega^2}$$

(5.60)

which coincide with (5.57) within the accuracy of the small terms (compare with Sect. 5.7). Substituting $J = J^0$ into the first equation of (5.60) we can observe that it defines regimes similar to those defined by the first equation of (5.57). Using the second

equation (5.60) or (5.57), the functions $J^i(\omega)$ may be obtained for the phases ϕ^i , $i = 1, 2, 3, 4$.

Note that when $R = 1$ the pair of additional solutions merges into one. The function $J(\omega)$ is defined by the value $\cos \omega\phi = 2 \cos^2 \omega\phi - 1$. In this particular case, from (5.59), we can see that additional different values of the phases produce the same value of $\cos \omega\phi$.

The appearance of additional solutions was noticed before, for example in monograph [191], where for the analysis of an oscillatory system the highest approaches of the method of harmonic linearisation were used (see also [78]).

5.14. Analysis of Superperiodic and Combination Resonances

In the case of a polyharmonic T -periodic external force the *superperiodic* and *combination resonances* may rise in a nonlinear system [107, 162, 186, 196]. The equations of energy balance, defining this vibration, may be obtained as described above. At the fundamental resonance, the frequency of the external force is ω and the resonance process is of the same frequency. In this case, the energy of the fundamental harmonic of excitation is received by the fundamental harmonic of the generative solution (regime of undamped free vibration), the second harmonic of

vibration of a conservative system when $\varepsilon = 0$ (see Sect. 6.3).

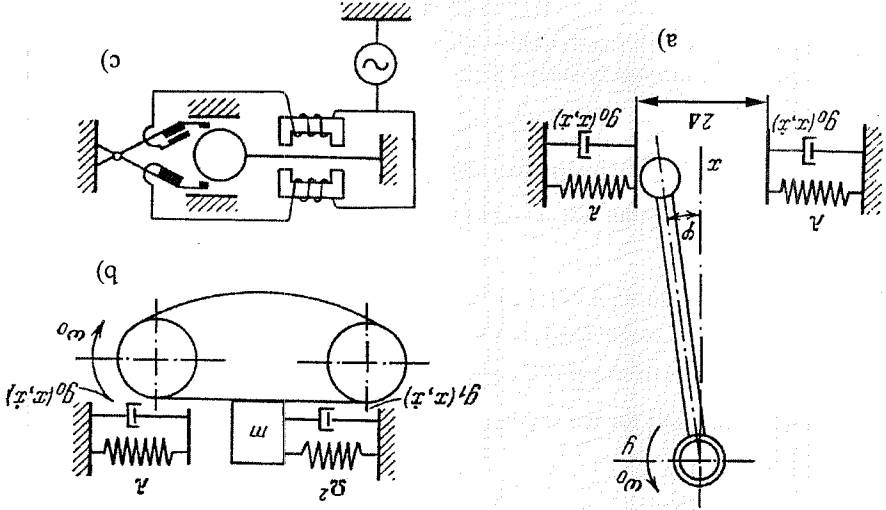


Fig. 6.1

In Fig. 6.1, two typical self-sustaining systems are sketched schematically. In Fig. 6.1a the Froude pendulum, which is mounted freely on a rotating shaft, is involved in the interaction with two obstacles of high stiffness. In this case, energy is supplied due to the shaft slipping in the pendulum bearing. In the system sketched in Fig. 6.1b, external energy is supplied due to the motion of the transporter band. In both cases energy is supplied due to friction.

Other mechanisms of self-sustaining vibration are also possible. In Fig. 6.1c, the scheme of the electromagnetic breaker, which is used in vibratory relays, electric bells etc., is shown. Here the self-sustaining vibration is due to the excitation of the vibratory system by electromagnets in accordance with the system motion. Other examples are discussed below.

Usually it is considered that the natural frequencies of a system and the frequencies of the energy source are incomparable. Otherwise, the equation of the energy source must be considered along with Eq. (6.1).

The typical approximations of the self-sustaining terms $g(x, x) = g_1(x, x) + g_2(x, x)$ in the reference problems are considered to be in the form $2bx - b_0 \operatorname{sgn} \dot{x}$, $b_0 \operatorname{sgn} \dot{x}$, $b_0(x - bx^3)$, where $b, b_0 > 0$. The Van der Pol form of the self-sustaining term $g(x, x) = (\alpha - \beta x^2)x$, $\alpha, \beta > 0$ is used widely.

The averaging method will be effectively applied to the study of autoresonant

§6 Vibration of Autonomous Systems

6.1. Preliminary Considerations

The conservative systems considered in §4 represent a particular case of the so-called autonomous systems, which are described using equations in the form

$$\ddot{x} + \Omega^2 x + \lambda \Phi(x) = \varepsilon [g_0(x, x) + g_1(x, x) + g_2(x, x)] \quad (6.1)$$

Here g_0 and g_1 are the forces of distributed and contact damping, and g_2 is the force produced by the source to sustain the motion (self-sustained vibration). Eq. (6.1) corresponds to the case of free damped vibration when $g_2 \equiv 0$ and to the free

excitation is received by the second harmonic of the generative solution, etc. In general, the energy of the k th harmonic of excitation is received by the k th harmonic of the generative solution, etc. In

At subperiodic resonance, the frequency of the external force is ω and the resonance process has the frequency ω/q , where q is an integer. In this case, the fundamental harmonic of excitation is received by the q th harmonic of the generative solution, the second harmonic of excitation is received by the $2q$ th harmonic of the generative solution, etc. Therefore, at the subperiodic resonance the amount of energy absorbed by the system is much lower than at fundamental resonance. It is also possible that the energy of the p th harmonic is received by the fundamental harmonic of the generative solution. Generally speaking, energy of the k th harmonic is received by the k th harmonic of the generative solution. The external force of the frequency ω causes the superperiodic vibration of the frequency $p\omega$.

Combination resonances have frequency $(p/q)\omega$. The energy of the p th harmonic of the excitation is received by the q th harmonic of the free undamped response of the system.

An analysis of all types of resonances is very similar to that described above. The phenomenon of bifurcation of solutions is also possible here if higher-order harmonic dominates. It is obvious that under feasible conditions such a phenomenon is "low-probable". Note that an analysis of resonant vibration under quasi-periodic excitation attracts considerable interest. Important results were obtained in [78].

The following example is mostly of theoretical interest. If the external force is represented by a generalised function of a high order of singularity, the phase function $\alpha(\varphi)$ may possess an infinite or even an uncountable number of zeros on the interval $[0, T]$, and, therefore, an infinite number of resonant solutions may appear.

vibration, which is close to the free vibration of an undamped system. Nowadays the chaotic vibration of self-sustaining systems is actively studied along with periodic motions (see references [125, 134, 163]). The relaxation vibration in nonlinear systems has been studied comprehensively, see references [5, 98, 115, 186]. The monograph of Landa [98] may be recommended for acquainting the reader with the modern theory of self-sustaining systems.

6.2. Analysis of Autonomous Systems using the Averaging Method

Since the averaging method is suitable for the analysis of the equation of motion in the form

$$\ddot{x} + \Omega^2 x + \lambda \Phi(x) = \varepsilon g(x, \dot{x}) \quad (6.2)$$

the formalism of Sect. 5.2 is applicable here. The generative solution takes the form

$$x_0(t) = X[A, \omega(A)(t - \phi)] = \sum_{k=0}^{\infty} X_k^k(A) \cos[\omega(A)(t - \phi)]$$

where $A(E)$ is a mutually single-valued function of energy, and the function $\omega(A)$ in the anisochronous case is supposed to be a continuous and mutually single-valued function of energy (the subscript "zero" is omitted). The transformation of variables

$$x = X(A, \psi), \quad \dot{x} = \omega X_\psi(A, \psi), \quad X(A, \psi + 2\pi) = X(A, \psi) \quad (5.5)$$

and calculations which are similar to those in Sect. 5.2, yield the system with single fast phase

$$\dot{A} = -\frac{\varepsilon}{2\pi} \frac{d_0(A)}{d_0(A)} g(X, \omega X_\psi) X_\psi, \quad \dot{\psi} = \omega(A) + \frac{\varepsilon}{2\pi} \frac{d_0(A)}{d_0(A)} g(X, \omega X_\psi) X_\psi \quad (6.3)$$

which is more simple than (5.7) since it does not explicitly contain the time. Here $d_0(A) = -\omega^{-1}(A)E_A$ (see (5.8)) and all functions are 2π -periodic with respect to the fast variable ψ , which takes the role of time.

The averaging of the right-hand parts of the above equations over the period $0 < \phi < 2\pi$ yields the system in the first approximation

$$\dot{A} = -\frac{\varepsilon}{2\pi} \int_0^{2\pi} g[X(A, \psi), \omega X_\psi(A, \psi)] X_\psi(A, \psi) d\psi$$

$$= \frac{\varepsilon}{T(A)} \int_0^{T(A)} g[X(A, t), X(A, t)] X(A, t) dt \equiv \varepsilon E_1(A) \quad (6.4)$$

It follows from above that

$$\dot{\xi} = \varepsilon E_{1A}(A^0) \xi \quad (6.7)$$

when $t > 0$ yields first equation of (6.4) for the small deviation $\xi(t)$ from the steady-state process A^0 that the initial value of the above perturbation is $\delta A^0 \equiv \xi(0)$. The linearisation of the to be constructed. Let the steady-state value A^0 be perturbed slightly in such a way For the analysis of the stability of the steady-state solution, an equation in variations is

$$\dot{\omega}_0 = \omega(A^0) \quad (6.6)$$

sustaining vibration within the accepted accuracy may be expressed in the form where $\omega(A^0) = \text{const} = \omega^*$. In the anisochronous case the frequency of the self-

$$\dot{\omega}_0 = \omega^* + \varepsilon E_2(A^0) \quad (6.5)$$

The frequency of the steady-state regime of the generative isochronous system is defines the phase of the steady-state regime.

$$\psi_0 = \omega(A^0)t + \varepsilon E_2(A^0)t$$

solution is not, in general, unique), and the quadrature transcendental equation $E_1(A) = 0$ defines the desired value $A = A^0$ (the above For the analysis of steady-state processes it is considered that $A = 0$. Then the small additional term, which may be neglected within the accepted accuracy [19].

For the phase is not feasible) the evaluation of the corresponding quadrature yields a anisochronous ($\omega(A) \neq \text{const}$, taking account of the ε -correction in the expression The above solution depends on the frequency $\omega(A)$. If the generative system is

$$x(t) = X[A(t), \psi(t)] + \varepsilon \dots$$

decomposition of the desired solution to Eq. (6.2) The functions $A(t)$ and $\psi(t)$ define the fundamental term of the asymptotic and the initial conditions $x(0)$, $\dot{x}(0)$.

where C_1 and C_2 are the constants of integration defined from transformation (5.5)

$$\int \frac{dA}{\varepsilon E_1(A)} + C_1 = t, \quad \psi = \int \left\{ \omega[A(t)] + \varepsilon E_2[A(t)] \right\} dt + C_2$$

which may be solved by means of two quadratures

$$\equiv \omega(A) + \varepsilon E_2(A)$$

$$\psi = \omega(A) + \frac{\varepsilon}{2\pi} \int_0^{2\pi} g[X(A, \psi), \omega X_\psi(A, \psi)] X_\psi(A, \psi) d\psi$$

Finally, for

$$\xi = \xi(0) e^{\varepsilon E_{1A}(A^0)t} \quad (6.8)$$

$$E_{1A}(A^0) > 0 \quad (6.9)$$

the steady-state solution will be asymptotically stable. The decrement $\gamma = \varepsilon |E_{1A}(A^0)|$ defines the decay rate of the transient process due to perturbation of the steady-state process A^0 .

The physical meaning of condition (6.9) is evident. The first shortened equation, when $A = 0$, represents the energy balance equation of nonconservative forces. For the stability of motion, the nonconservative forces must compensate the small deviations of motion from the steady-state process. In the case of an impact force (see Sect. 5.3), transformation (5.5) takes the form of (5.21),

$$\dot{x} = -J\chi[\psi, \omega(J)], \quad \dot{x} = -J\omega\chi_\psi[\psi, \omega(J)]$$

where the function $\omega(J)$ is given by (5.24). Thus, Eqs. (6.3) take the form

$$\begin{aligned} J &= -4\varepsilon\omega(J)g(-J\chi, -J\omega\chi_\psi) \\ \psi &= \omega(J) - 4\varepsilon\omega(J)J^{-1}g(-J\chi, -J\omega\chi_\psi)(-J\chi) \end{aligned} \quad (6.10)$$

The steady-state value of the impulse J^0 is defined from the equation

$$\frac{\omega(J)}{2\pi} \int_{-\pi}^{\pi} g\{-J\chi[\psi, \omega(J)], -J\omega(J)\chi_\psi[\psi, \omega(J)]\} d\psi = \frac{1}{T(J)} \int_0^T g[-J\chi(t), -J\chi(t)] \chi(t) dt = 0$$

The frequency of the self-sustained vibration in the anisochronous case may be obtained from (5.24) using formula (6.6), $\omega_0 = \omega(J^0)$.

The generative solutions in the form of the improved singular approximation

$$x_0(t) = -J\chi[t - \phi - \frac{1}{2}t_2(J)]$$

($t_2(J)$ is the time of interaction, see §4), may be also considered. The corresponding shortened equations may be similarly constructed, but an additional differentiation of the function $t_2(J)$ is required. Consequently, the second equation will be modified to

$$\dot{\psi} = \omega(J) \left[1 + 4\varepsilon J^{-1}g(-J\chi, -J\omega\chi_\psi) \right] \left[\chi + J\omega\chi_\psi \right] + \frac{1}{2}J \frac{\partial t_2}{\partial t_2}(J)$$

And finally, in the case of the vibro-impact system with the restitution ratio $R < 1$, the first equation of (6.10) will be modified. In accordance with (5.34) we have

6.3. Chatter

In the following calculations of particular autonomous systems we consider that the strong nonlinearity which is represented in the system is of vibro-impact type. The generalisation of the obtained results to systems with large threshold forces may be carried out without particular difficulties using the technique discussed above. Many important qualitative results, however, are obtained in the especially distinctive form for the limiting vibro-impact case. It is evident that the described procedure is suitable for the calculation of a wide class of autonomous strongly nonlinear systems of second order. An analysis of the damped free vibration of a vibro-impact system is an intractable problem. Thus, we will demonstrate an application of the discussed methods to the solution of this problem. Then, self-sustained vibration will be analysed.

where $\varepsilon R = 1 - R$ and $\delta_{2\pi}^-(\psi)$ is defined in Sect. 5.6.

$$J = -\varepsilon \left[rJ\omega(J)\delta_{2\pi}^-(\psi) + 4\omega(J)g(-J\chi, -J\omega\chi_\psi)\chi_\psi \right] \quad (6.11)$$

Consider the problem of attenuation of free vibration by the forces of contact and viscous damping. The contact damping will be accounted for using the restitution ratio R . The oscillator in Fig. 6.2 is involved in free vibration with collisions. The corresponding equation of motion takes the form

$$\ddot{x} + \Omega^2 x + \Phi_0(x, \dot{x}) = -\varepsilon [\Phi_2(x, \dot{x}) + 2b\dot{x}] \quad (6.12)$$

where the generalised function Φ_2 defines the losses of energy due to impacts (see Sect. 5.6). Assume that $x(0) = \Delta$ when $t = 0$, and the impact impulse $J = J_0 \neq 0$. With provision for (6.11), the first averaged equation of (6.10) takes the form

$$J = \varepsilon E_1(J) = -\varepsilon \left\{ rT^{-1}(J)J + Jb \frac{1 - [T(J)\Omega]^{-1} \sin[T(J)\Omega]}{\sin^2[\frac{1}{2}T(J)\Omega]} \right\} \quad (6.13)$$

where the function $T(J)$ follows from the equation $J = -2\Omega\Delta \tan(\frac{1}{2}\Omega T)$.

The second equation of (6.10) is required only for the first approximation in the isochronous case (see Sect. 6.2). It may be shown that even for similar damping forces the solutions to Eq. (6.13) may differ qualitatively depending on the sort of anisochronism which is represented in the generative ($\varepsilon = 0$) system (6.12).

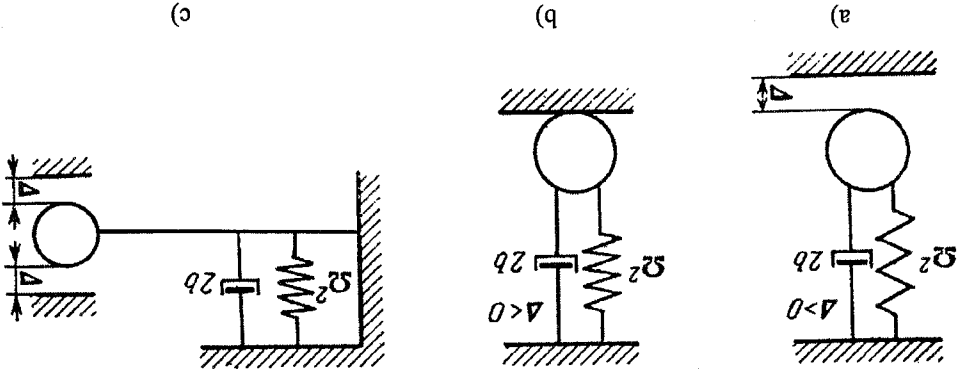


Fig. 6.2

Assume that in the generative system in Fig. 6.2a the strong anisochronism ($\Delta > 0$) takes place. The cases $b = 0$, $R < 1$ and $R = 1$, $b \neq 0$ are considered here in succession.

When $b = 0$ and $R < 1$, Eq. (6.13) takes the form

$$J = -\frac{\varepsilon J \Omega}{2[\pi - \text{atan}(J/(2\Omega\Delta))]} \quad (6.14)$$

where it is considered that $\Omega < \omega < 2\Omega$ for $\Delta > 0$. The above equation may be integrated using the fast converging series.

The calculations may be simplified. The trivial solution of (6.13) is $J^0 = 0$. This solution reflects the possibility of the existence of steady-state undamped harmonic vibration with the frequency Ω . The solution $J^0 = 0$ is asymptotically stable since for small deviations of ξ_0 from the stationary value, the equation in variations takes the form

$$\dot{\xi} = -\frac{1}{2}\varepsilon\Omega\xi$$

and yields the solution

$$\xi = \xi_0 \exp(-\frac{1}{2}\varepsilon\Omega t)$$

Therefore, the initial perturbations are decaying exponentially with the rate $\gamma = \frac{1}{2}\varepsilon\Omega$.

Let $R = 1$ ($r = 0$), but $b \neq 0$. Then

$$J = -\varepsilon J b \left(1 + \frac{4\Omega^2 \Delta^2}{J^2} \right) + \varepsilon J b \left(1 + \frac{J^2}{4\Omega^2 \Delta^2} \right) \sin \left[T(J)\Omega \right] \quad (6.15)$$

The above equation can not be integrated in the elementary functions and does not yield the stationary solution because its right-hand side is negative. For small initial

values of the impulse J_0 the dominating term may be determined because

$$T = O(2\pi\Omega^{-1}) \text{ and } J \text{ is small. Therefore, we can approximately write:}$$

$$J = -\frac{J}{4\varepsilon\Omega^2\Delta^2b}$$

For the "small initial conditions", $J_0 = O(\sqrt{\varepsilon})$, we find

$$J(t) = \sqrt{J_0^2 - 8\varepsilon\Omega^2\Delta^2b}t$$

The condition $J \geq 0$ yields the time of vibro-impact process completion,

$$t = t_* = \frac{J_0^2}{8\varepsilon\Omega^2\Delta^2b} \quad (6.16)$$

The phenomenon of the appearance of large number of impacts within the finite time interval is known as *chatter* (compare with Sect. 5.7). In this particular case the chatter time is defined by formula (6.16), and the number of collisions in the strongly anisochronous system is finite.

The general case $b \neq 0$, $R \neq 1$ may be analysed similarly and the time of chatter existence may be defined approximately using a formula which is similar to (6.16).

Thus, the dying out of the vibro-impact regime in the system, which is strongly anisochronous when $\varepsilon = 0$, is governed by the viscous damping.

For $\Delta < 0$ the generative system in Fig. 6.2b will be softly anisochronous and the situation will be completely different. Again, let $b = 0$. With provision for

$2\Omega < \omega < \infty$ we find

$$J = \frac{1}{2}\varepsilon\Omega J [\text{atan}(J/(2\Omega\Delta))]^{-1}$$

and the steady-state process is impossible. Since the value of J_0 is considered to be small, the above equation may be linearised to

$$J = \varepsilon\Omega^2\Delta = -\varepsilon\Omega^2|\Delta|$$

The solution takes the form

$$J(t) = J_0 - \varepsilon\Omega^2|\Delta|t, \quad J_0 = O(\varepsilon)$$

and the chatter time may be approximated by the formula

$$t_* = \frac{\varepsilon\Omega^2|\Delta|}{J_0} \quad (6.17)$$

As distinguished from the case of clearance, $\lim_{J \rightarrow 0} T(J) = 0$. Hence, an infinite number of collisions will take place within the finite time defined by (6.17).

Recall that in the case of clearance $\Delta > 0$, the steady-state regime $J^0 = 0$ exists only if $R < 1$ and $b = 0$. Similar to this, in the case of preload $\Delta < 0$ such a regime may exist only if $R = 1$ and $b \neq 0$. It may be easily verified from Eq. (6.15) by considering $(\sin \nu)/\nu \sim 1$ when $\nu \sim 0$. The decrement of vibration here is

In the first case of clearance we have finite chatter the duration of which is defined by formula (6.16). In the second case of preload with $\Delta < 0$, the infinite chatter takes place, after completion it transforms into the static process $x = \Delta$. The duration of the chatter may be assessed using formula (6.17). In the case of $\Delta = 0$, the exponential decay of the process takes place.

For all three cases the envelopes of the vibro-impact regimes correspond to the obtained dependencies $J(t)$ and the processes are defined by the formula

$$x(t) = -J(t)\chi[t, \omega(t)]$$

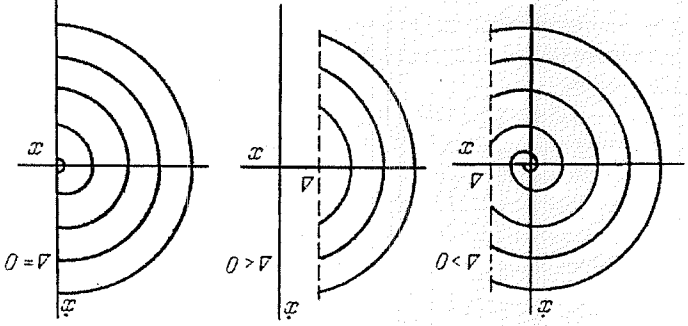


Fig. 6.4

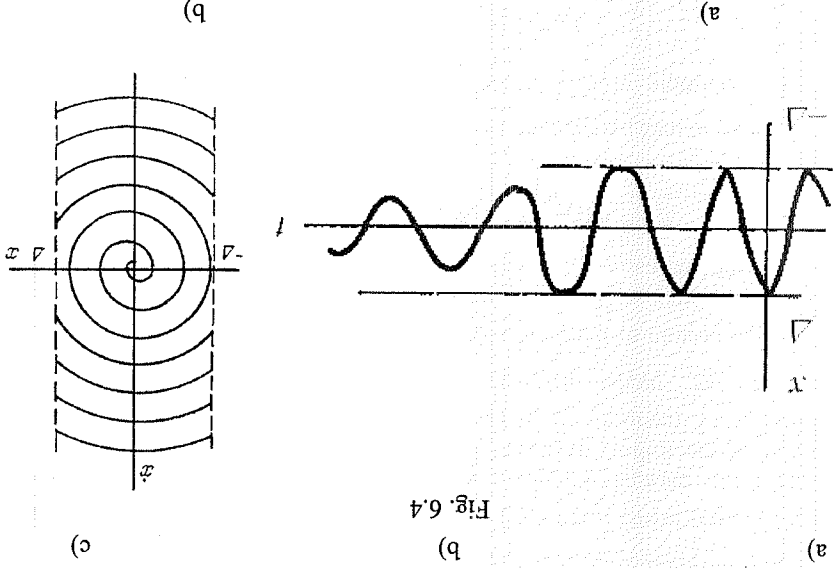


Fig. 6.5

In Fig. 6.4 the phase portraits of the considered systems are represented. Figure 6.5 corresponds to the symmetric case.

approximated as $\frac{2}{3}b\varepsilon$.

In the general case, when $\Delta < 0$, $R \neq 1$ and $b \neq 0$, infinite chatter exists in the system. Therefore, if the generative system is softly anisochronous, the decay of the vibro-impact regime is governed by the energy losses at the impacts.

If the generative system is isochronous, the period of vibration is $T = \pi\Omega^{-1}$ and Eqs. (6.10) are of the simplest structure,

$$J = -\varepsilon(r\Omega\pi^{-1} + b)J, \quad \psi = 2\Omega$$

Thus,

$$J(t) = J_0 e^{\gamma t}, \quad \gamma = \varepsilon(r\Omega\pi^{-1} + b), \quad \psi(t) = 2\Omega t$$

The time to full decay is infinite. It is seen that the forces of viscous and impact damping have the same influence on the process.

The system with the symmetric obstacles in Fig. 6.2,b is strongly anisochronous when $\varepsilon = 0$. Therefore, the time of chatter is finite. An analytical study of such a system may be carried out similarly.

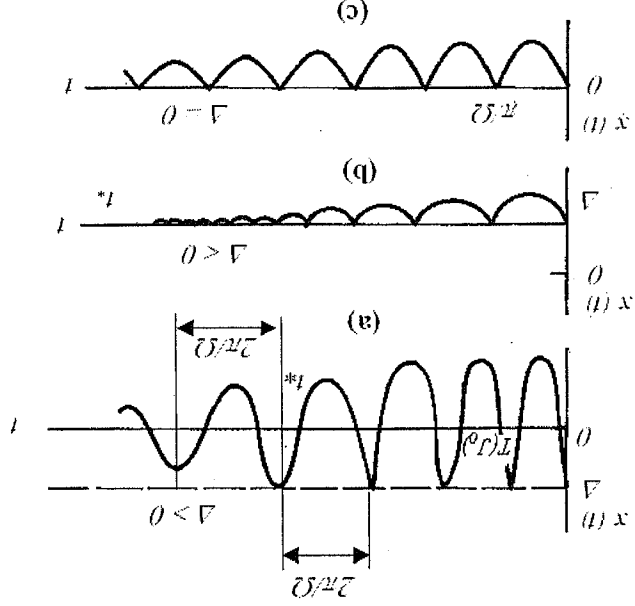


Fig. 6.3

In Fig. 6.3a-c the typical time histories $x(t)$ for $\Delta > 0$, $\Delta < 0$ and $\Delta = 0$ are shown.

The problems of chatter were studied in detail in references [42,43,122]. The results obtained here are in qualitative agreement with those obtained in the cited references. In conclusion, it may be noted that the appearance of the infinite-impact process of finite duration is a result of our idealisation which assumes the possibility of momentary impact. In reality, chatter is always of a finite-impact nature. This may be verified both experimentally and theoretically by considering a more realistic model of impact interaction. Such problems are not studied here.

6.4. Analysis of the Autoresonant System

Consider an example of the autoresonant system [14]

$$\ddot{x} + \Omega^2 x + \Phi_0(x, \dot{x}) = \varepsilon(\alpha - \beta x^2) \dot{x} \quad (6.18)$$

where $\alpha, \beta > 0$, Φ_0 is the nonlinearity which describes the impact, and, for simplicity, the restitution ratio is considered to be unity. Averaging the first equation of (6.10) yields

$$\dot{J} = \frac{\varepsilon J}{2 \sin^2(\frac{1}{2} \Omega T)} \left[\alpha \left(1 - \frac{\sin \Omega T}{\Omega T} \right) - \frac{\beta J^2 [1 - (\sin 2\Omega T)/(2\Omega T)]}{16 \Omega^2 \sin^2(\frac{1}{2} \Omega T)} \right] \quad (6.19)$$

where $J = -2\Omega \Delta \tan(\frac{1}{2} \Omega T)$, $\pi/\Omega < T < 2\pi/\Omega$ for $\Delta > 0$; $0 < T < \pi/\Omega$ for $\Delta < 0$.

The function $J(T)$ defines the period T^0 of steady-state self-sustained vibration, which corresponds to the solution J^0 of the transcendental equation obtained from

(6.19) assuming $J = 0$.

Equation (6.19) is nonintegrable in elementary functions. A transcendental equation in J^0 can not be solved in an explicit form. We are not focusing here on possible numerical solutions.

Some simplifying procedures which allow for approximate solutions follow.

Suppose, for example, $\Delta < 0$. Taking into account that $y^{-1} \sin y \approx 1 - \frac{1}{6} y^2$,

$\sin^{-2}(\frac{1}{2} \Omega T) = 1 + 4J^{-2} \Omega^2 \Delta^2$, Eq. (6.19) may be written in the form

$$\dot{J} = \frac{1}{3} \varepsilon J (\alpha - \frac{1}{4} \beta J^2 \Omega^{-2} - \beta \Delta^2)$$

It may be integrated in elementary functions. The equation in variations

$$\dot{\xi} = \frac{1}{3} \varepsilon \left[\alpha - \beta \Delta^2 - 3\beta (J^0/2\Omega)^2 \right] \xi$$

corresponds to the two steady-state solutions $J^0 = 0$ and $J^0 = 2\Omega(\alpha\beta^{-1} - \Delta^2)^{1/2}$,

$$\alpha\beta^{-1} > \Delta^2.$$

It follows from above that the regime with impacts is stable if $\alpha > \beta \Delta^2$. For $\alpha < \beta \Delta^2$ the impactless motion is stable, i.e. for $\Delta < 0$ self-sustained vibration does not exist at all. The undefined period of self-sustained vibration T^0 is given by the expression

$$T^0 = 2\Omega^{-1} \arctan [J^0/(2\Omega|\Delta|)]$$

If the stable periodic regime exists ($\alpha - \beta \Delta^2 > 0$), the general solution of the simplified equation may be obtained by inversion of the elementary function

$$\frac{3e^{-t}}{J^2 \beta} \ln \left| \frac{2(\alpha - \beta \Delta^2)}{4(\alpha - \beta \Delta^2) \Omega^2 - J^2 \beta} \right| + \text{const} = t$$

We will describe qualitatively the studied process. For the development of the regime with impacts an initial "wiggling" is required, i.e. a certain amount of energy must be supplied to the system to overpower the force of preload $\Omega^2 |\Delta|$ (*hard start*). If the condition $\alpha < \beta \Delta^2$ holds, then the viscous force damps the vibration. After exponential decay (chatter) the system transforms to the static state $x = \Delta$. The introduction of the impact damping results in an appearance of an infinite-impact process of finite duration (compare with the previous section). The stable self-sustained vibration arises if the condition $\alpha > \beta \Delta^2$ holds. The rate of the build-up of self-sustained vibration is governed by the decrement $\gamma = \frac{2}{3} \varepsilon (\alpha - \beta \Delta^2)$.

The existence condition of the stable vibration is independent of the natural frequency Ω . This is because of the hard start, after which the preload force ceases to affect the motion of the system. In the case $\Delta > 0$, self-exciting vibro-impact processes may be analysed similarly with respect to the strong anisochronism of the generative system. Note that, similar to the case of resonant forced vibration, the self-sustaining vibration exists at relatively low levels of damping and excitation, when the resulting vibration is close to the free vibrations. In another case the developing vibration may be of a qualitatively different character.

6.5. Quasi-isochronous Approximation

It was shown in the previous section that in a common case the transcendental equation for the undefined stationary value of an impulse turns out to be very complicated. It can not be solved in an explicit form and, as a result, it can not be analysed analytically. In this context, it is reasonable to distinguish the class of systems allowing simple and explicit solutions to the above equation.

The nonlinear conservative system is said to be *quasi-isochronous* if the potential force $\Phi_\Delta(x)$ depends on the parameter $\Delta = \varepsilon\Delta_1$ in such a way that as $\varepsilon \rightarrow 0$ the system becomes isochronous.

In particular, a vibro-impact system turns out to be quasi-isochronous if the clearance (preload) is defined as $\Delta = \varepsilon\Delta_1$, where $\Delta_1 = O(1)$.

The above quasi-isochronous systems may be used as the generative systems for simplification of the calculations. The necessary technique is demonstrated below.

The equation of motion for all systems considered above takes the form

$$\ddot{x} + \Omega^2 x + \Phi_0(x, \dot{x}) = \varepsilon g(x, \dot{x}, t)$$

where $\Phi_0(x, \dot{x}) \equiv \Phi_0(x - \Delta, \dot{x})$ (see §4). The substitution $x = y + \varepsilon\Delta_1$ is feasible for a quasi-isochronous system. The above equation in the coordinate y may be rewritten in the form

$$\ddot{y} + \Omega^2 y + \Phi_0^{(0)}(y, \dot{y}) = \varepsilon \left[g(x, \dot{x}, t) - \Omega^2 \Delta_1 \right]$$

where the superscript index of the function $\Phi_0^{(0)}$ emphasises the isochronism ($T = \pi\Omega^{-1}$) of the generative system possessing the solution

$$x(t) = -J\chi(t - t_0) = -\frac{J}{2\Omega} \sin[\Omega(t - t_0)]$$

where the latter equality is correct only if $0 \leq t - t_0 < \pi\Omega^{-1}$, $\chi(\psi) = (2\Omega)^{-1} \sin(\frac{1}{2}\psi)$ for $0 \leq \psi < 2\pi$ (see (5.21)).

We will consider the problem mentioned in the previous section using the anisochronous approach. The equation for the impulse

$$J = \frac{1}{2} \varepsilon J \left(\alpha - \frac{\beta J^2}{16\Omega^2} \right)$$

follows from (6.19) (the mean value of $\Omega^2 \Delta_1 \chi_\psi$ is zero). The second equation (6.10)

$$\psi = 2\Omega + 8J^{-1} \varepsilon J^2 \left(\alpha - \beta J^2 \chi_\psi^2 \right) \left(-2\Omega J \chi_\psi - \Omega^2 \Delta_1 \chi \right)$$

has to be considered in the isochronous case. Averaging over the period 2π yields

$$\psi = 2\Omega \left(1 - 2J^{-1} \varepsilon \Delta_1 \Omega \pi^{-1} \right)$$

The equation for the impulse is integrable in elementary functions and an unknown phase may be defined using the single quadrature.

Now we will define the stationary points J^0 . Equating the right-hand side of the equation for impulse to zero yields $J^0 = 0$, $J^0 = 4\Omega\sqrt{\alpha/\beta}$. The equation in

variations

$$\dot{\xi} = \frac{1}{2} \varepsilon \left[\alpha - \frac{3\beta}{16\Omega^2} (J^0)^2 \right] \xi$$

yields the instability of the impaceless solution J^0 for all $\alpha > 0$, and the stability of the regime with impulses J^0 (the decrement of vibration $\gamma = \varepsilon\alpha$).

The phase of vibration is $\psi^0 = 2\Omega t \left[1 - \frac{1}{2} \varepsilon \sqrt{\beta/\alpha} \Delta_1 \pi^{-1} \right]$. Therefore, the frequency of vibration is $\omega^0 = 2\Omega \left[1 - \frac{1}{2} \varepsilon \sqrt{\beta/\alpha} \Delta_1 \pi^{-1} \right]$, and as it was expected for $\Delta_1 > 0$, the

autoresonant vibration takes place in the frequency band located to the left of the frequency $\omega = 2\Omega$. For $\Delta_1 < 0$ it takes place in the frequency band located to the right of the above frequency. Notice that in the first approximation the condition of the system operation $\alpha > \beta\Delta^2$ for $\Delta < 0$ (see Sect. 6.4) is not revealed. It may be explained as follows. Under the assumption $\Delta = \varepsilon\Delta_1$, the above condition contains a term $\propto \varepsilon^2$. The next approximation may reveal such tiny effects.

Notice that when $J^0 = 0$ the equation for the phase is senseless. This reflects the fact that in the isochronous approximation the typical frequencies of the process are arranged close to 2Ω , i.e. in the first approximation it is feasible to consider the autoresonant processes only.

It is worth noting that for the problem of damped free vibration (Sect. 6.3) the shortened equations in the quasi-isochronous approximation take the form

$$\begin{aligned} \dot{J} &= -\varepsilon \left(\gamma \pi^{-1} \Omega + b \right) J \\ \psi &= 2\Omega \left(1 - 2\varepsilon \Delta_1 J^{-1} \pi^{-1} \right) \end{aligned}$$

These equations may lead to qualitatively wrong results. It can be verified easily by constructing their solutions and comparing these with the solutions of Sect. 6.3. The aforementioned equations do not describe the chattering for $\Delta \leq 0$ due to the nonresonant nature of such regimes. For example, using these equations it is impossible to make a conclusion about the existence of

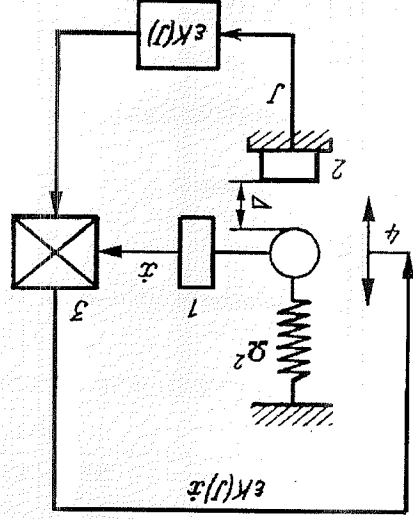


Fig. 6.6

infinite vibro-impact processes of finite duration.

Now, another autoresonant system will be analysed. Consider the electromechanical device with the feedback in Fig. 6.6.

The linear oscillator is involved in a vibratory motion which is accompanied by collisions against the stationary obstacle 2, where the sensor of impact impulses is mounted (the output signal of the sensor is proportional to the value of J). The above signal is then transformed into the function $\varepsilon K(J)$ which is multiplied by the signal $\dot{x}$ produced by the velocity sensor 1. The obtained product is then fed to the input of the vibration exciter 4.

Let the losses of the energy at impact be accounted for using the restitution ratio R such that $1 - R = \varepsilon r$. Assume also that all converters and transducers are ideal elements with zero inertia. Under the above assumptions the equation of motion takes the form

$$\ddot{x} + \Omega^2 x + \Phi_0(x - \Delta, x) = -\varepsilon [\Phi_0(x, x) + K(J)\dot{x} + 2bx]$$

For definiteness, let $K(J) = K_0 J^{-1}$, $K_0 > 0$. The solution procedure of the above problem in the quasi-isochronous approximation $\Delta = \varepsilon \Delta_1 > 0$ follows.

The averaged equations are

$$\dot{J} = -\varepsilon \left[r \pi^{-1} \Omega + b \right] J - \frac{1}{2} K_0, \quad \dot{\psi} = 2\Omega \left(1 - 2\varepsilon \Delta_1 J^{-1} \pi^{-1} \right)$$

There exists the single stationary regime $J^0 = K_0 \left[2 \left(r \pi^{-1} \Omega + b \right) \right]^{-1}$. The general solution of the first of the above equations takes the form

$$J = J^0 \left\{ 1 - \exp \left[-\varepsilon \left(r \pi^{-1} \Omega + b \right) t \right] \right\} + J(0) \exp \left[-\varepsilon \left(r \pi^{-1} \Omega + b \right) t \right]$$

This indicates the self-excitation of the system and the stability of the stationary solution with the decrement $\gamma = \varepsilon \left(r \pi^{-1} \Omega + b \right)$. The frequency of the autoresonant process is

$$\omega_0 = 2\Omega \left[1 - \frac{4\varepsilon \Delta_1 \left(r \pi^{-1} \Omega + b \right) \Omega}{\pi K_0} \right]$$

Note that the above system differs from the Van der Pol system. In this system an increase in the natural frequency is reflected in a decrease in the stationary value of the impulse. A reciprocal situation occurs in the Van der Pol system. The decrement γ of this system is defined by the damping $\left(r \pi^{-1} \Omega + b \right)$. In the Van der Pol system the level of excitation α defines the decrement. At the same time, the two considered schemes possesses many similar features.

When $J = 0$ the system degenerates into the impactless linear system. The second equation, therefore, is senseless.

6.6. Symmetric Systems

method of Van der Pol.

The method of excitation analysed above allows for a wide arbitrariness in the choice of the functions $K(J)$. Therefore, in similar schemes other values related to the total energy of the system E may be used as observers. Note that the equations of the quasi-isochronous approximation are not much more complicated as compared with the shortened equations obtained using the classical method of Van der Pol.

Consider the autoresonant system with double-side symmetric stops. The symmetric nonlinearity of impact type Φ_0 is represented in Eq. (6.18) by expression (4.20): $\Phi_0 \rightarrow \Phi_0^*$.

The generative solution (4.30) takes the form

$$x_0 = -J \chi^*(t - \varphi)$$

(6.20)

where $\chi^*(t + T/2) = -\chi^*(t)$ is the symmetric PGF

$$\chi^*(t) = \frac{2\Omega}{1} \frac{\cos(\Omega T/4)}{\sin[\Omega(t - T/4)]}, \quad 0 \leq t < \frac{T}{2}$$

The impact impulse J is related to the period of motion T as $J = 2\Omega \Delta \cot(\Omega T/4)$, $0 < T < 2\pi/\Omega$

The transformation of variables

$$x = -J \chi^*(\psi), \quad \dot{x} = -J \omega(J) \chi^*(\psi)$$

(6.21)

yields equations in the fast phase which are similar to (6.10); the substitution $\chi \rightarrow \chi^*$ is required everywhere. By virtue of symmetry, the averaging may be carried out over half of the period of motion. As a result, the first shortened equation for system (6.18) with symmetric nonlinearity is obtained in the form

$$\dot{J} = \frac{\varepsilon J}{1 + \frac{2 \sin(\Omega T/2)}{\Omega T}} \left\{ \alpha \left[1 + \frac{2 \cos^2(\Omega T/4)}{8 \Omega^2 \cos^2(\Omega T/4)} \right] - \beta J^2 \left[1 - \frac{\sin(\Omega T)}{\Omega T} \right] \right\}$$

An analysis of the above equation is similar to that for the system with asymmetric nonlinearity; the single limit cycle is stable. An analysis of some models necessitates consideration of the smallness of the clearance $\Delta \propto \varepsilon$. In this case the potential force $\Omega^2 x$ has little effect on the motion. In the first approximation this force may be neglected in the equation of motion:

$$\ddot{x} + \Phi_0^*(x, x) = \varepsilon g(x, x) - \varepsilon \Phi_2^*(x, x)$$

where Φ_0^* and Φ_2^* are the elastic and dissipative components of a symmetric force of

impact.

As $\Omega \rightarrow 0$, solution (6.20) yields the generative solution

$$x_{01} = -J\chi_1^*(t - \phi), \quad \chi_1^*(t) = \frac{1}{2}t - \frac{1}{8}T, \quad 0 \leq t < T/2$$

and the impulse-to-period dependence is given by the equation

$$J = 8\Delta/T$$

which follows from the condition $x(\phi) = \Delta$.

Consideration of the Van-der-Pol excitation $(\alpha - \beta x^2)x$ is senseless due to the smallness of the deflections $|x(t)| \leq \Delta = O(\varepsilon)$. Hence, the self-sustaining term in the form of "cubic friction" is considered here,

$$g(x, \dot{x}) = \alpha\dot{x} - \beta x^3, \quad \alpha, \beta > 0$$

The shortened equation for the impulse in the case of the elastic impact takes the form

$$J = -\varepsilon \left(\frac{1}{4}\beta J^3 - \alpha J \right)$$

The equation in variations

$$\dot{\xi} = -\varepsilon \left[\frac{3}{4}\beta(J_0^2 - \alpha^2) - \alpha \right] \xi$$

corresponds to stationary values of the impulse

$$J_0^1 = 0, \quad J_0^2 = 2\sqrt{\alpha/\beta}$$

It follows from above that the impactless regime is unstable with the increment

$$\gamma_1 = \varepsilon\alpha \text{ and the vibro-impact regime is stable with the decrement } \gamma = 2\varepsilon\alpha.$$

We notice that in this case the shortened equation for the impulse allows an explicit solution. To verify this, the equation of motion will be considered along with the impact conditions,

$$\ddot{x} - \alpha\dot{x} + \beta x^3 = 0, \quad \dot{x}_+ = -\dot{x}_-, \quad |x(t)| \leq \Delta$$

The substitution $y = \dot{x}$ reduces the order of the equation, hence

$$\dot{y} - \alpha y + \beta y^3 = 0, \quad y_+ = -y_-$$

The latter equation yields three stationary solutions,

$$y_0 = 0, \quad y_- = -\sqrt{\alpha\beta^{-1}} \text{ and } y_+ = \sqrt{\alpha\beta^{-1}}$$

If the impact occurrence is ignored, then the above equation defines the straight-line motion of a depicting point under the force $\alpha y - \beta y^3$. The value y_0 corresponds to the unstable equilibrium position, $y_{\pm}$ corresponds to the uniform motion to the right or to the left of this position. The conditions of impact combine solutions y_- and y_+ . These two stationary solutions yield one solution, namely $y(t) = y_-$ (motion toward the left-hand side obstacle) and $y(t) = y_+$ (motion toward the right-hand side obstacle). An integration of the dependence $y(t)$ yields the law of motion represented

by a saw-tooth function which is expressed in terms of the PGF $\chi_1^*(t)$.

By the definition of the impulse, $J = 2|\gamma_-|$, we find the same stationary values as

obtained using the averaging method. The considered problem in a stochastic approach is analysed below in Sect. 8.3.

If the "cubic friction" term is represented in the form $g(x, \dot{x}) = \beta x^3 - \alpha\dot{x}$, $\alpha, \beta > 0$, the equation for the impulse is modified to

$$J = -\varepsilon \left(\alpha J - \frac{1}{4}\beta J^3 \right)$$

The stationary solutions remain the same, but the impactless regime is stable. The obtained results are in full agreement with the results of the analysis of corresponding impactless models in [98, 186].

The influence of the energy dissipation at the impacts on the self-sustained vibration may be accounted for as follows. The shortened equations may be thought of as equations of balance of work done by nonconservative forces on the generative

solution $x_{01}(t)$. The work done by the force $g(x, \dot{x})$ is calculated in motion through the small displacement $|x| \leq \Delta$, $\Delta = \varepsilon\Delta_1$, $\Delta_1 \ll 1$. The work done by the momentary force $\Phi_2^*(x, \dot{x})$ is independent of displacement. In order to account for the actions of the momentary and nonmomentary forces in the first approximation suppose that

$$1 - R = \varepsilon^2 r. \text{ If an excitation is represented by the "negative friction"}$$

$$g(x, \dot{x}) = \alpha\dot{x}, \quad \alpha > 0$$

we find that

$$J = -\varepsilon \left(\frac{rJ^2}{4\Delta_1} - \alpha J \right), \quad \varepsilon\Delta_1 = \Delta$$

where we took into account that $T = 8\Delta_1 r^{-1}$.

For the stationary process we have $J_0^1 = 0$ and $J_2 = \frac{4\alpha\Delta_1}{r}$. The first stationary solution is unstable, and the second is stable with the decrement $\gamma = \alpha$.

Calculations in the combined case of impact and cubic friction are similar.

The influence of the potential force $\Omega^2 x$ may be accounted for in the calculation of the higher-order approximations. Introducing this force into the equation of the first approximation does not change the results significantly.

When $\Delta = 0$ the motion is impossible and the symmetric system is degenerating. Hence, the system with small clearances is much more sensitive to fluctuations. Such sensitivity is a typical feature of many dynamic systems. While approaching the degenerating configuration, systems lose the structural stability which forms a

protection against the fluctuations.

In particular, if the level of fluctuation is comparable with the value of the clearance, few consequent impacts against the same obstacle are possible in the considered

models. It can not be accounted for in the framework of the accepted assumptions. Transform (6.21) assumes consequent collisions with each of the obstacles. It turns out to be reasonable to consider the problems with small clearances in a *stochastic* approach (see §8).

§7 Parametric Vibration

7.1 Preliminary Considerations

Numerous problems of mechanics may be reduced to the integration of equations of motion which contain parametric forces like $c(t)x(t)$, $c(t)x(t)$, etc., where $c(t)$ is a given function of time.

From §3, it may be seen that the theory of parametric systems is much more complicated than the theory of stationary systems with constant parameters. This is due to the variety of physical effects which may be observed in the systems with time-variable parameters. An analysis of the parametric phenomena in the strongly nonlinear systems which are described by the equation

$$\ddot{x} + \Omega^2 x + \lambda \Phi(x) = \varepsilon g(x, \dot{x}, t)$$

may be carried out using equations (5.7) given by the averaging method. It is feasible to take the generative solutions in the form of an improved singular approximation (Sect. 4.9).

In this section another method of analysis is based on the closeness of strongly nonlinear systems to vibro-impact systems. The extreme case, $\lambda \rightarrow \infty$, which corresponds to the singular force of momentary impact, attracts special attention. Such an approach gives significant computational simplification. Moreover, the above method may be generalised easily to systems with multiple-degrees-of-freedom. The method is based on the application of the principle of energy balance (see Sect. 5.10) and approximation of motion using parametric periodic Green functions (PPGF) (see §3).

The regimes studied below are periodic. The procedure for the analysis of transient processes is based on the averaging method. For the following analysis we consider that, in general, the threshold $\Delta \neq 0$ is not small. Therefore, the systems typically possess at least two periodic resonant regimes. Note that the systems, which are close to the isochronous case, typically possess a single resonant regime. This may be analysed using the procedures of Sect. 6.5 and §5. It is also worth noting also that explicit solutions may be obtained, in principle, without the application of such assumptions. The interested reader may find a systematic interpretation of the modern theory of

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parametric vibration in the monographs by Bolotin [25] and Schmidt [155], containing solutions of numerous problems, along with the useful literature survey (see also reference [186]). General mathematical problems related to the considered systems are analysed in monographs [35, 193]. Also, similar topics are mentioned in [3, 32, 191]. The material of the present paragraph mainly corresponds to references [16, 91, 92].

7.2 Resonant Regimes outside the Zones of Instability of

Consider the Hill equation with strong threshold nonlinearity

$$\ddot{x} + \Omega^2 x - \varepsilon \mu c(t)x = \varepsilon g_0(x, \dot{x}) - \lambda \Phi(x) \quad (7.1)$$

Here the parametric excitation $\mu c(t)$ is supposed to be T -periodic,

$$c(t) = \sum_{k=1}^{\infty} c_k \cos(k\omega t + \varphi_k)$$

where μ is the level of modulation and $g_0(x, \dot{x})$ is the damping force. Without loss of generality, we consider that $\Omega = 1$ for computational convenience. When $\varepsilon = 0$, system (7.1) is conservative. The solution for large impact impulses is given by the approximate formula

$$x_0(t) = -J\chi(t - \tau_\lambda)$$

where $\tau_\lambda = \frac{t_\lambda}{2}$, t_λ is the interaction time (see Sect. 4.9). The transcendental equation

$$\Delta = -J\chi(\tau_\lambda), \quad \tau_\lambda = \tau_\lambda(J)$$

defines the magnitude of the impact impulse as a function of the vibration frequency of a conservative system. Since $\Omega = 1$,

$$\chi(t) = \frac{\cos\left(t - \frac{\pi}{\omega_0}\right)}{\cos\left(\frac{\omega_0}{2\pi}\right)}, \quad 0 \leq t < \frac{2\pi}{\omega_0} \equiv T_0$$

For the perturbed system we find that

$$x \approx -J^0 \chi(t - \tau_\lambda; \omega) \quad (7.2)$$

where it is considered that the frequency of the parametric excitation belongs to the interval of natural frequencies of the conservative system and the impulse of the interaction $J^0 = J(\omega)$ is taken at the frequency $\omega_0 = \omega$. The existence condition of the above processes is found from the equation of energy balance (5.47) of work done by the nonconservative forces in motion (7.2)

$$E(J^0, \phi) = -\varepsilon \left[\int_T^0 \mu x(t) c(t + \phi) \dot{x}(t) dt + \int_T^0 g_0(x, x) \dot{x}(t) dt \right] = 0$$

where the initial instant of the interaction coincides with the time-zero reference. The

undefined phase ϕ is related to the parametric excitation.

The first integral from the energy balance equation

$$E_1(\phi) \equiv \mu \int_T^0 (J^0)^2 \chi(t - \tau_\lambda) \chi(t - \tau_\lambda) c(t + \phi) dt = \mu \int_T^0 (J^0)^2 \chi(t - \tau_\lambda) \chi(t - \tau_\lambda) c(t + \phi + \tau_\lambda) dt$$

is calculated, after substitution of $c(t)$, as

$$E_1(\phi) = \frac{1}{2} (J^0)^2 \mu \sum_{k=1}^{\infty} \frac{c_k k \omega}{4 - k^2 \omega^2} \cot \frac{\omega}{\pi} \sin[\phi_k + k\omega(\phi + \tau_\lambda)] \quad (7.3)$$

Since the function $g_0(x, x)$ is specified, the second integral E_2 may be calculated (see

§5).

Hence, the energy balance equation $E_1(\phi) + E_2 = 0$ yields unknown values for the

phase and existence conditions of regimes (7.2) which may be reduced to the condition

of existence of at least one root of the above equation on the interval $\phi \in [0, T]$,

$$T = 2\pi\omega^{-1}.$$

If, on the above interval, the function $E_1(\phi)$ possesses few extreme points, the

appearance of the few pairs of resonant regimes becomes possible (bifurcation of

solution, see Sect. 5.12).

Consider the extreme case of momentary interaction, $\tau_\lambda = 0$. The dissipation of energy

at the impact is defined using the restitution ratio R , and the energy dissipation due

to the motion is accounted for by a viscous friction. Hence,

$$E_2 = - \left[(J^0)^2 \frac{1-R}{2(1+R)} + \frac{1}{4} b (J^0)^2 \frac{\sin^2(T/2)}{T - \sin T} \right], \quad T = \frac{2\pi}{\omega}$$

and the energy balance equation after rearrangements takes the form

$$\frac{1}{2} b \frac{\sin^2(T/2)}{T - \sin T} + \frac{1+R}{1-R} = \mu \cot \frac{2}{T} \sum_{k=1}^{\infty} \frac{c_k k \omega}{4 - k^2 \omega^2} \sin(\phi_k + k\omega\phi) \quad (7.4)$$

The obtained equation is similar to that studied in §5. If the excitation is represented by a single-frequency component the existence condition of the resonant regime may

¹ Remember that the interacting body has unit mass, $m = 1$, $1 - R = O(\mu)$, $b = O(\mu)$.

be obtained immediately: $|\sin \phi| \leq 1$. The more complicated situation in the case of two or more harmonics was discussed in Sect. 5.12. If the excitation frequency is $\omega = 2$, the periodic regime of the same frequency has infinite amplitude because $-2\Delta \tan \pi\omega^{-1} = J^0 \rightarrow \infty$ as $\omega \rightarrow 2$ (for systems with large but finite λ , the frequency of the infinite amplitudes is close to 2). From (7.4), it is seen that nonzero energy delivery at this specific frequency is possible only if

$$4 - k^2 \omega^2 = 0 \quad (7.5)$$

For $k > 1$ the frequencies $\omega = \frac{k}{2}$ are outside the interval of the natural frequencies of the conservative system. Hence, in this case the existence of the resonant regimes may be assessed using the analysis of the power supply, which is required for sustaining the infinite amplitudes at the frequency $\omega = 2$ only. We proceed in (7.4) to the limit

$\omega \rightarrow 2$ and consider that $|\sin \phi| \leq 1$. The existence condition of the resonant process

$$|\mu c_k| \geq 4 \left[b + \frac{\pi(1+R)}{2(1-R)} \right] \quad (7.6)$$

and the resonant process may exist due to the action of the first harmonics of

parametric excitation only.

Consider now the symmetric system. In equation of motion (7.1) we consider that

$$\Phi(x) = -\Phi(-x) = \Phi^*(x).$$

Hence, $x \approx -J\chi^*(t - \tau_\lambda)$, where $\chi^*(t)$ is the symmetric PGF

$$\chi^*(t) = \frac{\sin[t - \pi/(2\omega_0)]}{2 \cos[\pi/(2\omega_0)]}, \quad 0 \leq t < \pi\omega_0^{-1}$$

The transcendental equation for the impulse $J(\omega_0)$ takes the form $\Delta = -J\chi^*(\tau_\lambda)$,

$$\tau_\lambda = \tau_\lambda(J). \text{ In particular, for the vibro-impact system}$$

$$J(\omega_0) = 2\Delta \cot[\pi/(2\omega_0)]$$

Here it is considered that the frequency of excitation ω belongs to the interval of

natural frequencies of the conservative system, hence $J = J^0 = J(\omega)$.

The expression for the work done by the forces of parametric excitation in this case is

$$E_1(\phi) = - (J^0)^2 \mu \sum_{k=1}^{\infty} \frac{c_k k \omega}{4 - k^2 \omega^2} \tan \frac{2\omega}{\pi} \sin[\phi_{2k} + 2k\omega(\phi + \tau_\lambda)] \quad (7.7)$$

i.e. energy is delivered through even harmonics only.

In the extreme case of momentary interaction, the infinite impulses are possible only if $\omega \rightarrow \infty$, $J \propto \cot[\pi/(2\omega)]$. From (7.7), for sustaining such processes it is necessary

that $\mu = \mu(\omega) = O(\omega^\alpha)$ for $\alpha > 1$. In the prelimit cases the frequencies of infinite

impulses are high but finite, so the above condition still holds. Note, that such a

situation may happen very often and for numerous systems the condition $\mu = O(\omega^2)$

is feasible (see Sect. 7.7 below)¹.

The considered resonant processes have simple physical meaning. The system is involved in such a motion, which is close to free undamped vibration, and the

parametric forcing just sustains this motion.

In the asymmetric system, therefore, the above resonant processes are sustained by the first harmonic of the parametric forcing, in the symmetric system they are sustained by the second.

From the structure of expression (7.4), considerable energy input may take place not only at the harmonics of motion $\omega = 2/k$ (see Sect. 7.5) but also at the frequencies

$\omega = 1/n$, when $\cot(\pi/\omega)$ is infinite (k, n are integer numbers). Thus, we can expect the appearance of intensive vibration with other (different from $\omega = 2$) frequencies.

The typical case² $\omega \sim 1$ will be studied in detail.

Multiply both parts of Eq. (7.4) by $(J^0)^2$ and proceed to the limit $\omega \rightarrow 1$. After

rearrangements we find

$$4b = -\mu c_2 \sin(2\omega\varphi + \varphi_2)$$

i.e. energy is supplied through the *second harmonic of excitation*. The existence condition of resonant processes at $\omega \approx 1$ takes the form

$$4b \leq |\mu c_2| \quad (7.8)$$

Note that $J^0 \rightarrow 0$ as $\omega \rightarrow 1$. The same condition and the same behaviour of the impact impulse may be obtained for the symmetric system.

Such a paradoxical conclusion may be corrected by accounting for higher approximations³.

The energy balance equations are constructed similarly and for subperiodic resonances with frequencies $\omega_1 = \omega l^{-1}$ (l is an integer).

¹ Speaking rigorously, for $\mu = O(\omega^\alpha)$, $\alpha > 1$ for large ω , the parametric term in the high-frequency span contains a large parameter. Therefore our calculations must be refined.

² Other cases correspond to low-frequency vibration, which involves multiple impacts and may be studied numerically only.

³ The behaviour of the nonlinear parametric system under linear parametric resonance may be characterised by high instability. This problem is discussed in Sect. 7.7.

7.3. Integral Equation of Parametric Vibration

It is assumed here that the reader is familiar with §3 of this monograph. Construct representation (3.16) for Eq. (7.1) using the formal substitution $g = g_0(x, \dot{x}) - \lambda\Phi(x)$. Hence, the equation

$$x(t) = \varepsilon \int_T^0 \kappa(t, s; \mu_0) g_0[x(s), \dot{x}(s)] ds - \lambda \int_T^0 \kappa(t, s; \mu_0) \Phi[x(s)] ds \quad (7.9)$$

defines the T -periodic regimes in the considered system. The PPGF is given by formula (3.17) or (3.18)¹; $\mu_0 = \varepsilon\mu$.

Let the system be undamped and $\lambda \rightarrow \infty$. This corresponds to the case of a vibro-impact system. The nonlinear conservative term may be represented as $J\delta^T(t - \varphi)$

(see (4.16)) and we assume that during the period of motion the single impact occurs in the unknown moment of time φ . Equation (7.9) by virtue of the properties of the δ -function yields the dual-parametric representation for the desired periodic solution

(compare with Sect. 5.7)

$$x(t) = -J\kappa(t, \varphi; \mu_0) \quad (7.10)$$

where, as usual, $x(\varphi) = \Delta$, $\dot{x}_-(\varphi) = -\dot{x}_+(\varphi)$ (elastic impact), $J \geq 0$, $x(t) \leq \Delta$.

The representation for the impulse takes the form

$$J = -\frac{\kappa(\varphi, \varphi; \mu_0)}{\Delta} \quad (7.11)$$

The relationship $\dot{x}_- = -\dot{x}_+$ serves for the calculation of the phase φ ; it is equivalent to the condition of zero energy input of the parametric source into the system.

For the calculation of $\kappa(t, s; \mu_0)$, representation (3.18) is used where the decomposition is truncated by the terms $O(\mu_0) = O(\varepsilon)$. For (3.18) we find that

$$\kappa(t, s; \mu) = \frac{\chi(t-s) - \mu_0[\chi(t-s)B_1 - \kappa_2(t, s)] + \dots}{1 - \mu_0B_1 + \dots}$$

where, from (3.12),

$$B_1 = \int_T^0 \chi(0)c(t)dt$$

because in this case the kernel $K(t, s) = \chi(t-s)c(s)$. However, in the considered case,

¹ If the damping is "linear", then the function $g_0(x, \dot{x})$ may be accounted for in the "linear part" of the system. The corresponding PPGF may be constructed with respect to this term. However, this results in bulky expressions.

the parametric excitation $c(t)$ has zero mean value. Hence, $B_1 = 0$ and the above decomposition takes the form

$$\kappa(t, s; \mu_0) = \chi(t - s) + \mu_0 \kappa(t - s) + \dots$$

By definition

$$\kappa_2(t, s) = \int_0^T \chi(t - \tau) c(\tau) \chi(\tau - s) d\tau$$

for this particular problem $s = \varphi$. For calculation convenience it is reasonable to account for the phase of the excitation in the parametric term by taking the impacts to occur at $t = lT$, $l \in \mathbf{Z}$. Then the second repetitive kernel may be calculated as

$$\int_0^T \chi(t - \tau) c(\tau + \varphi) \chi(\tau) d\tau = \int_0^T \chi(t - \tau_1 + \varphi) c(\tau_1) \chi(\tau_1 - \varphi) d\tau_1 = \kappa_2(t + \varphi)$$

In agreement with the above, Eq. (7.10) takes the form

$$\dot{x}(t) = -J[\chi(t) + \mu_0 \kappa_2(t + \varphi, \varphi)] \quad (7.12)$$

The condition $x(0) = \Delta$ yields representation (7.11).

If the damping $g_0 \neq 0$ the approximate solution of the full Eq. (7.9) may be obtained based on representation (7.12), where it is considered that the phase of motion is defined from the energy balance principle as applied to the work done by the nonconservative forces on motion (7.12)

$$E_1 + \int_0^T g_0[x(t), \dot{x}(t)] dx(t) = 0$$

and the impact impulse is obtained in accordance with (7.11). For practical calculations the "shortened" equations of energy balance from Sect. 7.2 may be used Let $\lambda < \infty$ in Eq. (7.9). Then, similar to above, the improved singular approximation to the desired solution may be constructed on the basis of formula (7.2). As a result, the transcendental equation for the impulse of interaction appears to be too tedious for analysis. For the calculation convenience it is feasible to represent the solution in the form

$$x(t) = -J[\chi[t - \tau_\lambda(J)] + \mu_0 \dots]$$

where the value of the impulse J is taken from the limiting relationship (7.11)

(compare with Sect. 5.9).

Note that if $\Phi(x) = \Phi^*(x)$ and $c(t)$ is of period $\frac{T}{2}$, for the definition of symmetric T -periodic solutions under the assumption $g_0(-x, -\dot{x}) = -g_0(x, \dot{x})$ an equation

similar to (7.9) may be constructed on the half-period time interval

The symmetric PPGF may be obtained on the basis of the symmetric PGGF $\chi^*(t)$ using formulae similar to (3.17) or (3.18).

$$x(t) = \varepsilon \int_{T/2}^0 \kappa^*(t, s; \mu_0) g_0[x(s), \dot{x}(s)] ds - \lambda \int_{T/2}^0 \kappa^*(t, s; \mu_0) \Phi^*[x(s)] ds \quad (7.13)$$

7.4. Resonant Regimes in the Zone of Instability of Linear Systems

For the calculation of the repetitive kernel $\kappa_2(t, s)$ from (7.12) we have used the

formula

$$\kappa_2(t + \varphi, \varphi) = \int_0^T \chi(t - \tau) c(\tau + \varphi) \chi(\tau) d\tau$$

Here the PGGF $\chi(t - \tau)$ is written in accordance with expression (2.20) accounting for the relationship between t and τ . Therefore, we find

$$\begin{aligned} \kappa_2(t + \varphi, \varphi) &= \int_0^T \frac{4 \sin^2(T/2)}{\cos(\tau - T/2) \cos(t - \tau - T/2)} c(\tau + \varphi) d\tau \\ &+ \int_0^T \frac{4 \sin^2(T/2)}{\cos(\tau - T/2) \cos(t - \tau - T/2)} c(\tau + \varphi) d\tau \end{aligned} \quad (7.14)$$

Substitution of the series representing $c(\tau)$ and taking the quadratures yields, after

simple rearrangements,

$$\kappa_2(t + \varphi, \varphi)$$

$$\begin{aligned} &= \sum_{n=1}^{\infty} \frac{c_n}{2n\omega} \sin \frac{2n\omega \sin(T/2)}{2} \cos \frac{n\omega t}{2} + \varphi_n + n\omega\varphi \left(\sin \left(t - \frac{T}{2} \right) \right. \\ &\left. + \frac{c_n}{2} \cos \left(\frac{n\omega}{2} \right) \cos \left(\frac{n\omega}{2} + 1 \right) t - \frac{T}{2} \right) \left(\cos \left[\frac{n\omega}{2} + 1 \right] t - \frac{T}{2} \right) \left(\cos \left[\frac{n\omega}{2} - 1 \right] t + \frac{T}{2} \right) \end{aligned} \quad (7.15)$$

$$\left. \cos \left[\frac{n\omega}{2} - 1 \right] t + \frac{T}{2} \right) \left(\cos \left[\frac{n\omega}{2} - 1 \right] t + \frac{T}{2} \right)$$

where $0 \leq t < T$.

Impact impulse (7.11) is given by the expression

$$J = -\Delta \left[\chi(0) + \mu_0 \kappa_2(\varphi, \varphi) \right]^{-1} = -\frac{\Delta}{\chi(0)} \left[1 - 2\mu_0 \sum_{n=1}^{\infty} \frac{c_n \cos(\varphi_n + n\omega\varphi)}{n^2 \omega^2 - 4} \right]^{-1} \quad (7.16)$$

Let $\omega \approx 1$. From the above expression, only the single term corresponding to $n = 2$

contributes significantly to the expression in the square brackets. Therefore, when

$$J \approx -\frac{\Delta}{\omega} \left[1 - 2\mu_0 \sum_{n=1}^{\infty} \frac{c_2 \cos(\varphi_2 + 2n\omega\varphi)}{4\omega^2 - 4} \right]^{-1}, \quad 2\chi(0) = \cot \frac{T}{2} \quad (7.17)$$

It may be seen that $J \rightarrow \infty$ for $\omega = 1 + O(\mu_0)$. This may be explained by considering the significant energy input when $\omega = 1$ (see Sect. 7.2). If condition (7.8) holds,

of the parametric excitation may arise in the system. The solutions corresponding to formulae (7.17) are discussed in Sect. 7.7.

Such resonant regimes take place in the vicinity of *fundamental linear parametric resonance* (frequency of the motion equals unity, frequency of the excitation equals two). They are defined by the properties of the external source of energy.

In the asymmetric system the vibration outside the zone of instability of a linear system is sustained by the first harmonic of the external excitation. The vibration inside the above zone is sustained by the first harmonic (fundamental parametric resonance).

For symmetric systems described by Eq. (7.13), the representation

$$x(t) = -J \left[\chi^*(t) + \mu_0 \kappa^*(t + \varphi, \varphi) \right] \quad (7.18)$$

is defined by the kernel κ^* which can be calculated similarly to (7.14)

$$\kappa^*(t + \varphi, \varphi) = \int_0^t \frac{4 \cos^2(T/4) \sin(\tau - T/4) \sin(\tau - T/4)}{c(\tau + \varphi) d\tau}$$

As a result of calculations for $0 \leq t < T/2$

$$\kappa^*(t + \varphi, \varphi) =$$

$$= \sum_{n=1}^{\infty} \frac{c_{2n} \cos(n\omega t + \varphi_{2n} + 2n\omega\varphi)}{4 \cos(T/4)} \left\{ \frac{1}{(-1) \sin n\omega t \cos(t - T/4)} \right\} \quad (7.19)$$

$$+ \frac{1}{2(n\omega - 1)} \sin \left[(n\omega - 1)t + \frac{4}{T} \right] + \frac{1}{2(n\omega + 1)} \sin \left[(n\omega + 1)t - \frac{4}{T} \right]$$

The condition of impact $x(0) = \Delta$ yields, for the impact impulse,

$$J = -\frac{\Delta}{\omega} \left[1 - 2\mu_0 \sum_{n=1}^{\infty} \frac{c_{2n} \cos(\varphi_{2n} + 2n\omega\varphi)}{4n^2\omega^2 - 4} \right]^{-1}, \quad 2\chi^*(0) = -\tan \frac{T}{4} \quad (7.20)$$

i.e. significant growth in the value of the impulse takes place when $\omega \approx 1$ due to the action of the second harmonic.

7.5. Parametric Systems with Force Excitation

However, in the symmetric system the second harmonic of the parametric excitation defines periodic regimes existing close to linear parametric resonance as well as far from it. The calculation of the consequent approximations allows the description of the resonance regimes existing in the zones of the higher (lateral) linear parametric resonances.

Let the external force be represented by the term $sh(t)$ in equation of motion (7.1)

$$\ddot{x} + \Omega^2 x - \varepsilon \mu c(t)x = \varepsilon g_0(x, \dot{x}) - \lambda \Phi(x) + \varepsilon h(t) \quad (7.21)$$

Suppose that $h(t)$ has the same period as $c(t)$, therefore,

$$h(t) = \sum_{k=1}^{\infty} h_k \cos(k\omega t + \psi_k)$$

We will study briefly the qualitative changes in the behaviour of the system due to the action of an external force. The energy balance principle yields

$$\int_0^T \left\{ g_0[x(t), \dot{x}(t)] + \mu c(t)x(t) + h(t) \right\} \dot{x}(t) dt = 0$$

If we consider again that $x(t) = -J\chi(t - \tau_2)$, the equation of energy balance may be rewritten in the form

$$E_2 = 0 = \sum_{k=1}^{\infty} \left\{ \frac{1}{2} (J_0)^2 \mu \frac{c_k k \omega}{4 - k^2 \omega^2} \cot \frac{\omega}{\pi} \sin[\varphi_k + k\omega(\varphi + \tau_2)] - \frac{1 - k^2 \omega^2}{h_k k \omega} \sin[\psi_k + k\omega(\varphi + \tau_2)] \right\} \quad (7.22)$$

where the parametric and force excitations are thought of to be phase-locked. Hence, the undefined phase of the process φ is involved in both expressions for work. From the above equation, in particular frequency bands the force and parametric sources may be reflected in *domination* by one of them or in *mutual neutralisation*. The structure of Eq. (7.22) indicates also that the appearance of the additional force may result in additional bifurcation of solutions (see Sect. 5.12 and 5.13). The PPGF $\kappa(t, s, \mu_0)$, which is calculated in the previous section, defines (from

(3.16)) the motion of a linear system under the force $h(t)$,

$$f(t) = \int_T^0 \kappa(t, s; \mu_0) h(s) ds$$

An exact representation of the motion of a vibro-impact system may be obtained. In (7.21) let $\lambda \rightarrow \infty$ and the function $g(x, \dot{x})$ describe the energy dissipation at impact through the restitution ratio R .

The integral equation (containing the impact nonlinearity $\Phi_1(x, \dot{x})$) of T -periodic

regimes (see Sect. 5.6) takes the form

$$x(t) = f(t) - \int_T^0 \kappa(t, s; \mu_0) \Phi_1[x(s), \dot{x}(s)] ds$$

and yields, by virtue of the features of δ -functions, the solution¹

$$x(t) = f(t) - J\kappa(t, \varphi)$$

where $x(\varphi) = \Delta$, $J = \dot{x}_-(\varphi)(1+R)$, $\dot{x}_- > 0$, $x \leq \Delta$. The transcendental equations in

J and φ result from the above

$$\Delta = f(\varphi) - J\kappa(\varphi, \varphi), \quad J = (1+R) \left[f(\varphi) - J \frac{\partial \kappa(t, \varphi)}{\partial t} \right]_{t=\varphi=0}$$

If the function $f(t)$ is defined approximately by the single harmonic component, the

above system may be solved similarly to that in the stationary case (see (5.45)).

It is extremely rare to obtain an exact representation for the solution of a nonlinear

parametric problem. In Sect. 7.11 an example of such a solution is shown.

7.6. Energy Condition of Instability

The obtained solutions must be analysed with respect to stability².

In the framework of the simplified approach, the criterion of instability will be

obtained and unstable solutions will be excluded from consideration [17].

The scenario of derivation of the solutions mentioned above follows. First, we assume

that the solution is of the form $x = x(t; J, \varphi)$. Second, the relationships

$$E(J^0, \varphi^0) = 0, \quad J^0 = J(\varphi^0) \quad (7.24)$$

are used for finding the undetermined impulse and phase. It is supposed here that all roots

¹When it is not necessary we omit writing all the arguments of PPGF $\kappa(t, s; \mu_0)$.

²The asymptotic stability is considered, since the systems under consideration possess full dissipation.

of the first equation are isolated. The energy function $E(J, \varphi)$ may be considered to

be a function of two variables in the form

$$E(J, \varphi) = E_1(J, \varphi) - E_2(J, \varphi)$$

where E_1 is the work done by the external source, and E_2 is the work done by the friction forces. If the regime with parameters $J = J^0$, $\varphi = \varphi^0$ is unstable, for any

small deviation in the value of J^0 , the damping forces will not be capable of

stabilisation of the system due to an additional intake of external energy. Therefore, the total derivative of the energy function $E(J, \varphi)$ with respect to the impulse J

when $J = J^0$, $\varphi = \varphi^0$ is

$$\left. \frac{dE}{dJ} \right|_{J=J^0} = \left[\frac{\partial E}{\partial J} + \frac{\partial E}{\partial \varphi} \frac{d\varphi}{dJ} \right]_{J=J^0}^{\varphi=\varphi^0} > 0 \quad (7.25)$$

The derivative φ_J is calculated by virtue of the second relationship (7.24).

Condition (7.25) may be verified easily and unstable solutions may be excluded

immediately from consideration.

It was shown through examples that the stable regimes which were obtained through

exact methods [17, 74] yield negative total derivative of the energy function, $E(J, \varphi)$

$$\left. \frac{dE}{dJ} \right|_{J=J^0}^{\varphi=\varphi^0} < 0.$$

The limitations of this method as applied to an analysis of stability have not yet been estimated. The proof of the validity of the sufficiency of condition (7.25) for the instability is given in reference [14].

7.7. Mathieu Equation with Strong Nonlinearity

The above procedure may be applied to the dynamic analysis of different mechanical systems, in particular, for the SDOF mechanisms with variable stiffness and nonlinear position functions of the driven components [191], and also for the mechanisms

We will consider in detail the particular case of system (7.1) when the parametric

excitation $c(t)$ is given by the sine function. The dynamic model of the system is

represented in Fig. 7.1a. Here $\Omega^2(t) = 1 + \varepsilon \mu \cos(\nu + \varphi)$, $g(x, \dot{x})$ is the damping in the contact zone. Equation of motion (7.1) takes the form

$$\ddot{x} + [1 + \varepsilon \mu \cos(\nu + \varphi)] \dot{x} = \varepsilon [-2b\dot{x} + g_2(x, \dot{x})] - \lambda \Phi(x) \quad (7.26)$$

If $\mu = \mu_1 v^2 > 0$, then Eq. (7.26) corresponds to the system in Fig. 7.1 b, where x is small and $u(t) = -\varepsilon u_0 \cos(v + \phi)$, $\mu_1 = u_0 l^{-1}$. The derivation of Eq. (7.26) for this system may be found in numerous references (see [17], for example). The left-hand side of the equation corresponds exactly to the equation of a pendulum with vibrating joint (see (3.20)). The parametric term is defined by the translational force of inertia $\propto [-\ddot{u}(t)]$, which is accounted for while considering the relative motion $x(t)$.

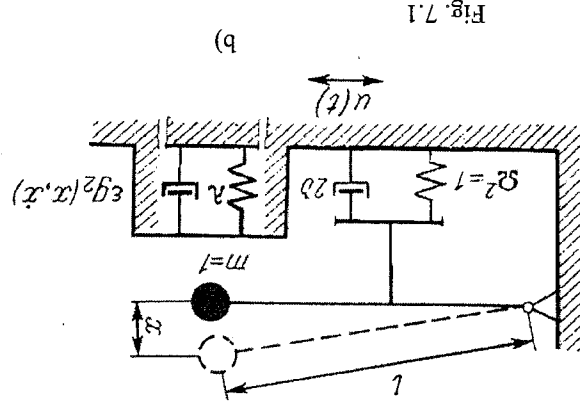
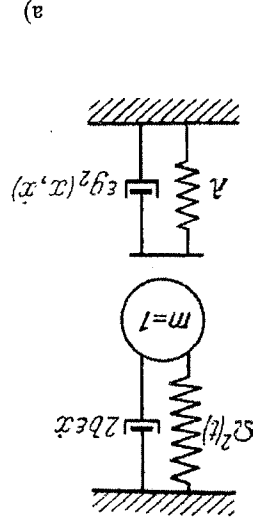


Fig. 7.1

Let $\lambda\Phi(x) = \lambda(x - \Delta)^a \eta(x - \Delta)$, where $a \geq 1$, $\Delta \neq 0$. The periodic solution to problem (7.26) takes the form

$$x(t) = -J\chi[t - \tau_2(J)] \quad (7.27)$$

where $\tau_2(J) = \frac{2}{\lambda}$, i.e. half of the interaction time, and the PGF $\chi(t)$ depends upon the frequency of the motion, ω .

For the definition of the unknown interaction impulse we use the solution corresponding to the case of momentary impact $\tau_2(J) = 0$. The contact damping is accounted for through the restitution ratio R , which characterises the properties of a limiter.

The approximate form of the motion in the vibro-impact case, $\lambda \rightarrow \infty$ (in (7.26)), is

¹ A similar problem was analysed using other methods in references [17, 76, 109]. The

Given by representation (7.12) using formulae (7.15) and (7.16). Initially let $v = \omega$. Then in (7.15) and (7.16) $c_1 = -1$, $c_n = 0$, $n > 1$, $\mu_0 = \varepsilon\mu$. Neglecting damping which may be accounted for in the equation of energy balance as $\lambda \rightarrow \infty$ and $0 \leq t \leq T = 2\pi\omega^{-1}$, we find

$$x(t) = -\frac{J}{\cos\left(t - \frac{2}{T}\right)} \left\{ \cos\left(\frac{2\sin(T/2)}{T}\right) \sin\left(t - \frac{2}{T}\right) - \varepsilon\mu \left[\omega^{-1} \sin \frac{\omega t}{2} \cos\left(\frac{\omega t}{2} + \phi\right) \sin\left(t - \frac{2}{T}\right) + \frac{1}{2} \cos\left(\frac{\omega t}{2} + \phi\right) \right] \right\} \left[\cos\left(\frac{1}{2}\omega + 1\right)t - T/2 \right] - \frac{\omega + 2}{\cos\left(\frac{1}{2}\omega - 1\right)t + T/2} \right] \left[\cos\left(\frac{1}{2}\omega - 1\right)t + T/2 \right] \right\}$$

where in accordance with (7.16),

$$J = -2\Delta \left(\frac{4 - \omega^2}{4 - \omega^2} \tan \frac{\pi}{\pi} \right) \left(\frac{4 - \omega^2 - 2\varepsilon\mu \cos \phi}{4 - \omega^2} \right)$$

For definition of the undefined phase ϕ , apply the equation of energy balance in the form of (7.4). After evident simplifying it takes the form

$$\frac{1}{2} b \frac{1}{T - \sin T} + \frac{1 - R}{1 - R} = \frac{\omega^2 - 4}{\mu \omega \sin \phi} \cot \frac{\omega}{\pi}, \quad T = \frac{\omega}{2\pi} \quad (7.29)$$

The right-hand side of (7.29) for $0 \leq \phi \leq \pi$ is positive due to the fact that the intervals of the resonant frequencies for $\Delta \leq 0$ are $1 < \omega < 2$ and $2 < \omega < \infty$. Equation (7.29) yields two values for the phase

$$\phi_1 = \arcsin \left\{ \frac{\mu \omega}{\omega^2 - 4} \left[\frac{1}{2} b \frac{1}{T - \sin T} + \frac{1 + R}{1 - R} \tan \frac{\omega}{\pi} \right] \right\}, \quad \phi_2 = \pi - \phi_1 \quad (7.30)$$

The existence condition of resonant regimes is found in the form

$$\left| \frac{\omega^2 - 4}{\omega^2 - 4} \left[\frac{1}{2} b \frac{1}{T - \sin T} + \frac{1 + R}{1 - R} \tan \frac{\omega}{\pi} \right] \right| \leq 1 \quad (7.31)$$

The condition for sustaining maximal amplitudes when $\omega \approx 2$

$$\mu \geq 4 \left[b + \frac{\pi(1 + R)}{2(1 - R)} \right] \quad (7.32)$$

is similar to (7.6).

The energy condition of instability obtained from (7.25) yields the inequality

$$\Delta^{-1} \cot \phi < 0 \quad (7.33)$$

which indicates the instability of the processes with the smaller value of impulse.

In order to obtain (7.33), we construct the function $E(J, \phi)$ reflecting the energy

present results were obtained in reference [90].

balance

$$E(J, \phi) = J^2 \frac{\epsilon \mu \omega \sin \phi}{2(\omega^2 - 4)} \cot \frac{\omega}{\pi} - J^2 \left[\frac{b\epsilon}{2} \frac{T - \sin T}{T} + \frac{4}{\epsilon} \frac{4 \sin^2 \frac{T}{2}}{2(1+R)} + \frac{1-R}{2(1+R)} \right], \quad T = \frac{\omega}{2\pi}$$

The expression defining the dependence $\phi(J)$ may be taken from (7.28). By virtue of (7.29) and the homogeneity of the function with respect to the argument E ,

$$\left. \frac{\partial E}{\partial J} \right|_{\phi=\phi_0} = 0$$

for all J . Making allowance for $1 < \omega < 2$ when $\Delta > 0$ and $2 < \omega < \infty$ when $\Delta < 0$, we find that

$$\frac{\partial E}{\partial \phi} = J^2 \frac{\epsilon \mu \omega \cos \phi}{\pi} \cot \frac{\omega}{\pi} = A^2(\omega) \cos \phi$$

with the evident notation $A^2(\omega)$. Further,

$$\frac{dJ}{d\phi} = 2\Delta(4 - \omega^2) \frac{2\epsilon \mu \sin \phi}{(4 - \omega^2 - 2\epsilon \mu \cos \phi)^2} \tan \frac{\omega}{\pi}$$

The inversion of the latter equality yields

$$\frac{dJ}{d\phi} = -B^2(\omega) \Delta \sin^{-1} \phi$$

Condition (7.25) is, therefore, reduced to the form of (7.33), as seen above.

The analysis carried out using the method of harmonic linearisation [17] and the stitching method [76] revealed that in the considered frequency spans the inequality which is reciprocal to (7.31) describes the stable processes¹

Fig. 7.2a, b shows the dependencies $J(\omega)$ calculated using (7.28) and (7.30). The

maximal width of the zone of resonant vibration in the vicinity $\omega \approx 2$ occurs in the absence of damping when $\cos \phi = 1$. This may be assessed in the form

$$\omega_{1,2}^* \approx 2 \pm 0.25\epsilon \mu.$$

The dependencies $J(\omega)$ for $\Delta > 0$ are sketched using the dot-and-dash curve; the

backbone curves are sketched using the dotted curves, and the thick dotted curves are used to represent unstable regimes. Figure 7.2a shows the case when $\Delta < 0$ and condition (7.32) holds. In Fig. 7.2b $\Delta < 0$ and condition (7.32) does not hold. The

¹ Solution (7.27) has to be analysed numerically with respect to satisfying the

condition $x(t) \leq \Delta$. For small μ this condition holds in the frequency spans which are close to the natural frequencies of a conservative system, $1 < \omega < 2$ for $\Delta > 0$ and $2 < \omega < \infty$ for $\Delta < 0$.

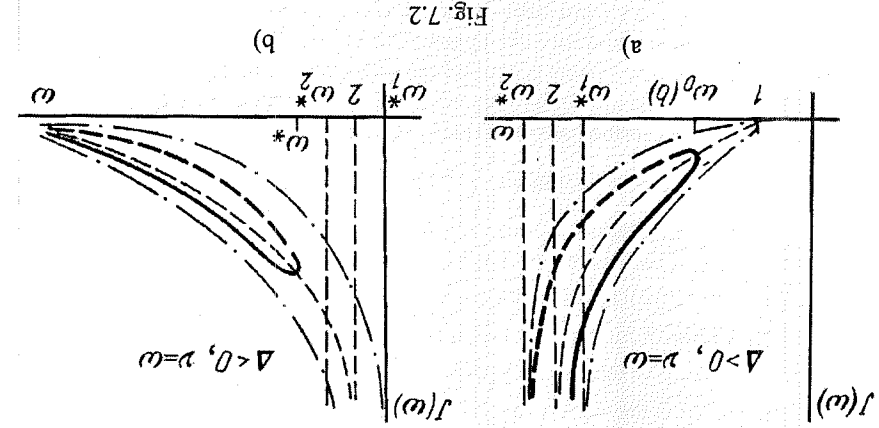


Fig. 7.2

if $\mu \neq O(\omega^\alpha)$, $\alpha \geq 2$.

For $\Delta < 0$ (see Fig. 7.2b) similar restriction is possible at high frequencies only brackets increases infinitely. realisation of existence condition (7.31). When $\omega \approx 1$ the first addend in the square value of the above frequency is defined for $1 < \omega < 2$ by the left boundary of the (7.31). In Fig. 7.2a the additional frequency $\omega_0(b) \rightarrow 1$ as $b \rightarrow 0$ is marked. The frequency ω_* of the downward jump corresponds to the case of equality in condition

The realisation of the periodic regime for $\Delta < 0$ is possible with hard excitation of the system. Description of the approximate solutions of Eq. (7.27) is possible now. The value of the impulse is defined using (7.28). Take the expression for the interaction time in the form (4.52), $t_1 = D_2 J^r$, where $r = \frac{1-\alpha}{1+\alpha}$ and the value D_2 is defined using Γ -functions. The solution of (7.27) takes the form

$$(7.34) \quad x(t) = -J\chi \left[t - \frac{1}{2} D_2 J^r \right]$$

where the impulse of the interaction is taken from (7.28) and the period of the PGF, $2\pi\omega^{-1}$, equals the period of the parametric excitation. Solution (7.34) may be amended using numerical iterative procedure [57]. We will study the subperiodic resonant motions corresponding to the fundamental parametric resonance in the linear case. Let $\nu = 2\omega$, where ω is the frequency of the motion. We are looking again for the solution in the form of (7.27), but now the period of the PGF $\chi(t)$ is twice as large as that of the excitation.

As $\lambda \rightarrow \infty$ the solution of the vibro-impact problem (7.12) may be obtained again from (7.15) and (7.16). Here $c_1 = 0$, $c_2 = -1$, $c_n = 0$, $n > 2$; $\mu_0 = \epsilon \mu$. Therefore,

$$x(t) = -\frac{J}{2\sin(T/2)} \left\{ \cos\left(t - \frac{T}{2}\right) - \epsilon\mu \left[\frac{1}{2\omega} \sin \omega t \cos(\omega t + \varphi) \sin\left(t - \frac{T}{2}\right) + \cos(\omega t + \varphi) \right] \cos\left[(\omega + 1)t - T/2\right] - \frac{4(\omega + 1)}{\cos[(\omega - 1)t + T/2]} \right\} \quad (7.35)$$

and, in accordance with (7.16),

$$J = -4\Delta \left(\frac{1 - \omega^2}{2 - 2\omega^2 - \epsilon\mu \cos \varphi} \right) \tan \frac{\omega}{\pi} \quad (7.36)$$

The energy balance equation is

$$\left[\frac{b}{T} - \sin T + 1 - R \right] = \frac{2\omega^2 - 2}{\mu\omega} \cot \frac{\omega}{\pi} \sin \varphi, \quad T = \frac{\omega}{2\pi} \quad (7.37)$$

thus,

$$\pi \leq \varphi \leq 2\pi \text{ if } \Delta > 0, 1 < \omega < 2 \text{ and } \cot \frac{\omega}{\pi} < 0$$

$$0 \leq \varphi \leq \pi \text{ if } \Delta > 0, \omega > 2 \text{ and } \cot \frac{\omega}{\pi} > 0$$

Equation (7.37) defines two values of the unknown phase (the formulae are similar to (7.30)) and the existence condition of a resonant regime takes the form

$$\left| \frac{2\omega^2 - 2}{bT - \sin T} \left(\frac{1}{1 - R} + \frac{1}{1 + R} \right) \tan \frac{\omega}{\pi} \right| \leq 1 \quad (7.38)$$

We will study in detail the problem of the existence of vibration with large amplitudes. When $\omega = 2$ (as seen from (7.36)), $J \rightarrow \infty$, but the existence of infinite amplitudes is feasible in the absence of damping (when $b \neq 0$ and $R \neq 1$ condition (7.38) can not

be satisfied as $\omega \rightarrow 2$).

When $\omega = 1$ (only the case of $\Delta > 0$ is considered here), from (7.37) follows the

condition for sustaining vibration

$$\mu \geq 4b \quad (7.39)$$

which is independent of the energy dissipation at impact since, in accordance with (7.36), $J(\omega = 1) = 0$. This corresponds to the case of *grazing*. From (7.36), $J(\omega) \rightarrow \infty$ in the vicinity of $\omega = 1$ for $1 < \omega < \sqrt{1 + 0.5\epsilon\mu}$, i.e. in the zone of the fundamental

linear parametric resonance, existing at $\mu \geq 2b$ [99, 187], the grazing vibration and

other regimes with large amplitudes are possible. This indicates the instability of the

vibration.

The small frequency variation in the vicinity of $\omega \approx 1$ may result in a cessation of the single-impact resonant regime as well as a sharp increase in its intensity. This indicates that the regimes of motion of a nonlinear system which correspond to the

case of linear parametric resonance in the frequency band $\omega \approx \sqrt{1 \pm \epsilon\mu}$, $\nu = 2\omega$ are not robust.

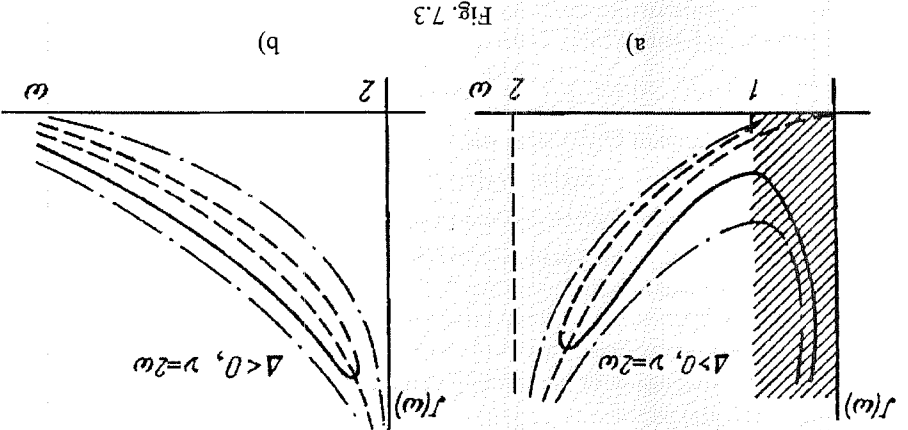


Fig. 7.3

More detailed analysis, which involves the calculation of parametric periodic Green functions and numerical simulation, reveals that the regimes of motion in the above frequency band may vary from stable periodic to chaotic vibration. On the boundary of the zone of parametric instability, the appearance of grazing regimes is quite possible. An analysis of these tiny effects is beyond the scope of this monograph. The energy condition of instability may be obtained as a result of calculations which are similar to those described above. Condition (7.25) takes the form of (7.33).

$\Delta^{-1} \cot \varphi < 0$, and, therefore, phases $\frac{2}{3}\pi < \varphi < 2\pi$ and $0 < \varphi < \frac{2}{3}\pi$ correspond to

unstable processes in cases $\Delta > 0$ and $\Delta < 0$, respectively. In both of these cases "unstable" values of the phase correspond to a smaller value of the impulse.

The dependencies $J(\omega)$ for $\Delta \leq 0$ are sketched in Fig. 7.3 a,b. The line types are in accordance with Fig. 7.2a. The hatched area corresponds to the fundamental

parametric resonance of the linear system. The behaviour of the function $J(\omega)$ for

$\Delta < 0$ and large ω is the same as at $\nu = \omega$. Representation (7.27) may be constructed for the case $\lambda < \infty$ on the basis of solutions obtained (see (7.34)). The graphs $J(\nu)$ (see Fig. 7.4a,b) define the dependence of the impulse of interaction on the excitation

frequency for the processes with one single interaction over the single period of excitation. This was obtained after combining Figs. 7.2 and 7.3. Condition (7.32) is

assumed to be fulfilled, therefore at $\nu \approx 2$ the appearance of infinite amplitudes is

likely.

In reference [17], the results of the experimental analysis of the parametrically excited vibro-impact systems are described. In particular, the obtained results indicate the possibility of the existence of periodic regimes with impacts in the region of linear

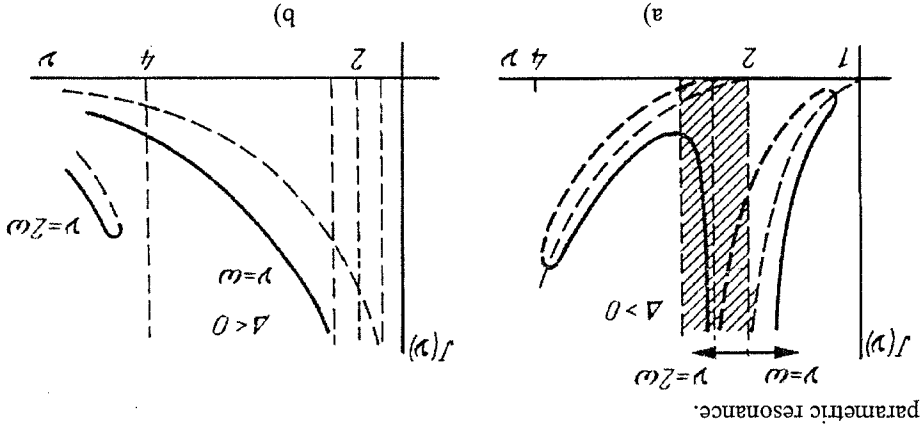


Fig. 7.4

At the same time, the available experimental data indicates also the presence of unstable chaotic motions in the above region. Therefore, in the asymmetric system the resonant processes inside and outside the zone of linear instability are governed by the different harmonics of excitation. The dynamics of the symmetric system is much more simple.

7.8. System with Symmetric Nonlinearity

Let the equation of motion takes form (7.26), as follows

$$\ddot{x} + [1 + \varepsilon \mu \cos(vt + \varphi)]x = \varepsilon [-2bx + g_2(x, \dot{x})] - \lambda \Phi^*(x) \quad (7.40)$$

but the function $-\Phi^*(x) = \Phi^*(-x)$ is symmetric. In Fig. 7.5 the dynamic model of the system at $\lambda \rightarrow \infty$ is sketched. In this case $\mu \propto \nu^2$.

We are looking for a solution to Eq. (7.40) in the form

$$x(t) = -J\chi^*[t - \tau_\lambda(J)] \quad (7.41)$$

where $\chi^*(t)$ is a symmetric PGF and τ_λ is half of the interaction time. The magnitude of the impulse is to be found from the limiting model at $\lambda \rightarrow \infty$. As seen from Eq. (7.7), within the accepted accuracy of the approximation, only the even harmonics of the parametric excitation contribute to the energy input. For the considered sine excitation $\nu = 2\omega$, where ω is the frequency of motion.

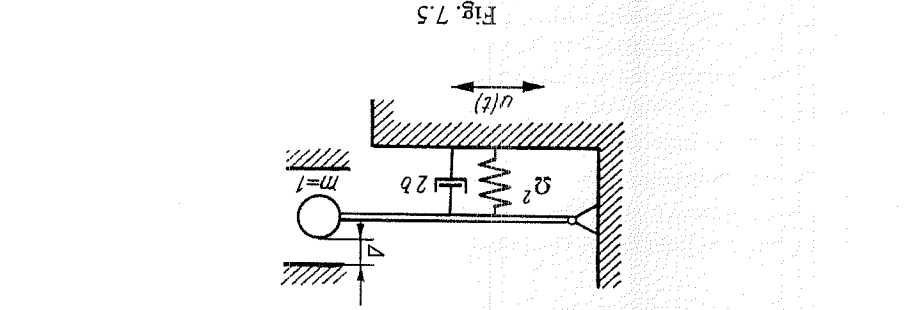


Fig. 7.5

The above-mentioned scheme of analysis is applied to the study of Eq. (7.40). Representation (7.18) is constructed using formula (7.18) with $c_2 = -1$, $\mu_0 = \varepsilon \mu$ and $c_{2n} = 0$ for $n > 1$. Similar calculations were carried out in Sect. 7.7 (see (7.27) and (7.35)) for the asymmetric system. Relationship (7.20) yields

$$J = \Delta \frac{2 - 2\omega^2}{1 - \omega^2 - 0.5\varepsilon \mu \cos \varphi} \cot \frac{\varphi}{4} \quad (7.42)$$

and the equation of energy balance takes the form

$$\frac{1}{T} b \frac{T/2 + \sin(T/2)}{1 - R} + \frac{1 + R}{\mu} \frac{2}{\omega} \tan \frac{\varphi}{4} \sin \varphi = \frac{2}{T} \frac{\cos^2(T/4)}{1 - R} \quad (7.43)$$

The condition of existence of the regime in the resonance zone $\omega > 1$ is as follows

$$\frac{\mu}{T} \frac{2}{\omega} \tan \frac{\varphi}{4} > \frac{1}{T} b \frac{T/2 + \sin(T/2)}{1 - R} + \frac{1 + R}{\mu} \frac{2}{\omega} \tan \frac{\varphi}{4} \sin \varphi, \quad T = \frac{\omega}{2\pi} \quad (7.44)$$

which indicates that at $\omega \rightarrow \infty$ (resonance regimes outside the zone of linear instability) the infinite magnitudes of the impulses are possible only if $\mu = O(\omega^n)$,

$n \geq 2$, or at $R = 1$. At the frequency $\omega = 1$ in the zone of the fundamental linear parametric resonance the condition of existence of the regime takes the form $\mu \geq 4b$.

The infinite values of the impulse correspond to the frequency interval

$$1 < \omega < \sqrt{1 + 0.5\varepsilon \mu}; \quad \pi \leq \varphi < 2\pi$$

The motion in the zone of the fundamental linear resonance is not robust (see comments in Sect. 7.7): the small perturbation may change significantly the entire character of the regime. However, there exist experimental data indicating the possibility of the existence of stable and robust resonant regimes in this zone [17]. The condition of energy balance (7.25) yields that lower values of J correspond to unstable regimes.

At $\mu = O(\omega^2)$ and large ω the level of modulation becomes high and our calculation must be corrected.

The hatched zone corresponds to the resonant frequencies of the linear system. The condition for sustaining the resonant regime is assumed to be fulfilled. Now, by analogy with Sect. 7.7 the representation (7.41) may be constructed.

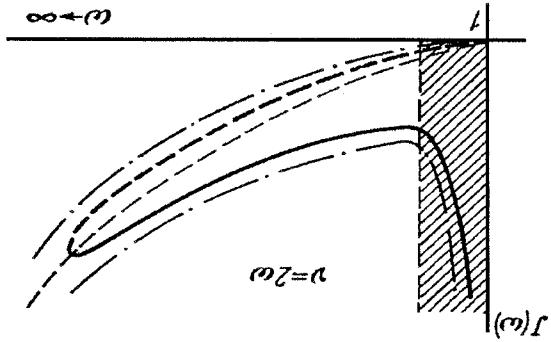


Fig. 7.6

The dependence $J(\omega)$ is shown in Fig. 7.6.

7.9. Calculations for Systems Under Combined Excitation

In this section we study the dynamic effects due to the simultaneous action of two external sources, namely force and parametric excitation. We consider systems which may be described using *Mathieu equations* with strong nonlinearity and a right-hand term represented by a sine function of time,

$$\ddot{x} + [1 + \varepsilon \mu \cos(v_1 t + \phi)] \dot{x} = \varepsilon [-2bx + g_2(x, \dot{x}) + h \cos(v_2 t + \phi_0)] - \lambda \Phi(x) \quad (7.45)$$

where v_1, v_2 are the frequencies of excitation and ϕ_0 is the phase lag. We also assume that $v_1 = v_2 = v$.

Equation (7.45) describes the dynamics of the mechanisms in Fig. 7.1b or 7.5 under the condition that the direction of action of the vibration $u(t)$ is not collinear to the axis of the mechanism's rod. Linearisation of the equation of motion about the steady-state angle ψ_0 (see Fig. 7.7) yields the decomposition of the function $\sin \psi$,

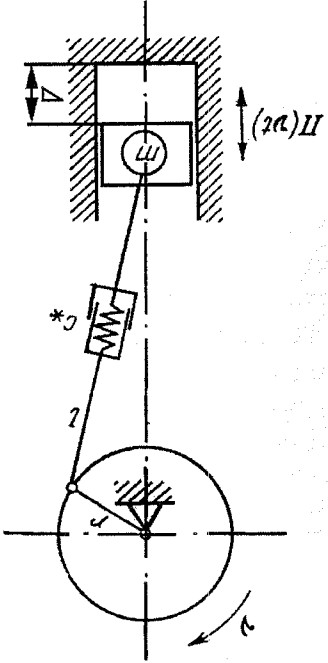
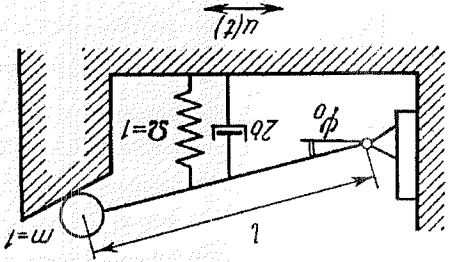
$$\sin \psi = \sin \psi_0 + (\psi - \psi_0) \cos \psi_0 \quad (\text{compare with (3.20)}).$$


Fig. 7.7

For the preloaded mechanism in Fig. 7.7a in Eq. (7.45) we find $\phi_0 = 0$,

$$\mu \sim u_0 v^2 \cos \psi_0, \quad h \sim u_0 v^2 (l \sin \psi_0 - \Delta \cos \psi_0), \quad \text{where } u_0 \text{ is the amplitude of harmonic vibration, } \Delta = \psi_0 l, \text{ and the quantities } \psi_0 \text{ and } l \text{ are shown in Fig. 7.7a.}$$

As $\psi_0 \rightarrow 0, h \rightarrow 0$ Eq. (7.45) describes the parametric vibration. As $\psi_0 \rightarrow \pi/2, \mu \rightarrow 0$ this equation defines forced vibration.

Equation (7.45) may be obtained as a result of simplified analysis of some mechanisms with nonlinear position-function and variable stiffness [191]. Consider, for example, the mechanism in Fig. 7.7b where both the striker of mass m and the obstacle are involved in the programmed motion in accordance with the nonlinear position function $\Pi(v_2 t) \approx h_1 \sin(v_2 t + \beta)$, where h_1 and β are the parameters. Accounting for the compliance of the links of the crank mechanism yields the effective stiffness as a periodic function of time. After simplifications the problem may be reduced to Eq. (7.45). The phase shift ϕ_0 may be an arbitrary value, and also

$$h \propto \omega^2.$$

In the above examples the limiter was considered to be absolutely rigid and stationary.

therefore $\lambda \rightarrow \infty$ in Eq. (7.45). The compliance of the limiter may be accounted for similarly to the previous example.

We will find the approximate formulae describing the periodic regimes in system (7.45) where the parameters μ and h are independent of frequency and $\lambda \rightarrow \infty$.

Under the assumption $v_1 = v_2 = v$, the frequency of the vibration will be ω , and the regimes with $\omega = v$ will be analysed. Consider two possible phase shifts $\varphi_0 = 0$ and $\varphi_0 = \pi$. The case $0 < \varphi < \pi$ may be analysed similarly.

The motion $x(t)$ is described by representation (7.23), where the T -periodic solution to the linear problem $f(t)$ is calculated using bulky quadrature. To simplify our analysis we will apply the method of formal trigonometric decomposition (a special case of the *Galerkin method*). The solution of the linear Mathieu equation with the right-hand side

$$(7.46) \quad f + [1 + \varepsilon \mu \cos(\omega t + \varphi)]f = \pm \varepsilon h \cos(\omega t + \varphi)$$

where the positive sign corresponds to the phase shift $\varphi_0 = 0$ and negative sign

corresponds to the phase shift $\varphi_0 = \pi$, may be represented as

$$(7.47) \quad f(t) = a_0 + a_1 \cos(\omega t + \varphi) + a_2 \cos(2\omega t + 2\varphi) + \dots$$

The substitution of the above solution into the initial equation (7.46) yields, for the undefined magnitudes,

$$(7.48) \quad \begin{aligned} a_0 &= \pm \frac{1}{\varepsilon^2 \mu h (1 - 4\omega^2)} \left(2 \left(1 - \frac{1}{2} \varepsilon^2 \mu^2 - \omega^2 \right) \left(1 - 4\omega^2 \right) - \frac{1}{4} \varepsilon^4 \mu^2 \right) \\ a_1 &= \pm \frac{\varepsilon h (1 - 4\omega^2)}{\left(1 - \frac{1}{2} \varepsilon^2 \mu^2 - \omega^2 \right) \left(1 - 4\omega^2 \right) - \frac{1}{4} \varepsilon^4 \mu^2} \\ a_2 &= \pm \frac{\varepsilon^2 \mu h}{\left(1 - \frac{1}{2} \varepsilon^2 \mu^2 - \omega^2 \right) \left(1 - 4\omega^2 \right) - \frac{1}{4} \varepsilon^4 \mu^2} \end{aligned}$$

Neglecting the $O(\varepsilon^2)$ terms yields $a_0 \approx 0$, $a_2 \approx 0$, $a_1 = \pm \varepsilon h (1 - \omega^2)^{-1}$ and the

solution is similar to that of the stationary problem. It may be explained as follows. The regimes excited by the first harmonic of the excitation correspond to the *first interval* linear parametric resonance, which appears as a second-order effect in the frequency band $\propto \varepsilon^2$. It can be excited only if the damping is of the same order of smallness. Actually, this resonance, and especially, *higher order* resonances, are rarely observed in experiments. This can be neglected within the assumed accuracy. Hence, the first approximation for the solution of the linear problem is found, the PPGF $\kappa(t, \varphi)$ may be approached using the repetitive kernel (7.15) and, thus, all functions from (7.23) are defined. The undefined phase and impulse may be obtained

from the impact conditions $x(\varphi) = \Delta$, $J = (1 + R)x_-$. For this purpose we may simply consider $f(t) = a_1 \cos(\omega t + \varphi) + O(\varepsilon^2)$ (see (7.23) and (5.45)).

However, using the first impact condition and the energy balance equation leads to more simple solutions. Coinciding the instance of impact and the time zero reference in equation (7.23) we find

$$(7.49) \quad x(t) = f(t + \varphi) - J\kappa(t + \varphi, \varphi)$$

The condition $x(0) = \Delta$ yields (see (7.28))

$$(7.50) \quad J = -\frac{\Delta - f(\varphi)}{\kappa(\varphi, \varphi)} = -2\Delta_0(\varphi) \left(\frac{4 - \omega^2 - 2\varepsilon \mu \cos \varphi}{4 - \omega^2} \right) \tan \frac{\omega}{\pi}$$

where $\Delta_0(\varphi) = \Delta - [a_0 + a_1 \cos \varphi + a_2 \cos 2\varphi + \dots] = \Delta - a_{10} \cos \varphi + O(\varepsilon^2)$, $a_{10} = \pm h(1 - \omega^2)^{-1}$.

The energy balance equation (7.22), after multiplication by J^2 , becomes

$$(7.51) \quad \left(\frac{1}{2} J^2 \frac{\mu \omega}{h \omega} \cot \frac{\omega}{\pi} \pm J \frac{\omega}{h \omega} \right) \sin \varphi = \frac{1}{2} J^2 \left(\frac{2}{b} \frac{\sin^2(T/2)}{T - \sin T} + \frac{1}{1 + R} \right)$$

The approximate substitution $J \approx J^0 = -2\Delta \tan(\pi/\omega)$ yields the equation in the undefined phase

$$\sin \varphi = \frac{\frac{1}{2} J^0 \left(\frac{2}{b} \frac{\sin^2(T/2)}{T - \sin T} + \frac{1}{1 + R} \right)}{\frac{1}{2} J^0 \frac{\mu \omega}{h \omega} \cot \frac{\omega}{\pi} \pm \frac{\omega}{h \omega}}$$

and the existence condition

$$(7.52) \quad \left| \Delta - \frac{1}{b} \frac{\sin^2(T/2)}{T - \sin T} + \frac{1}{1 + R} \right| \tan \frac{\omega}{\pi} \leq \left| \frac{\mu \Delta \omega}{\omega^2 - 4} \pm \frac{h \omega}{1 - \omega^2} \right|$$

Therefore, we have obtained all the representations for the parameters of motion. The above representations must be analysed with respect to stability (see below), and the geometrical conditions $x(t) \leq \Delta$ ($J \geq 0$) may be verified numerically. For small values of μ the above conditions are satisfied in the resonance vicinities $1 < \omega < 2$ for $\Delta > 0$ and $2 < \omega < \infty$ for $\Delta < 0$.

Now we will analyse the existence conditions (7.52). The right-hand side part of this inequality contains the terms corresponding to the energy inputs from the parametric and force sources. We do not consider now the resonant regimes due to the excitation of the fundamental linear parametric resonance and similar regimes due to the excitation of the first lateral parametric resonance. As $\omega \rightarrow 2$ (resonant frequency which corresponds to the large magnitudes of impulses) the parametric source

dominates. Hence, in the considered relationships we can neglect the term containing h and the problem may be reduced to that already solved. The variation in the frequency of motion from 2 to 1 yields the decrease in the energy input from the parametric source. Hence, the vibration with the frequency $\omega \approx 1$ is possible. The positive sign prefixing the term defining the force source corresponds to the matched case, when the parametric and the force excitation are acting in the same phase. Here $0 < \varphi < \pi$. (The mechanism mounted over the vibrating base is subjected to the matched action of both sources.)

The negative sign prefixing the "force" term corresponds to the phase shift $\varphi_0 = \pi$ and the unmatched case. When

$$\omega = \omega^* = \sqrt{\frac{\mu\Delta + 4h}{h + \mu\Delta}}$$

the condition (7.52) can not be satisfied for any damping. This is reflected in the mutual neutralisation of the sources¹

If $\Delta > 0$ then we always have $1 < \omega^* < 2$. If $\Delta < 0$ then the latter equality takes the form

$$(7.53) \quad \omega = \omega^* = \sqrt{\frac{4h - \mu|\Delta|}{h - \mu|\Delta|}}$$

The above radical is feasible if $h > \mu|\Delta|$ or if $\mu|\Delta| > 4h$. In both cases, in the vicinity of the frequency $\omega \approx 2$, the parametric source dominates, but for $h > \mu|\Delta|$ the vibration in the high-frequency band may be due to the action of the force excitation.

At $\mu|\Delta| > 4h$ this is due to the parametric excitation².

The energy condition of instability in the form (7.25) is now used. For this purpose Eq. (7.51) is rearranged and takes the form

$$E(J, \varphi) = \left(\frac{1}{2} J^2 \frac{\omega^2 - 4}{\mu\omega} \cot \frac{\omega}{\pi} \pm J \frac{h\omega}{1 - \omega^2} \right) \sin \varphi - \frac{1}{2} J^2 \left(\frac{1}{T} b \frac{\sin^2(T/2)}{T - \sin T} + \frac{1 - R}{1 + R} \right)$$

The corresponding derivatives E_J , E_φ and φ_J are calculated as

¹ If the force source dominates, we have $\pi < \varphi < 2\pi$.

² For small Δ , $\omega^* \approx 2$, hence, the parametric effects can not be developed significantly. For $\mu|\Delta| > 4h$ the frequency $\omega^* < 2$ and mutual neutralisation of the two sources does not take place because $\omega > 2$.

$$(7.54) \quad E_J = \left[J \frac{\mu\omega}{1 - \omega^2} \cot \frac{\omega}{\pi} \pm J \frac{h\omega}{1 - \omega^2} \right] \sin \varphi - J \left[\frac{1}{2} b \frac{\sin^2(T/2)}{T - \sin T} + \frac{1 + R}{1 - R} \right] T = \frac{\omega}{2\pi}$$

The derivative is calculated at the stationary points where condition (7.51) holds.

Hence,

$$E_J = \pm \frac{1 - \omega^2}{h\omega} \sin \varphi$$

Further, we find

$$(7.55) \quad E_\varphi = \left[\frac{1}{2} J^2 \frac{\varepsilon\mu\omega}{1 - \omega^2} \cot \frac{\omega}{\pi} \pm \frac{h\omega}{1 - \omega^2} \right] \cos \varphi$$

In order to obtain φ_J we differentiate condition (7.51),

$$\varphi_J = -2 \tan \frac{\omega}{\pi} \frac{4 - \omega^2}{4 - \omega^2 - 2\varepsilon\mu \cos \varphi} \left(\pm h \frac{1 - \omega^2}{4 - \omega^2} - 2\Delta\varepsilon\mu \right) \sin \varphi$$

Hence,

$$(7.56) \quad \varphi_J = \frac{(4 - \omega^2 - 2\varepsilon\mu \cos \varphi)^2 \cot(\pi/\omega)}{2 \left(\pm h \frac{1 - \omega^2}{4 - \omega^2} - 2\Delta\varepsilon\mu \right) (4 - \omega^2) \sin \varphi}$$

Now we can construct condition (7.52) for the definition of unstable regimes, as

$$\frac{dE}{dJ} = E_J + E_\varphi \varphi_J > 0$$

It is worth noting that in the case of matched regimes the phase $\varphi \in [0, \pi]$. The phase $\varphi \in [\pi/2, \pi]$ corresponds to the unstable regime independently of the dominating source. In Fig. 7.8a,b the possible dependencies $J(\omega)$ for $\Delta \geq 0$ for the different levels of modulation and excitation are depicted (condition (7.32) does or does not hold).

The unstable regimes are sketched using dotted lines and solid lines are used for indicating the regimes which do not satisfy the instability condition. The hatched area corresponds to the zone of the first lateral parametric resonance. The dash-dotted line indicates the unmatched regimes. It appears to be difficult to analyse their stability due to the changes of the interval of the phase variation. Rigorous analysis may be carried out using the averaging method (see Sect. 5.5). A detailed analysis of all possible cases appearing for different combinations of parameters leads to tedious relationships.

If the coefficients c_1 and c_2 satisfy the condition

$$(7.61) \quad \left| \frac{c_1(1-\omega^2)}{\omega^2-4} c_2 \right| \leq 1$$

the bifurcation of the periodic regimes becomes possible. As a result, two pairs of resonant regimes arise. For $\omega \approx 1$ or $\omega \approx 2$ the above regimes may merge into a single pair of regimes inside or outside the zone of instability. Equation (7.60) describes a variety of possible solutions. This can be illustrated graphically in any way similar to that in Fig. 5.9. For the regimes of motion, the bifurcation diagrams similar to Fig. 5.10 may be plotted. The energy condition of instability (7.52) takes the form

$$(7.62) \quad \frac{A_1^2(\omega) \cos \varphi + A_2^2(\omega) \cot(\pi/\omega) \cos 2\varphi}{A_3^2(\omega) \Delta \sin \varphi + A_4^2(\omega) \Delta \tan(\pi/\omega) \sin 2\varphi} > 0$$

where $A_i^2(\omega)$ are functions of system parameters, which may be obtained using regular calculations. It is seen that domination of the first or second harmonics of the parametric excitation results in the transformation of condition (7.62) into the conditions of Sect. 7.7 (see (7.33)).

We will consider briefly the problem given by (7.45) for $v_1 = 2\omega$ and $v_2 = \omega$. In order to find out the regimes of the frequency ω , representation (7.23) is to be constructed. The function $f(t)$ defining the forced vibration of the linear system satisfies the equation

$$f'' + [1 + \varepsilon h \cos(2\omega t + 2\varphi)] f = \pm \varepsilon h \cos(\omega t + \varphi)$$

where again it is considered that $\varphi_0 = 0$ or $\varphi_0 = \pi$. The period of the external force is $T = 2\pi/\omega$ and the period of the parametric excitation is $T/2$. Hence, we can expect that $f(t)$ will be a symmetric function represented by odd harmonics only,

$$(7.63) \quad f(t) = a_1 \cos(\omega t + \varphi) + a_3 \cos(3\omega t + 3\varphi) + \dots$$

The substitution of this expression into the initial equation yields

$$a_1 = \frac{\pm \varepsilon h (1 - 9\omega^2)}{(1 + \frac{1}{2} \varepsilon h - \omega^2) (1 - 9\omega^2) - \frac{1}{4} \varepsilon^2 h^2}, \quad a_3 = \frac{\frac{1}{2} (1 + \frac{1}{2} \varepsilon h - \omega^2) (1 - 9\omega^2) - \frac{1}{4} \varepsilon^2 h^2}{\pm \varepsilon^2 h}, \quad a_5 = \frac{1 + \frac{1}{2} \varepsilon h - \omega^2}{\pm \varepsilon h}, \quad a_7 = 0$$

Hence, within the accuracy of $O(\varepsilon^2 h^2)$,

The increase of the first harmonic in the vicinity of the frequency $\omega = 1$ is due to the fundamental parametric resonance. Representation (7.23) may be constructed now. Taking the initial condition of the phase to be the impact gives

$$(7.64) \quad J(\omega) = \frac{-4\Delta_0(\varphi)(1-\omega^2)}{\pi} \tan \frac{\omega}{2-2\omega^2-\mu \cos 2\varphi}$$

where $\Delta_0(\varphi) \approx \Delta \pm \varepsilon h (1 + \frac{1}{2} \varepsilon h - \omega^2)^{-1} \cos \varphi$. The energy balance equation takes the form

$$\frac{1}{J} \left[\frac{1}{T} - \sin T \right] + \frac{1-R}{1-R} = J \left[\frac{2}{T} \frac{\sin^2(T/2)}{\sin^2(T/2)} + \frac{1+R}{1-R} \right] = J \left[\frac{4\omega^2-4}{\mu\omega} \cot \frac{\omega}{\pi} \sin 2\varphi \pm \frac{1-\omega^2}{h\omega} \sin \varphi \right]$$

which is of the same structure as (7.60). The bifurcation condition of the periodic regimes when $b = 0$ and $R = 1$ relates the levels of modulation and of the force excitation, as

$$\left| \frac{J\mu}{2h} \tan \frac{\omega}{\pi} \right| \leq 1$$

Outside the zone of the linear parametric resonance where $J \approx J^0 = -2\Delta \tan(\pi/\omega)$ the latter condition takes the form

$$\frac{\mu|\Delta|}{h} \leq 1$$

i.e. the bifurcation takes place if the parametric excitation dominates. The energy condition of instability (7.52) is constructed in accordance with the above examples.

Note that in the undamped system the pair of additional regimes merges into one, and the equation for the unknown phase takes the form

$$A_1 \sin 2\varphi + A_2 \sin \varphi = 0$$

where A_1 and A_2 are unknown coefficients. The solution to the above equation is $\sin \varphi = 0$ which yields two values for the phase, namely $\varphi_{1,2} = 0, \pi$, and $\cos \varphi = -(A_2/2A_1)$ producing one additional solution because (7.59) and (7.64) are single-valued functions of $\cos \varphi$. Similar situation occurs when the bifurcation of the regimes is due to the force excitation (Sect. 5.13).

The appearance of the additional regimes is due to the presence of different sources of the external energy contributing to different harmonics. Such regimes arise not only when the system transits from fundamental to subperiodic regimes, but also at the transition from purely force excitation to purely parametric excitation, and vice versa.

7.11. Explicit Solutions to a Specific Class of Model Problems

The latter analysis was based on the approximate representation of the PGF $\kappa(t, s; \mu)$ of the linear part of a parametric system. The knowledge of the PPGF is equivalent to the knowledge of the particular integral of a corresponding linear problem. In general, the approximate formulae obtained in §3 do not require the assumption of the smallness of the modulation level. The series for $\kappa(t, s; \mu)$ converge uniformly for all regular values of μ . Hence, the decompositions for the PPGF are not *asymptotic* but are *uniform*. In practical calculations this fact is of rather theoretical interest. First, the requirement that μ is small is essential for calculations. Second, in the first approach the difference in the type of convergence is not important. Third, practical interest is attracted, in general, to the resonance regimes caused by small forces.

However, there exists a specific class of model problems allowing simple periodic solutions (in the case of impact interaction) without the application of a series and without any assumptions concerning the value of the parameter μ . This class represents the systems allowing the closed form representation of corresponding PPGF [91]. The periodic parametric excitation is described in such systems using special functions, like a linear combination of the Dirac δ -functions¹. Such linear systems were analysed in Sect. 3.4.

The possibility of obtaining the exact analytical solutions to a nonlinear parametric problem arises rarely. The results obtained below are valuable for modelling. Figure 7.9 represents the different vibro-impact systems corresponding to the linear systems in Figs. 3.1 and 3.2. In the cases represented by Fig. 7.9a,b the equations of motion are of the form

$$\ddot{x} + [1 + \mu \delta_{T/2}(t)]x + \Phi_0(x, x) = 0$$

where $\Phi_0(x, x)$ is the force of impact interaction (symmetric or asymmetric),

$$\delta_{T/2}(t) = (4/T) \sum_{k=0}^{\infty} \cos[(2k+1)\omega t]$$

For the cases in Fig. 7.9c,d we have

$$\ddot{x} + [1 - \mu + \mu T \delta^T(t)]x + \Phi_0(x, x) = 0 \quad (7.65)$$

An example of the calculation of PPGF for such a system in a closed form may be found in Sect. 3.4 (see (3.24)). The corresponding calculations may be carried out

¹ See also paper [76], where a system with a piecewise constant parametric term was analysed numerically using the stitching method. The application of the stitching method requires knowledge of the general, but not particular, integral of a linear problem.

accounting for viscous friction, but for simplicity it is considered that energy dissipation takes place at impacts only. Consider Eq. (7.65) with $\mu < 1$. To calculate the primary and subperiodic single-impact regimes of the asymmetric system in Fig. 7.9c, the knowledge of two T -periodic PPGF is required. Namely $\kappa_1(t, s; \mu)$ is the response of the system to the force sequence $\delta_T^T(t-s)$, and $\kappa_2(t, s; \mu)$ is the response of the system containing a parametric term of period $\frac{1}{2}T$ to the same force sequence (the regimes of the frequency ω are supported by the excitation of the frequency 2ω). In the first case we find from (3.24) that

$$\kappa_1(t, s) = -\frac{\mu T \chi(t) \chi(s)}{1 + \mu T \chi(0)} + \chi(t-s) \quad (7.66)$$

where, for $0 \leq t < T$,

$$\chi(t) = \frac{2\sqrt{1-\mu} \sin(\sqrt{1-\mu} T/2)}{1 - \cos[\sqrt{1-\mu}(t-T/2)]}$$

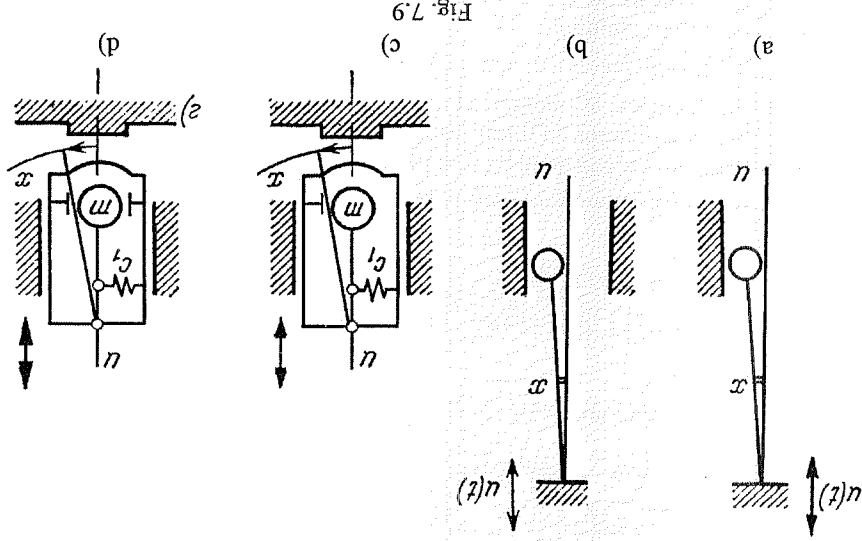


Fig. 7.9

In the second case we find the integral equation of the T -periodic vibration (see Sect. 3.4)

¹ This assumption is not essential. The PPGF defining the desired law of motion may be calculated in a closed form for all regular values of μ .

$$\kappa_2(t, s) = -\mu T \int_T^0 \chi(t-t) \delta_{T/2}^1(t) \kappa_2(t, t) dt + \int_T^0 \chi(t-t) \delta_T(t) \delta_T^1(t-s) dt$$

where $\delta_{T/2}^1(\tau) = \sum_{q=-\infty}^{\infty} \delta(\tau - qT/2)$. Hence,

$$\kappa_2(t, s) = -\mu T \chi(t) \kappa_2(t, 0) - \mu T \chi(t - T/2) \kappa_2(t, T/2) + \chi(t-s)$$

and, assuming consequently $s = 0$ and $s = T/2$, for the unknown values $\kappa_2(t, 0)$ and $\kappa_2(t, T/2)$ we find the system of linear algebraic equations possessing the solutions

$$\begin{aligned} \kappa_2(t, s) &= \chi(t-s) - K(\mu) [1 + \mu T \chi(0)] [\chi(t) \chi(s) + \chi(t-T/2) \chi(s-T/2)] \\ &\quad - \mu T \chi(T/2) [\chi(t) \chi(s-T/2) + \chi(t-T/2) \chi(s)] \end{aligned} \quad (7.67)$$

where $K(\mu) = T\mu [1 + T\mu \chi(0) + (T\mu)^2 [2(1 - \mu T)]^{-2}]^{-1}$

For the symmetric system in Fig. 7.8d, in the case of the fundamental regimes we have to calculate the symmetric PPGF $\kappa^*(t, s)$ which may be obtained as a response of the linear part of the system to the excitation sequence $\delta_{T/2}(t-s) = \sum_{q=-\infty}^{\infty} (-1)^q \delta(t - 1/2 qT - s)$ under the condition that the parametric term

takes the form $\delta_{T/2}^1(t)$. Symmetric regimes may exist in such a system if the nonlinearity is symmetric and the parametric term is $T/2$ -periodic. Simple calculations yield

$$\kappa^*(t, s) = -\frac{1 + T\mu \chi^*(0)}{T\mu \chi^*(s) \chi^*(t-s)} + \chi^*(t-s) \quad (7.68)$$

where the symmetric PGF is

$$\chi^*(t) = \frac{2\sqrt{1-\mu} \cos\sqrt{1-\mu} T/4}{1 - \sin[\sqrt{1-\mu}(t-T/4)]}$$

The closed-form solutions to the problems with impact interaction are found as follows. For the asymmetric system in the case of the fundamental and subperiodic regime, from the condition of impact $x(\phi) = \Delta$, we find

$$x(t) = -J\kappa_1(t, \phi), \quad J = -\Delta\kappa_1^{-1}(\phi, \phi) \quad (7.69)$$

$$x(t) = -J\kappa_2(t, \phi), \quad J = -\Delta\kappa_2^{-1}(\phi, \phi) \quad (7.70)$$

$$x(t) = -J\kappa^*(t, \phi), \quad J = -\Delta[\kappa^*(\phi, \phi)]^{-1} \quad (7.71)$$

and for the solution in the symmetric case we find

The value of the unknown phase ϕ for the above solutions may be obtained from the relationship $J = (1+R)x^-(\phi) > 0$, and representations (7.69)-(7.71) must be analysed with respect to the geometrical conditions $x \leq \Delta$, $(|x| \leq \Delta)$, $J \geq 0$ and stability. The relationship $J = (1+R)x^-(\phi)$ may be replaced by energy balance equation which yields the exact value of the phase¹ if based on exact representations (7.69)-(7.71). We will analyse solution (7.69) in detail in the frequency ranges $\sqrt{1-\mu} \leq \omega < 2\sqrt{1-\mu}$ for $\Delta > 0$ and $2\sqrt{1-\mu} < \omega < \infty$ for $\Delta < 0$, corresponding to the natural frequencies of motion of the stationary part of system (7.65). The conditions for self-excitation of linear parametric resonance causing the transition to a vibro-impact regime are governed by condition (3.22) here.

yields

$$x(t) = \frac{\Delta(1 + \mu T \chi(0))}{\mu T \Delta \chi(\phi)} \chi(t - \phi) - \frac{\chi(0) + \mu T [\chi_2^2(0) - \chi_2^2(\phi)]}{\mu T \Delta \chi(\phi)} \chi(t) =$$

$$J = -\Delta(1 + \mu T \chi(0)) / \left\{ \chi(0) + \mu T [\chi_2^2(0) - \chi_2^2(\phi)] \right\}$$

$$= \frac{-\Delta \left[4(1 - \mu) \sin^2 \left(\sqrt{1 - \mu} T/2 \right) + \mu T \sqrt{1 - \mu} \sin \left(\sqrt{1 - \mu} T \right) \right]}{\sqrt{1 - \mu} \sin \left(\sqrt{1 - \mu} T \right) \sin \left[\sqrt{1 - \mu} (\phi - T) \right]}$$

The value of the phase ϕ may be obtained using the energy balance principle, as

$$-\mu T J^2 \int_T^0 \kappa_1(t, \phi) \kappa_1(t, \phi) \delta_T^1(t) dt - J^2 \frac{2(1+R)}{1-R} E_1(J, \phi) - E_2(J) = 0$$

(7.73) The calculation of the work done by the excitation force is accompanied by some formal difficulties. In accordance with Sect. 1.2 we find that

$$-\mu T J^2 \int_T^0 \kappa_1(t, \phi) \kappa_1(t, \phi) \delta_T^1(t) dt = \mu T J^2 \int_T^0 \kappa_1(t, \phi) \kappa_1(t, \phi) [\delta_+^1(t) + \delta_-(t-T)] dt$$

Therefore, taking into account (7.66) and (7.73)

¹ Note that in the considered system the simple single-impact regimes may coexist with the complex multiple-impact regimes.

$$E_1(J, \varphi) = -\mu T J^2 \int_0^T \left[\chi(t - \varphi) - \frac{\mu T \chi(t) \chi(\varphi)}{1 + \mu T \chi(0)} \right] [\delta_+(t) + \delta_-(t - T)] dt$$

Integration with the functions $\delta_+(t)$ and $\delta_-(t)$ which are normalised by the value of $1/2$, $\chi(-\varphi) = \chi(\varphi)$, $\chi_\pm(-\varphi) = -\chi_\pm(\varphi)$, $\chi_\pm(0) = \pm 1/2$, yields, after simplifications

$$E_1(J, \varphi) = \mu T J^2 \frac{1 + \mu T \chi(0)}{\chi(\varphi) \chi(\varphi)}$$

where $\chi(\varphi) = [\dot{\chi}_+(\varphi) + \dot{\chi}_-(\varphi)]/2$.

For $\varphi = kT$, $k \in \mathbb{Z}$, the derivative $\dot{\chi}(\varphi)$ is to be additionally defined as zero. The physical sense of this additional definition will be discussed later. Taking into account the above rearrangements the energy balance equation takes the final form

$$(7.74) \quad \frac{\mu T \chi(\varphi) \dot{\chi}(\varphi)}{1 - R} - \frac{1 + \mu T \chi(0)}{2(1 + R)} = 0$$

The above equation may be rewritten as

$$\alpha(\varphi) = \frac{1 - R}{2(1 + R)} \frac{\mu T}{\chi(0)}$$

where $\alpha(\varphi) \equiv \chi(\varphi) \dot{\chi}(\varphi)$. In accordance with the definition of the PGF $\chi(t)$ we find

$$\alpha(\varphi) = -\frac{4\sqrt{1-\mu}}{1} \frac{\sin^2\left(\sqrt{1-\mu}T/2\right)}{\sin\left[2\sqrt{1-\mu}(\varphi-T/2)\right]}, \quad 0 < \varphi < T$$

and $\alpha(\varphi) = 0$ for $\varphi = 0; T$. Such an additional definition is warranted since the appearance of discontinuities of the phase function is due to the application of the hypothesis of momentary energy supply into the system. At any other "prelimit" form of the parametric term, the phase function is continuous and $\alpha(kT) = 0$. The above additional definition means that when $\varphi = kT$, energy is supplied to the system and dissipated by the system simultaneously. Note that using such an additional definition Eq. (7.74) may be obtained using the impact condition $J = (1 + R)\dot{\chi}_-(\varphi)$. In fact differentiating (7.69), taking into account the above condition and $\dot{\chi}_-(0) = -1/2$, yields

$$\frac{2}{1} \frac{1 + R}{1} = -\frac{\mu T \dot{\chi}_-(\varphi) \chi(\varphi)}{1 + \mu T \chi(0)}$$

The formal additional definition of this equation by the zero at $\varphi = kT$, $k \in \mathbb{Z}$ yields

(7.74) once again. The above-obtained function $\alpha(\varphi)$ yields the existence condition of the periodic regimes

$$(7.75) \quad \left| 2\sqrt{1-\mu} \frac{1-R}{1} \left(\frac{\mu T}{1} + \frac{2\sqrt{1-\mu}}{1} \cot \frac{\sqrt{1-\mu}T}{2} \right) \sin^2 \frac{\sqrt{1-\mu}T}{2} \right| \leq 1$$

The possible plots of the phase functions are shown in Fig. 7.10a for $\Delta > 0$ and Fig. 7.10b for $\Delta < 0$. It is seen that for $\Delta > 0$ two pairs of regimes are possible in the system.

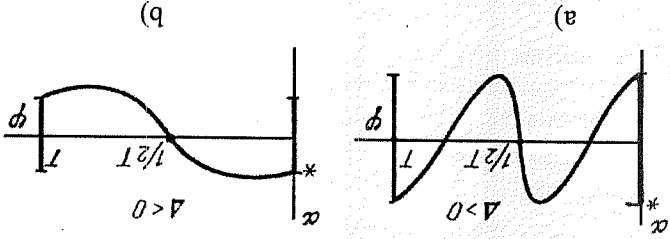


Fig. 7.10

The instability condition (7.25), accounting for (7.73), takes the form

$$\frac{dE}{dE_1} = \frac{dJ}{dE_2} - \frac{dJ}{dE_2} > 0$$

By virtue of (7.74), $E_J = 0$. Further,

$$E_\varphi = \frac{J^2 \mu T \alpha'(\varphi)}{1 + \mu T \chi(0)}, \quad J_\varphi = -\frac{\left\{ \chi(0) + \mu T [\chi_2'(0) - \chi_2''(0)] \right\}^2}{2\Delta[1 + \mu T \chi(0)] \alpha(\varphi)}$$

Hence, (7.25) takes the form

$$(7.76) \quad A_2(\varphi, J) \alpha'(\varphi) \frac{\Delta \alpha(\varphi)}{\alpha(\varphi)} > 0$$

thus revealing unstable regimes.

For an illustration of the obtained result, we will plot the dependencies $J(\omega)$ for $\Delta > 0$ based on the second formula of (7.72). For the undamped system the two stationary values of the phase $\varphi_1 = 0$ and $\varphi_2 = 1/2 T$ yield two values of the impulse

$$J_1(\omega) = -2\Delta\sqrt{1-\mu} \tan \frac{\omega}{2\pi\mu\Delta}, \quad J_2(\omega) = -2\Delta\sqrt{1-\mu} \tan \left(\frac{\omega}{\sqrt{1-\mu}\pi} + \beta \right), \quad \beta \in \left[0, \frac{\pi}{2} \right]$$

8.1. Preliminary Considerations

In this paragraph we consider the vibration of the strongly nonlinear SDOF systems under force or parametric random excitation. We suppose that the reader is familiar with the theory of random processes in the framework, for example, of the monograph by Sveshnikov [169] (see also monograph by Svetitskiy [170]).¹ In the monographs by Bolotin [27] and by Dimentberg [37] the methods of statistical dynamics are exposed in detail. We use some results obtained in the cited references. The main tools for our analysis are the methods of the theory of the Markov processes based on exact or approximate solutions of the Fokker-Planck-Kholmogorov equations. Some approximate methods follow the ideas of Stratonovich [164] and Khas'minskiy [72,143], in particular the averaging methods of stochastic differential equations are also used here. The present material is based on papers [13,14,17,112-114]. In monograph [17] similar systems are studied using the method of statistical linearisation. Detailed references to the problems of statistic dynamics may be found in [27,73,127,144,146,147,172,184,186].

8.2. Some Exact Solutions

We begin our analysis with an interpretation of some well-known concepts for the characteristics of random processes describing the dynamics of nonlinear oscillatory systems. Let the external force applied to such a system be represented by the δ -correlated random process (white noise) $\xi_0(t)$ of the intensity S_0 . For an analysis of the equation of motion

$$\ddot{x} + \Omega^2 x + \lambda \Phi(x) + 2bx = \xi_0(t) \quad (8.1)$$

we can use the methods of the theory of diffusive Markov processes [27,37,172], because the process described by the vector of the phase coordinates $\{x(t), \dot{x}(t)\} \equiv y(t)$ is completely defined for $t \geq t_0$ by the vector $\{x(t_0), y(t_0)\}$ for all t_0 . The behaviour of the random process $\{x(t), \dot{x}(t)\}$ may be characterised by the

where β is obtained from the equality $\tan \beta = \mu\pi / (\omega\sqrt{1-\mu})$. The dependencies $J_{1,2}(\omega)$ define the boundaries of all resonance curves for $R < 1$. The asymptotic values for the functions mentioned above are the straight lines $\omega_{1*} = 2\sqrt{1-\mu}$ and $\omega_{2*} = 2\sqrt{1-\mu}/(\pi-2\beta)$. In Fig. 7.11 the obtained dependencies are sketched using the dot-and-dash curves. The possible shapes of the resonant curves for $R < 1$ are represented here for two cases indicating the possibility and impossibility of frequency pulling up to infinite values of impulses. The most intensive regimes in the close vicinity of the frequency $\omega = 2\sqrt{1-\mu}$ exist (as follows from (7.75)) if

$$\frac{1}{\pi\mu} \frac{21-\mu}{1-R} \geq \frac{1+R}{1-\mu} \quad (\text{compare with (7.6) for small } \mu).$$

As $\mu \rightarrow 1$ the resonance frequency $\omega_{1*} \rightarrow 0$, and the resonant frequency $\omega_{2*} \rightarrow \infty$. For $\mu \geq 1$ the structure of the system changes, the corresponding PPGF must be reconfigured.

In the considered system, the phase $\phi_1 = 0$ if the damping is lower than the level signified by the asterisk in Fig. 7.10b. The subsequent increase in the value of damping results in the increase in the value of the phase up to the existence boundary of the regime corresponding to the maximum of the phase function $\alpha(\phi)$. In our case $\alpha(\phi) > 0$ and $\Delta < 0$. Hence, the unstable regimes are defined by the increasing branches of the phase function. The considered regimes must be analysed with respect to the fulfilment of geometrical conditions. Systems that are more complex can be analysed by means of numerical procedures only.

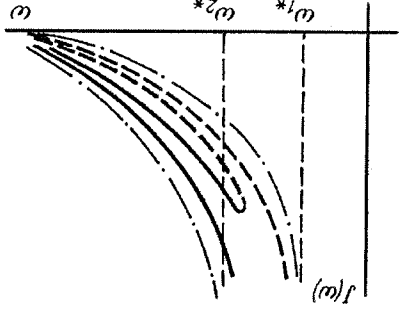


Fig. 7.11

¹ For texts in English covering similar material, the reader may consult Clarke AB, Disney RL, Probability and random processes for engineers, Wiley, 1970, or O'Flynn M, Probabilities, random variables and random processes, Harper & Row, 1982.

stationary joint probability density $P(x, y)$ satisfying the Fokker-Planck-Kholmogorov (FPK) equation,

$$y \frac{\partial P(x, y)}{\partial y} = \frac{\partial}{\partial y} \left[\left[\Omega^2 x + \lambda \Phi(x) + 2by \right] P(x, y) \right] + \frac{1}{2} S_0 \frac{\partial^2 P(x, y)}{\partial y^2}$$

The exact solution to this equation is known [172] to be

$$P(x, y) = C \exp \left[-\frac{4b}{S_0} \left(\frac{y^2}{2} + \Pi(x) \right) \right] \quad (8.2)$$

where $\Pi(x) = \frac{1}{2} x^2 \Omega^2 + \lambda \int_0^x \Phi(x_0) dx_0$ is the potential energy of the system and the constant C is defined from the normalisation condition

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} P(x, y) dx dy = 1$$

Function (8.2) satisfies the boundary conditions

$$P(-\infty, y) = P(\infty, y) = 0, \quad P(x, -\infty) = P(x, \infty) = 0$$

This reflects the fact that the components of the process $\{x, y\}$ take large values with

small probability.

If the damping in a system tends to zero, or intensity S_0 tends to infinity, infinite energy is developing. If the damping tends to infinity or the intensity S_0 tends to zero,

$P(x, y) \rightarrow \delta(x)\delta(y)$, i.e. the process becomes purely deterministic. Solution (8.2) may be rewritten using the potential function for strong nonlinearity

$$\lambda U(x) = \lambda \int_x^0 \Phi(x_0) dx_0$$

$$P(x, y) = C \exp \left[-\frac{S_0}{2b} y^2 \right] \exp \left[-\frac{S_0}{4b} \left(\Omega^2 x^2 + \lambda U(x) \right) \right]$$

Let $\Phi(x)$ be an asymmetric threshold force. Then $U(x) \equiv 0$ for $x \leq \Delta$, and instead of

the latter equality we find

$$P(x, y) = C \exp \left[-\frac{S_0}{2b} y^2 \right] \exp \left[-\frac{S_0}{4b\lambda} \Omega^2 x^2 \right] \exp \left[-\frac{S_0}{4b\lambda} U(x) \eta(x - \Delta) \right] \quad (8.3)$$

where $\eta(x)$ is the Heaviside function.

Therefore, the considered distribution is *normal* with respect to the velocity y ,

$$P(y) = \left\{ C \int_{-\infty}^{\infty} \exp \left[-\frac{S_0}{4b} \Pi(x) \right] dx \right\} \exp \left(-\frac{S_0}{2b} y^2 \right)$$

where $P(y)$ is the probability density for the velocity. The above distribution $P(x)$ will be "normal" for $x \leq \Delta$ and fast decreasing ($\lambda \gg 1$) for $x \geq \Delta$. Hence, the probability of the coordinate x to occur in the zone of strong nonlinearity is small. In the case of symmetric nonlinearity, $U(x) = U(x)\eta(|x| - \Delta)$, and the distribution of the probability is symmetric. The density $P(x)$ may be found similarly to $P(y)$ by integrating the two-dimensional density $P(x, y)$,

$$P(x) = C \sqrt{\frac{\pi S_0}{2b}} \exp \left(-\frac{S_0}{2b} \Pi(x) \right) \quad \text{where} \quad \int_{-\infty}^{\infty} e^{-a^2 x^2} dx = \frac{\sqrt{\pi}}{2a}$$

The obtained relationships allow the study of problems with arbitrary strong nonlinear potential forces.

We consider in detail two examples: the asymmetric system with a piecewise linear restoring force, and the case of singular symmetric nonlinearity with $\lambda = \infty$ in Eq. (8.1) [13, 100].

Let the equation of motion (8.1) be given by

$$\ddot{x} + \Omega^2 x + \lambda(x - \Delta)\eta(x - \Delta) + 2bx = \xi_0(t) \quad (8.4)$$

Expression (8.3) for the two-dimensional density $P(x, y)$ takes the form

$$P(x, y) = C \exp \left[-\frac{S_0}{2b} y^2 \right] \exp \left\{ -\frac{S_0}{2b} \left[\Omega^2 x^2 + \lambda(x - \Delta)^2 \eta(x - \Delta) \right] \right\} \quad (8.5)$$

To calculate the constant C , the two-dimensional density $P(x, y)$ must be integrated

for all x and y . As a result, for large λ we find

$$C = \left\{ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} P(x, y) dx dy \right\}^{-1} = \frac{\pi S_0 \Omega^{-1} \left[\frac{1}{2} + \mathcal{P}(2\Omega\Delta\sqrt{b/S_0}) \right] + 1/2\pi S_0 \lambda^{-1/2}}{2b} \quad (8.6)$$

where, while integrating in the vicinity $x > \Delta$, we neglect the quantity Ω^2 in comparison with $\lambda \gg 1$. The values of the integral $\mathcal{P}(u) = (2\pi)^{-1/2} \int_u^0 \exp \left(-\frac{1}{2} \alpha^2 \right) d\alpha$

are tabulated in reference [139].

Expressions (8.5) and (8.6) define completely the desired two-dimensional probability density. The single-dimensional characteristics and other characteristics of the process may be calculated readily using the above two-dimensional probability density. Assume that, for example, we want to find the *probability* of the occurrence of the coordinate x above the threshold value $x = \Delta$. For the single-dimensional density

$P(x)$, we find, integrating (8.5) by y ,

$$P(x) = C \sqrt{\frac{\pi S_0}{2b}} \exp \left[-\frac{\pi S_0}{2b} \left(\Omega^2 x^2 + \lambda(x - \Delta)^2 \eta(x - \Delta) \right) \right]$$

The distribution function $W(x_0) = \int_{-\infty}^{\infty} P(x') dx'$, by definition, yields the probability

$P\{x \leq x_0\}$. The desired probability is given by the expression $P\{x > \Delta\} = 1 - W(\Delta)$.

The integration yields

$$P\{x > \Delta\} = 1 - C \sqrt{\frac{\pi S_0}{2b}} \left[\frac{1}{2} + \operatorname{erf} \left(2\Delta \sqrt{\frac{\pi S_0}{b}} \right) \right]$$

where for large λ the constant C is given by the formula (8.6). Finally,

$$P\{x > \Delta\} \approx \frac{1}{2} \Omega \left[\frac{1}{2} + \operatorname{erf} \left(2\Delta \sqrt{\frac{\pi S_0}{b}} \right) \right] \lambda^{-1/2}$$

and the desired probability is $O(1/\sqrt{\lambda})$.

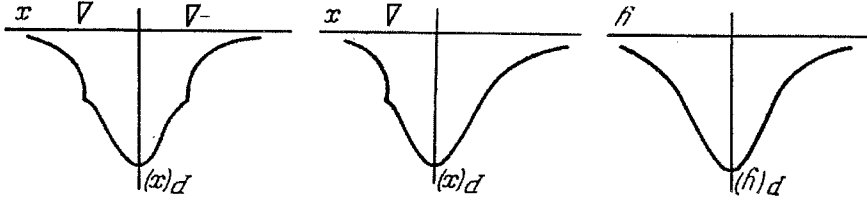


Fig. 8.1

In Fig. 8.1a the plot of the single-dimensional probability density $P(y)$ is sketched. It is similar in the symmetric and asymmetric cases (the velocities are distributed

normally). The plots of the probability densities $P(x)$ for the asymmetric case for

$\Delta > 0$ and for the symmetric case are plotted in Fig. 8.1b,c respectively. The expressions for $P(x)$ are qualitatively the same for other types of threshold

forces, for example $\Phi(x) = (x - \Delta)^\alpha \eta(x - \Delta)$, where $\alpha > 1$. Hence, it is reasonable for

large λ to consider the singular approach with $\lambda \rightarrow \infty$. In this case the potential

$U(x)$ takes the form of a barrier whose height equals the kinetic energy of the system

(see §4). The probability of the coordinate x to occur above the threshold value is

zero.

Consider in detail the symmetric system

$$\ddot{x} + \Omega^2 x + 2bx + \Phi_0^*(x, x) = \xi_0(t) \quad (8.7)$$

where notation are defined above, and $\Phi_0^*(x, x)$ is the impact force.

The desired function of the two-dimensional probability density satisfies the boundary

conditions

$$P(x, -\infty) = P(x, \infty) = 0$$

$$P(\Delta, y) = P(\Delta, -y), \quad P(-\Delta, y) = P(-\Delta, -y)$$

where the latter pair of conditions follows directly from the conditions of elastic

impact:

$$\dot{x}_+ = -\dot{x}_- \text{ at } |x| = \Delta, \quad |x| \leq \Delta$$

From (8.5) as $\lambda \rightarrow \infty$ it follows that for $|x| \leq \Delta$

$$P(x, y) = C \exp \left[-\frac{S_0}{2b} (y^2 + \Omega^2 x^2) \right] \quad (8.8)$$

The above solution satisfies the FPK equation if $|x| < \Delta$ and the considered boundary

conditions at the boundary imply $|x| = \Delta$.

The constant C in this case is defined from the equality

$$1 = C \int_{-\Delta}^{\Delta} \exp \left[-\frac{S_0}{2b} (\Omega^2 x^2 + y^2) \right] dy dx = \frac{\pi S_0}{2b} \operatorname{erf} \left(2\Delta \Omega \sqrt{\frac{S_0}{b}} \right) C$$

Hence

$$C = \frac{2b}{\pi S_0 \operatorname{erf} \left(2\Delta \Omega \sqrt{\frac{S_0}{b}} \right)}$$

Note, that in accordance with well-known solutions, the variance of the process in a system without limits takes the form [187]

$$\sigma_0^2 = \frac{4b\Omega^2}{S_0}$$

The latter equality may be rewritten as

$$C = \frac{4\pi S_0^2 \mathcal{P}(\Delta/\sigma_0)}{1} \quad (8.9)$$

Therefore, the desired two-dimensional probability density is defined. From the corresponding integration, the single-dimensioned probability densities may be obtained. $P(y)$ is represented by a "normal distribution", and $P(x)$ is represented by a "truncated normal distribution" (Fig. 8.1d, e correspond to the cases of asymmetric and symmetric systems, respectively). The infinite increase in the value of the clearance, $\Delta \rightarrow \infty$, results in the transition to the linear problem where the density $P(x)$ is represented by the "normal" curve.

We will apply the two-dimensional probability density for the solution obtained to a few problems where definition of the characteristics of the process $\{x(t), y(t)\}$ is required.

Assume that, for example, we want to obtain the variance of the coordinate and velocity. We find

$$\sigma_x^2 = C \int_{-\Delta}^{\Delta} x^2 \exp\left(-\frac{2bS_0^2 x^2}{S_0}\right) dx \int_{-\infty}^{\infty} \exp\left(-\frac{2by^2}{S_0}\right) dy = \sigma_y^2 \left[1 - \frac{\Delta \exp\left[-\Delta^2/(2\sigma_0^2)\right] \sqrt{2\pi\sigma_0^2} \mathcal{P}(\Delta/\sigma_0)}{2\sigma_0^2} \right]$$

In the dynamic analysis of machines, the characteristics of the impact impulse $J = 2|y_-|$ are very important. The statistical moments of this random quantity may be obtained by integrating (8.8) for $|x| \leq \Delta$. The density $P(y)$ takes the form (here the

$$P(y) = \sqrt{\frac{\pi S_0}{2b}} \exp\left(-\frac{\pi S_0}{2b} y^2\right) \quad \left(-\frac{S_0}{2b} y^2\right)$$

By virtue of the stationarity of the process, the randomly distributed quantity y_- possesses the same distribution. Using the general rule for the calculation of the probability density of a deterministic function of a randomly distributed argument [169] we find for the density $P(J)$

$$P(J) = \sqrt{\frac{\pi S_0}{2b}} \exp\left(-\frac{\pi S_0}{2b} J^2\right), \quad J \geq 0$$

Therefore, the values of the impulse are distributed in accordance with a truncated normal probability density function ($J \geq 0$).

For the calculation of the ordinary odd moments of the random quantity J we find that

$$m_{(2n+1)}^{J_f} = \int_{-\infty}^0 J^{2n+1} P(J) dJ$$

or, using the tables [56],

$$m_{(2n+1)}^{J_f} = \frac{\sqrt{2b}}{n!} \left[\frac{\pi S_0}{2} \right]^{n+1} 2 \left[b/(2S_0) \right]^{n+1}$$

In particular, when $n = 0$ we find the mean of the distribution

$$m_{J_f} = \sqrt{\frac{2b}{\pi S_0}}$$

where the uppercase index is omitted for simplicity. For the even ordinary moments we find similarly

$$m_{(2n)}^{J_f} = \frac{b^n}{(2n-1)! S_0^n}$$

In particular, when $n = 1$

$$m_{(2)}^{J_f} = \frac{b}{S_0} = 4\Omega^2 \sigma_0^2$$

The above result may be obtained immediately if we consider b and $J = 2|y_-|$.

The centered moments are obtained similarly. In particular, for the variance of the impulse we find

$$\sigma_{J_f}^2 = m_{(2)}^{J_f} - m_{J_f}^2 = S_0 b^{-1} (1 - 2\pi^{-1}) \approx 0.365 S_0 b^{-1}$$

Now, we will calculate the next important characteristics of the considered process: the ratio of the averaged "frequency of impacts" to the averaged "frequency of process". From [169], the averaged frequency ω_a of *overcrossing* by the coordinate $x(t)$ an arbitrary level $x = a$ and the two-dimensional density $P(x, t)$ of the process are related as

$$\omega_a = 2\pi \int_{-\infty}^0 y P(a, y) dy$$

It is feasible to define the "averaged frequency" of the process $x(t)$ as a mean value of the frequency of the *overcrossing* by a depicting point the level $a = 0$ when $x = y > 0$, i.e.

$$\omega_0 = 2\pi \int_{-\infty}^0 y P(0, y) dy$$

(8.10)

The reciprocal quantity to ω_0 , therefore, represents the averaged period of a process. Similarly, the averaged frequency of impacts is

$$\omega_{\Delta} = 2\pi \int_{-\infty}^0 y P(\Delta, y) dy$$

(8.11)

8.3. Random Vibration in Self-sustained System with Small Clearance

Systems allowing the exact solution to the FPK equation are rare. Hence, such systems may be treated as references for the estimation of the accuracy of approximate solutions. Consider the self-sustaining system which was studied in Sect. 6.6 in the deterministic approach. The study will be based on the FPK equation allowing the simple form of a solution.

Let the symmetric vibro-impact system with "cubic friction" be subjected to external wideband random excitation of white noise and let it oscillate within a small clearance. The differential equation and the conditions of elastic impact take the form

$$\ddot{x} - \alpha \dot{x} + \beta x^3 = \xi_0(t), \quad x_- = -x_+, \quad |x| \leq \Delta \quad (8.13)$$

where $\alpha, \beta > 0$. Here we neglect the potential force $\Omega^2 x$ since the clearance is small. This system was analysed in Sect. 6.6 when $\xi_0(t) \equiv 0$. Two stationary values of the impact impulse were obtained,

$$J_1^0 = 0, \quad J_2^0 = 2\sqrt{\alpha/\beta} \quad (8.14)$$

where the value J_1^0 corresponds to an unstable regime. It was noticed that the obtained solution was exact. The procedure for obtaining the simple and exact solution was based on the possibility of decreasing the order of the differential equation of motion. The same method will be used for obtaining the exact solutions for the system under stochastic excitation.

Introduce the new variable (energy) $E = \frac{1}{2} \dot{x}^2$. Then the impact condition $x_- = -x_+$ holds automatically: $x_-^2 = x_+^2$. The multiplication of Eq. (8.13) by $\dot{x}$ yields

$$E - 2\alpha E + 4\beta E^2 = \sqrt{2E} \xi_0(t) \quad (8.15)$$

The stationary FPK equation for (8.15) takes the form [164]

$$S_0 \frac{dP}{dE} [EP(E)] = \left[2\alpha E - 4\beta E^2 + \frac{1}{2} S_0 \right] P(E) \quad (8.16)$$

where $P(E)$ is the desired stationary probability density of energy. Integration of (8.16) yields

$$P(E) = \frac{2C}{\sqrt{2E}} \exp \left(\frac{2\alpha}{S_0} E - \frac{2\beta}{S_0} E^2 \right), \quad E > 0 \quad (8.17)$$

The constant C is defined from the normalisation condition

$$\int_0^\infty P(E) dE = 1$$

Taking into account [56]

and the quantity inversely proportional to ω_Δ yields the averaged time between impacts. In the case of a double-sided obstacle the frequency ω_Δ must be doubled. The substitution of (8.8) and (8.9) into the obtained equations yields

$$\omega_0 = \Omega \left[2\mathcal{P}(\Delta/\sigma_0) \right]^{-1}, \quad \omega_\Delta = \frac{\Omega}{2} \left(\frac{\Delta/\sigma_0}{\Delta^2} \exp \left(-\frac{2\sigma_0^2}{\Delta^2} \right) \right) \quad (8.12)$$

For the ratio of the above frequencies we find

$$\beta = \frac{\omega_0}{\omega_\Delta} = \exp \left(-\frac{\Delta^2}{2\sigma_0^2} \right) \quad (8.12)$$

Since the system approaches the linear case as $\Delta \rightarrow \infty$ this ratio increases infinitely. The value of β in Fig. 8.2a represents the ratio of the averaged time between impacts (low-probable for large Δ) to the averaged "period of a process".

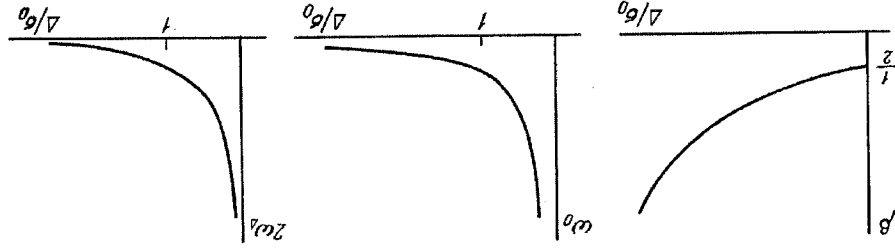


Fig. 8.2

The dependencies $\omega_0 \left(\frac{\sigma_0}{\Delta} \right)$ and $2\omega_\Delta \left(\frac{\sigma_0}{\Delta} \right)$ are shown in Fig. 8.2b,c. It is seen that an increase in the value of Δ results in a decrease in the values of the frequencies of the process and the impacts.

An increase of the clearance reduces the probability of the occurrence of impacts. However, when $\frac{\sigma_0}{\Delta} > 1$ the influence of further increasing the clearance as a method of preventing an impact against the stops is significantly lower.

When $\Delta \approx 0$, $\beta \approx 1$, and the "frequency" of impacts approaches the "frequency" of the process.

In Sect. 6.6 we analysed the vibration of the self-sustained system with a small clearance. We revealed that such a system is not "protected" from the fluctuating factors. It will be worth analysing this problem in the stochastic approach.

$$\int_0^\infty x^{\nu-1} \exp^{-\beta x^2 - \gamma x} dx = (2\beta)^{-\nu/2} \Gamma(\nu) \exp\left(\frac{\gamma^2}{8\beta}\right) D^{-\nu}\left(\frac{\gamma}{\sqrt{2\beta}}\right) \quad (8.18)$$

where $\Gamma(\nu)$ is the Euler Γ -function, and $D^{-\nu}(z)$ is the function of parabolic cylinder

[97], we find

$$1 = \sqrt{2} C \int_0^\infty E^{-1/2} \exp\left(-\frac{S_0}{2\beta} E^2 + \frac{S_0}{2\alpha} E\right) dE \\ = \sqrt{2} C \left(\frac{4\beta}{S_0}\right)^{-1/4} \Gamma\left(\frac{1}{2}\right) \exp\left(\frac{\alpha^2}{4\beta S_0}\right) D^{-1/2}\left(-\frac{\alpha}{\sqrt{\beta S_0}}\right)$$

Hence, because $\Gamma(1/2) = \sqrt{\pi}$,

$$C = \frac{\sqrt{2\pi} D^{-1/2}\left(-\alpha/\sqrt{\beta S_0}\right)}{(4\beta/S_0)^{1/4} \exp\left[-\alpha^2/(4\beta S_0)\right]} \quad (8.19)$$

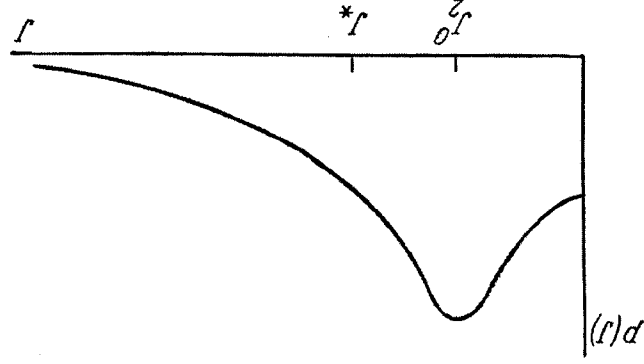


Fig. 8.3

Therefore, the probability density distribution of the energy is defined. The full representation for the probability density of the impulse $P(J)$ takes the form

$$P(J) = C \exp\left(\frac{\alpha J^2}{4S_0} - \frac{\beta J^4}{32S_0}\right), \quad J \geq 0$$

in accordance with the general rule for the calculation of the probability density of deterministic functions of random processes [169] ($J = 2|x|$, $E = 1/8 J^2$).

The function $P(J)$ is plotted in Fig. 8.3. The following points are marked: $J_0^2 = 2\sqrt{\alpha\beta^{-1}}$ is the distribution mode (the derivative of the exponential quantity is

zero); $J_* = 2\sqrt{2\alpha\beta^{-1}}$ (the exponential quantity is zero). It is seen that the distribution

mode corresponds to the stable stationary value of the impulse. For the following calculations it is more convenient to use the distribution in the form

of (8.17).

We will find the mean value of the energy. Equation (8.17) yields

$$m_E = \int_0^\infty E P(E) dE = \sqrt{2} C \int_0^\infty E^{1/2} \exp\left(\frac{S_0}{2\alpha} E - \frac{S_0}{2\beta} E^2\right) dE$$

Accounting for (8.18) and (8.19) and $\Gamma(3/2) = \Gamma(1+1/2) = 1/2 \Gamma(1/2) = 1/2 \sqrt{\pi}$, we find

$$m_E = \frac{1}{2} \sqrt{\frac{S_0}{\beta}} \frac{D^{-3/2}\left(-\alpha/\sqrt{\beta S_0}\right)}{D^{-1/2}\left(-\alpha/\sqrt{\beta S_0}\right)} \quad (8.20)$$

Similarly, using the formulae

$$\mu_n^E = \int_0^\infty (E - m_E)^n P(E) dE, \quad m_n^E = \int_0^\infty E^n P(E) dE$$

other centred or ordinary moments of the stationary process $E(t)$ may be derived. The corresponding calculations are carried out using integral (8.18).

We will analyse the expression for the mean value of the energy (8.20).

Let the quantities α and β be as small as $O(\varepsilon)$, and the intensity of the excitation is $S_0 = O(\varepsilon^2)$. Then the arguments of the functions $D_p(-\alpha/\sqrt{\beta S_0})$ are negative and

large in absolute value: $O(\alpha/\sqrt{\beta S_0}) = (\varepsilon)^{-1/2}$.

For simplification of (8.20), we will use the asymptotic expansion for the functions of a parabolic cylinder [128]. In the Whittaker nomenclature [192] it takes the form

$$D_p(x) = \text{const } x^{-p-1} \exp(\gamma^2 x^2) + \dots, \quad x \rightarrow -\infty$$

In the numerator of (8.20) we have $p = -3/2$, and in the denominator $p = -1$.

Therefore as $\alpha, \beta = O(\varepsilon)$ and $S_0 = O(\varepsilon^2)$

$$D_{-1/2}\left(-\alpha/\sqrt{\beta S_0}\right) \sim \exp(\gamma^2 \alpha^2 / \beta S_0) (\alpha/\sqrt{\beta S_0})^{1/2}, \quad \varepsilon \rightarrow 0 \\ D_{-3/2}\left(-\alpha/\sqrt{\beta S_0}\right) \sim \exp(\gamma^2 \alpha^2 / \beta S_0) (\alpha/\sqrt{\beta S_0})^{-1/2}, \quad \varepsilon \rightarrow 0$$

and, therefore, for small ε

$$m_E = (\alpha\beta^{-1})^{\text{const}+\dots}$$

(8.21)

This result is in full agreement with the expression for the stable stationary value of the impulse obtained in Sect. 6.6 using the averaging method

$$J_0^2 = 2(\alpha/\beta)^{1/2}, \quad E = 1/8 J_0^2$$

Therefore, for $S_0 = O(\varepsilon^2)$ this problem may be considered in a deterministic approach. If the level of the fluctuations increases (the value S_0 is not small, for example) accounting for the *fluctuation correction* to the value of (8.21) is required. To clarify the structure of the above correction, put $\alpha, \beta = O(\varepsilon)$ and leave the order of S_0 unspecified. Then the arguments of the functions $D_p(x)$ from (8.20) will be of order $\sqrt{\varepsilon}$. Representing the above functions obtained under the assumption that ε is small, we keep the first terms of the Taylor series only. Therefore, we find that

$$m_E = \frac{1}{\sqrt{S_0}} \frac{A_{3/2} \alpha(S_0 \beta)^{-1/2} - B_{3/2} \alpha(S_0 \beta)^{-1/2}}{\beta \frac{A_{1/2}}{4} - B_{1/2} \alpha(S_0 \beta)^{-1/2}}$$

where $A_{3/2}, B_{3/2}$ are the coefficients of the series, $\alpha(S_0 \beta)^{-1/2} = O(\varepsilon^{1/2})$. Then

$$m_E = \left[\frac{A_{3/2}}{4} + \left(\frac{A_{3/2} B_{1/2}}{B_{3/2}} - \frac{A_{1/2}}{4} \right) \frac{\sqrt{S_0} \beta}{\alpha} \right] \frac{1}{\sqrt{S_0}} \frac{1}{\beta} + O(\varepsilon)$$

or, finally, within the accuracy of the terms $O(\varepsilon)$,

$$m_E = C_1 \sqrt{\frac{S_0}{\beta}} + C_2 \frac{\beta}{\alpha}; \quad C_1, C_2 = \text{const} \quad (8.22)$$

where the sense of notation is evident. The value $C_1 \sqrt{\frac{S_0}{\beta}}$ defines the desired correction. Since $\beta = O(\varepsilon)$, the first term in (8.22) is dominant when $S_0 \sim 1$.

To obtain the values of the coefficients C_1 and C_2 we have to calculate the values of

$A_{3/2}$, etc. We find [97] that

$$A_{3/2} = D_{-3/2}(0) = \frac{2^{-3/4} \Gamma(1/2)}{dD_{-3/2}(0)} = \frac{\Gamma(5/4)}{\Gamma(3/4)}, \quad B_{3/2} = \frac{dD_{-3/2}(0)}{2^{-3/4} \Gamma(-1/2)} = \frac{\Gamma(3/4)}{2^{-3/4} \Gamma(-1/2)}$$

$$A_{1/2} = D_{-1/2}(0) = \frac{2^{-1/4} \Gamma(1/2)}{dD_{-1/2}(0)} = \frac{\Gamma(3/4)}{2^{-1/4} \Gamma(-1/2)}, \quad B_{1/2} = \frac{dD_{-1/2}(0)}{2^{-1/4} \Gamma(-1/2)} = \frac{\Gamma(1/4)}{\Gamma(-1/2)}$$

Hence,

$$C_1 = \frac{\Gamma(3/4)}{4\sqrt{2} \Gamma(5/4)}, \quad C_2 = \frac{\Gamma(-1/2) \Gamma^2(3/4)}{8\Gamma(5/4) \Gamma(1/4) \Gamma(1/2)} - \frac{8\Gamma(1/2)}{\Gamma(-1/2)}$$

The values C_1 and C_2 may be obtained from the above by using the tables of Γ -functions [81] or assessed approximately. Accounting for the features of the Γ -functions [97] we find

8.4. Contact Damping

Our involvement with the exact solutions is over. In our further representation, the approximate methods will be used. It is evident that the area of their application is much wider.

$$C_1 \approx 1/4, \quad C_2 \approx 1/8$$

approached as

yields the approximation $\Gamma^4(1/4) \approx (16\pi^2)^{1/4}$. The constants C_1 and C_2 may be

$$\Gamma^4(1/4) = 16\pi^2 \prod_{k=1}^{\infty} \frac{(4k-1)^2 [(4k+1)^2 - 1]}{(4k+1)^2 [(4k-1)^2 - 1]}$$

Accounting for equality [56]

$$C_1 = \frac{\pi}{\Gamma^2(1/4)}, \quad C_2 = \frac{1}{2\pi^2} - \frac{4}{\Gamma^4(1/4)}$$

Thus far, the interaction of a body with an elastic obstacle was considered as a fully elastic process. Now we will consider the problem of accounting for the energy losses at inelastic interactions. In the case of momentary impact, these losses may be considered using a restitution ratio. The dynamics of the inelastic vibro-impact system under white noise excitation was analysed in detail by means of the Zhuravlev transformation in [37]. Here we consider the model problem with nonmomentary interaction. This analysis is mainly based on the results obtained by Stratonovich [164].

Let the motion be described by the equation

$$\ddot{x} + \varepsilon h(x, \dot{x}) + \Phi_1(x) = \sqrt{\varepsilon} \xi_1(t)$$

where $\xi_1(t)$ is the stationary wideband random process with normal distribution and zero mean value (white noise). Considering the total energy of the system in the form

$$E = 1/2 \dot{x}^2 + \Pi(x), \quad \Pi(x) = \int_x^0 \Phi_1(x) dx$$

as a new variable yields, for the two-dimension Markov process $\{x, E\}$,

$$\dot{x} = \pm \sqrt{2(E - \Pi)}$$

$$\dot{E} = \pm \sqrt{2(E - \Pi)} h(x, \pm \sqrt{2(E - \Pi)}) \pm \sqrt{2(E - \Pi)} \xi_1(t)$$

Averaging the second equation over the fast variable x and the solution of the corresponding FPK equation for the probability density of the energy E yields the approximate formula

$$P(E) = C_0 T_0(E) \exp \left[-\frac{2}{E} \int_E^0 \frac{M(E_1)}{N(E_1)} dE_1 \right] \quad (8.23)$$

where $C_0 = \text{const}$, and $T_0(E)$ is the period of motion of the conservative ($\varepsilon = 0$) system,

$$M(E) = \int_{x_2(E)}^{x_1(E)} h(x) \sqrt{2[E - \Pi(x)]} dx + \int_{x_2(E)}^{x_1(E)} h(x) \sqrt{2[E - \Pi(x)]} dx$$

$$N(E) = 2 \int_{x_2(E)}^{x_1(E)} \sqrt{2[E - \Pi(x)]} dx$$

where $x_{1,2}(E)$ are the "halt points" where $\Pi(x_{1,2}) = E$ (see §4) and

$$S_1 = \int_{-\infty}^{\infty} \langle \xi_1(t) \xi_1(t + \tau) \rangle d\tau$$

is the intensity of the process $\xi_1(t)$ (the angle brackets $\langle \dots \rangle$ denote the procedure of

statistic averaging).

Relationships similar to (8.23) may be written also for the joint densities $P(y, E)$ and

$$P(x, y), \quad y \equiv \dot{x} \quad [164, 172].$$

It may be shown that if $\xi_1(t)$ is white noise, $h(x, \dot{x}) = 2bx$ and $\varepsilon = 1$, formula (8.23)

yields the exact representation.

Consider now the problem similar to that in Sect. 5.8 in a stochastic approach. The

system in Fig. 8.4 comprises the oscillator and the buffer installed with the clearance $\varepsilon\Delta$. Under wideband random excitation for the unknown coordinate z we have the

equation of motion

$$\ddot{z} + \Omega^2 z + 2b\dot{z} + \varepsilon k \operatorname{sgn} z \eta(z - \varepsilon\Delta) + \lambda(z - \varepsilon\Delta) \eta(z - \varepsilon\Delta) = \sqrt{\varepsilon} \xi_1(t)$$

$$\varepsilon = O(\lambda^{-1/2})$$

where $k = O(\varepsilon^{-1})$ due to the fact that in a contact zone the damping force arises for a

short time $\propto \lambda^{-1/2}$ (compare with Sect. 5.8).

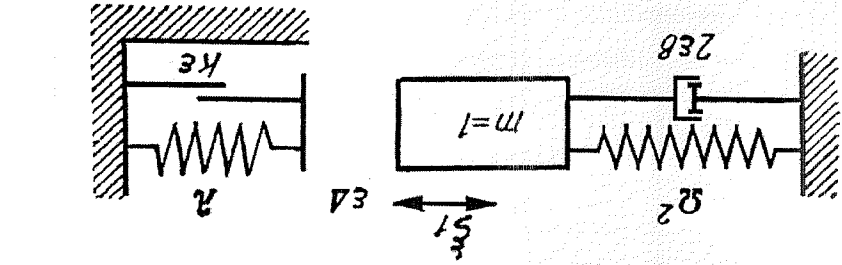


Fig. 8.4

Let $x = z - \varepsilon\Delta$. The equation of motion

$$\ddot{x} + \Omega^2 x + \varepsilon [2bx + \Delta\Omega^2] + \varepsilon k \operatorname{sgn} x \eta(x) + \lambda x \eta(x) = \sqrt{\varepsilon} \xi_1(t) \quad (8.24)$$

when $\varepsilon = 0$ corresponds to an isochronous system because there is only one possible

frequency of vibration given by the expression

$$\Omega_\lambda = 2\Omega \frac{\sqrt{\Omega^2 + \lambda}}{\sqrt{\Omega^2 + \lambda + \Omega}} \approx 2\Omega \left(1 - \frac{\sqrt{\lambda}}{\Omega} \right)$$

in accordance with (4.44).

Define the probability density of the energy $P(E)$ using (8.23). The functions $M(E)$

and $N(E)$ may be calculated as follows. We have

$$x_1(E) = -\Omega^{-1} \sqrt{2E}, \quad x_2(E) = (\Omega^2 + \lambda)^{-1/2} \sqrt{2E}, \quad h(x, \dot{x}) = (2bx + \Delta\Omega^2) + k \operatorname{sgn} x \eta(x)$$

It may be found easily that

$$M(E) = \int_{x_2(E)}^{x_1(E)} 4b \sqrt{2[E - \Pi(x)]} dx + 2kx_2(E) = 2[bN(E) + kx_2(E)]$$

$$N(E) = 2 \int_{x_2(E)}^{x_1(E)} \sqrt{2[E - \Pi(x)]} dx = \frac{2\pi E}{\Omega_\lambda}$$

because $N(E) = 2\pi I_0$, where I_0 is the action, and for an isochronous system

$E = \Omega_\lambda I_0$ [6]. Now all the quantities for (8.23) are specified, hence

$$P(E) = 2C_0 \frac{\Omega_\lambda}{\pi} \exp \left\{ -\frac{S_1}{2} \left[2bE + \frac{\pi(\Omega^2 + \lambda)}{2\sqrt{2}k\Omega_\lambda} E^{1/2} \right] \right\} \quad (8.25)$$

It may be shown [172] that the joint densities $P(x, E)$ and $P(x, y)$, $y \equiv \dot{x}$ are given

by the expressions

detailed analysis of the obtained expressions is not required. Moreover, in our future presentation different approaches allowing a more simple form for accounting energy losses at the interaction (see also reference [37]) are suggested. Now we will consider the computation methods based on the ideas of averaging stochastic equations without using the analysis methods of Markov processes.

8.5. Deviations from Solutions of Averaged Systems

A solution of the problems of nonlinear mechanics using the averaging method yields, in the first approximation, some deterministic function well approaching the exact solution on the asymptotically large or even infinite time interval. If the external excitation is small, it is reasonable to make an attempt to estimate the *fluctuation correction* to the solution obtained by the averaging method. The methods of deriving such a correction are developed by Stratonovich [164] and Khas'minskiy [72,143] (see also [184]).

In the present section we consider as an example the problem of the estimation of normal deviations from the averaged system [143]¹. The method is based on the possibility of obtaining the desired correction from some stochastic analogue of the equation in variations.

Let the system

$$\dot{z} = \varepsilon F(t, z), \quad z \in \mathbb{R}^n$$

be given, where the vector F represents the random process, satisfying the condition of *strong mixing*, which, in applications, may be understood as a requirement for the correlation time to be small. Let $z(t)$ be the solution to this equation. Let M be the mathematical expectation operator and $z^0(t)$ be the solution of the averaged system

$$\dot{z} = \varepsilon F_0(z), \quad F_0(z) = \lim_{T \rightarrow \infty} T^{-1} \int_{t_0+T}^{t_0+T+T} MF(t, z) dt$$

where t_0 is an arbitrary time instant. It is assumed also that the above limit exists uniformly in z and t_0 . After the transition to the slow time $\tau = \varepsilon t$ the solutions $z(\tau)$ and $z^0(\tau)$ are considered now on the finite interval $[0, L_0]$, $L_0 = \text{const} > 0$. It may be expected that the difference between z^0 and z will be small enough. For sufficiently smooth systems in [143] it is shown that the exact solution may be represented in the form

$$z(\tau) = z^0(\tau) + \sqrt{\varepsilon} y(\tau)$$

¹ Note that the problems considered here are beyond the scope of the theory validating the formal manipulations.

$$P(x, E) = C_1 \left[2\sqrt{E - \Pi(x)} \right]^{-1} \exp \left[-\frac{S_1}{2} Q(E) \right] \\ P(x, y) = C_2 \exp \left\{ -\frac{S_1}{2} Q \left[\frac{1}{2} y^2 + \Pi(x) \right] \right\}$$

where the constants C_1 and C_2 are defined from the normalization conditions, the

(8.25), and the potential $\Pi(x)$ corresponds to the conservative ($\varepsilon = 0$) system (8.24). With the help of the transformation $z = x + \varepsilon \Delta$, the initial coordinate may be

reconstructed

As an example of an analysis of the obtained expressions, we will find the probability density function for the half point of the peak-to-peak values of the vibration $A = 1/2 [x_2(E) - x_1(E)] = \sqrt{2E}/\Omega_\lambda$. In accordance with the general rules and taking

into account $E = 1/2 \Omega_\lambda^2 A^2$ we find from (8.25) that

$$P(A) = 2\pi C_0 \Omega_\lambda A \exp \left\{ -\frac{S_1}{2} \left[b \Omega_\lambda^2 A^2 + \frac{\pi(\Omega_\lambda^2 + \lambda)^{1/2}}{2k \Omega_\lambda^2} A \right] \right\} \quad (8.26)$$

After integration [56] with respect to $A \geq 0$ we have

$$C_0 = 2b' \left[\exp \left(\frac{k' b'}{8b'} \right) D^{-2} \left(\frac{\sqrt{2b'}}{k'} \right) 2\pi \Omega_\lambda \right]^{-1}$$

where $D_{-a}(z)$ is the function of the parabolic cylinder,

$$b' = \frac{S_1}{2b \Omega_\lambda^2}, \quad k' = \frac{S_1 \pi (\Omega_\lambda^2 + \lambda)^{1/2}}{2b \Omega_\lambda^2}$$

Similar integration yields the moments of the stationary process $A(t)$. For the ordinary moments we have

$$m_p^{(p)} = \int_0^\infty A^p P(A) dA = (2b')^{-p/2} \frac{D_{-2}(k'/\sqrt{2b'})}{D_{-p-2}(k'/\sqrt{2b'})} (p+1)!$$

Using the successive simplifications, this expression may be reduced to a more useful form (compare with Sect. 8.3). The structure of the latter formula indicates that accounting for the contact dissipation significantly complicates the basic

computational relationships, especially for nonsmall clearances. Nevertheless, for the systems mentioned above the qualitative distortions due to taking into account the contact dissipation are not very significant. The appearance of the additional addend $\propto E^{1/2}$ in the exponent in (8.25) results mainly in quantitative changes. Therefore, a

and as $\varepsilon \rightarrow 0$, the process $y^\varepsilon(\tau)$ yields the desired fluctuation correction. The above correction may be approached on the finite time interval by the function $y^0(\tau)$, which is the solution of the system of *Ito stochastic differential equations*

$$dy^0(\tau) = \frac{\partial F[z^0(\tau)]}{\partial z^0} y^0(\tau) d\tau + \sigma [x^0(\tau)] dw(\tau) \quad (8.27)$$

The correlation matrix σ may be found from the condition $\sigma\sigma^* = [A_{ij}]$, where the asterisk denotes the transposition operation,

$$A_{ij}(\mathbf{z}) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{t_0+T}^{t_0+T} ds \int_{t_0+T}^{t_0+T} dt [F_j(\mathbf{z}, t) - MF_j(\mathbf{z}, t)] [F_i(\mathbf{z}, s) - MF_i(\mathbf{z}, s)] \quad (8.28)$$

where $w(\tau)$ is the n -dimensional *Wiener process*.

In the case when the initial system is deterministic, Eq. (8.28) transforms into the equation in variations for the averaged system. The *drift factor* $\partial F_0/\partial z^0$ indicates the mean value of the fluctuation correction defined by the *diffusion factor* σ .

We will apply this method formally to the analysis of the vibro-impact system where $R=1$. This system was studied in Sect. 6.5 in the deterministic approach (see Fig. 6.5).

Let the system be subjected to the random force $\xi(t)$, defined by the centred stationary random process. The condition of strong mixing for the above process may be written with respect to the corresponding correlation function $K_\xi(\tau)$,

$$|K_\xi(\tau)| > M_0 e^{-\alpha\tau}, \quad M_0, \alpha = \text{const} > 0 \quad (8.29)$$

Consider the problem in the quasi-isochronous approximation ($\Delta = \varepsilon\Delta_1$). The equation of motion takes the form

$$\begin{aligned} \ddot{x} + \Omega^2 x + \Phi_0(x, \dot{x}) &= \varepsilon [-2bx - \Omega^2 \Delta_1 + K_0 J^{-1} \dot{x} + \xi(t)] \\ J &= -4\varepsilon [2bJ\chi(t - \phi) - K_0 \chi(t - \phi) + \xi(t) - \Omega^2 \Delta_1 [\chi(t - \phi) \\ &\quad \phi = -4\varepsilon J^{-1} [2bJ\chi(t - \phi) - K_0 \chi(t - \phi) - \Omega^2 \Delta_1 + \xi(t)] \chi(t - \phi) \end{aligned}$$

The right-hand sides of the system define the components of the vector $\mathbf{F} = \{F_1, F_2\}$ of the system in the standard form $\mathbf{z} = \varepsilon \mathbf{F}(\mathbf{z}, t)$, where $\mathbf{z} = \{z_1, z_2\}$, $z_1 = J_1$, $z_2 = \phi$. Since the generative system is isochronous, we find $T = \pi/\Omega$ and

$$\chi(t) = (2\Omega)^{-1} \sin \Omega t, \quad \chi(t) = 1/2 \cos \Omega t, \quad t \in [0, \pi/\Omega]$$

Averaging the right-hand sides of the obtained equations, with $M_\xi = 0$ yields, as in Sect. 6.5,

$$\begin{aligned} J &= -\varepsilon(bJ - 1/2 K_0) \equiv \varepsilon F_{01}(J) \\ \phi &= 2\varepsilon \Omega \Delta_1 J^{-1} \pi^{-1} \equiv \varepsilon F_{02}(J) \end{aligned} \quad (8.31)$$

The linear equation $F_{01}(J) = 0$ possesses the unique solution

$$J^0 = K_0/(2b)$$

defining the stable stationary regime with the phase $\phi^0(t) = 2\varepsilon \Delta_1 (J^0 \pi)^{-1} \Omega t$, where the frequency of self-sustained vibration is defined by the value $2\varepsilon \Delta_1 (J^0 \pi)^{-1} \Omega$ which is close to the value of 2Ω (see Sect. 6.5).

In order to define the normal deviation from the solution (J^0, ϕ^0) we refer to (8.27). The differentiation of (8.31) and transition to the slow time $\tau = \varepsilon t$ yields

$$\frac{\partial F_{01}}{\partial J^0} = -b, \quad \frac{\partial F_{02}}{\partial J^0} = -2\Omega \Delta_1 \pi^{-1} (J^0)^{-2}, \quad \frac{\partial F_{01}}{\partial \phi^0} = \frac{\partial F_{02}}{\partial \phi^0} = 0$$

The components of the correlation matrix (8.28) are calculated in accordance with the equations in a standard form, the condition $M_\xi^2(t) = 0$ and with provision for the periodicity of the PGF $\chi(t)$ and the derivative $\chi(t)$,

$$\begin{aligned} A_{11} &= \frac{J}{16} \int_0^T ds \int_0^T dt \chi(t) \chi(s) K_\xi(t-s) dt \\ A_{12} &= A_{21} = \frac{1}{16} \int_0^T ds \int_0^T dt \chi(t) \chi(s) K_\xi(t-s) dt \\ A_{22} &= \frac{1}{16} \int_0^T ds \int_0^T dt \chi(t) \chi(s) K_\xi(t-s) dt \end{aligned}$$

Hence, system (8.27) is defined. The substitution of corresponding correlation functions yields the desired solution in accordance with the known rules [27,53]. We will consider some particular cases allowing detailed analysis of the fluctuation correction to a stationary value of the impulse J^0 .

(1) Let $\xi(t)$ be white noise with intensity S_0 . Then $K_\xi(t) = 2\pi S_0 \delta(t)$, where $\delta(t)$ is the Dirac δ -function. In accordance with the form of the PGF $\chi(t)$, $A_{12} = A_{21} = 0$, and system (8.27) may be decomposed. The equation for the fluctuation correction $y_1^0(\tau)$ takes the form of the *Langevin equation*

$$\frac{dy_1^0}{d\tau} = -\gamma y_1^0 + \xi_0^0(\tau), \quad \tau = \varepsilon t \quad (8.32)$$

where $\xi_0^0(\tau)$ is white noise with intensity $4\pi S_0$, $\gamma = b$ is a decrement of the vibration of the deterministic system. The solution of the above equation yields the correlation

function of the process $y_0^1(\tau)$

$$K_y(\tau) = \sigma_y^2 \exp(-\gamma|\tau|)$$

where dispersion $\sigma_y^2 = 4\pi^2 S_0 \gamma^{-1}$.

(2) Now let $\xi(t)$ be a process which is different from white noise but with a small

correlation time, hence in (8.29), $\alpha T \ll 1$. For the fluctuation correction $y_0^1(\tau)$ the

Langvin equation with a different diffusion factor may be obtained again.

To verify this, we find the representations for the components of the matrix A_{ij} in the

form of the numerical series. For this purpose the PGF and its derivative are

represented as Fourier series

$$\chi(t) = \frac{1}{T} \sum_{k=-\infty}^{\infty} \frac{\exp(ik\omega t)}{\Omega^2 - k^2 \omega^2}; \quad \chi(t) = \frac{1}{T} \sum_{k=-\infty}^{\infty} \frac{\exp(ik\omega t)}{\Omega^2 - k^2 \omega^2}$$

For the coefficient A_{12} after integration with respect to t we find

$$\int_{-\infty}^{\infty} \chi(t) K_{\xi}(t-s) dt = \frac{1}{T} \sum_{k=-\infty}^{\infty} \frac{\exp(ik\omega s)}{\Omega^2 - k^2 \omega^2}$$

where the spectral density for frequency $k\omega = 2k\Omega$ is

$$S(k\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} K_{\xi}(\tau) e^{ik\omega \tau} d\tau$$

The substitution of the obtained series into the formula for A_{12} and integration with

respect to s yields

$$A_{12} = \frac{16}{T} \sum_{k=-\infty}^{\infty} \frac{2\pi k \omega S(k\omega)}{(\Omega^2 - k^2 \omega^2)^2}$$

The above series may be summed up approximately using the method used in Sect. 4.9 for the improved singular approximation (see (4.39) and (4.40)). Take into consideration the auxiliary function

$$v_0(t) = \frac{16}{T} \sum_{k=-\infty}^{\infty} \frac{2\pi k \omega S(k\omega) \exp(ik\omega t)}{(\Omega^2 - k^2 \omega^2)^2}, \quad v_0(0) = v_0'(0) = A_{12}$$

The above function may be written in a closed form on the time interval $[0, T]$ using the calculation of the residuals of the function

$$Z(p) = \frac{p L_0(p)}{(p^2 + \Omega^2)^2}$$

where $L_0(p)$ is the Laplace transform of the correlation function. If the correlation time is small, the residuals $L_0(p)$ are disposed far from the imaginary axis and the

8.6. Quasi-resonant Regimes

the linear part of the system. Similarly, other particular cases may be analysed. The methods of calculation of fluctuation corrections may be used also in a different form (see Sect. 8.7).

Thus, the Langevin equation (8.32) for the deviation $y^0(\tau)$ contains white noise $\xi_0(\tau)$ with intensity $4\pi S(\Omega)$ and variance $\sigma_y^2 = 4\pi^2 S(\Omega) \gamma^{-1}$. The above result may be explained physically as follows. Independently of the type of wideband excitation $\xi(t)$, the fluctuations are "sensed" by the system mainly at the resonant frequency of

$$A_{11} \approx 4\pi S(\Omega)$$

Hence, we find that within the accuracy of the terms of higher order of smallness, $A_{12} = 0$. Similar reasoning yields

$$\int_0^T \chi(t) \chi(t) dt = \frac{1}{T} \sum_{k=-\infty}^{\infty} \frac{\exp(ik\omega t)}{\Omega^2 - k^2 \omega^2} = \frac{1}{8\Omega} \int_{-\pi/2}^{\pi/2} \sin 2\Omega t dt = 0, \quad \omega = 2\Omega$$

but the infinite series equals zero because

$$A_{12} = \frac{16}{T} \sum_{k=-\infty}^{\infty} \frac{2\pi k \omega S(\Omega)}{(\Omega^2 - k^2 \omega^2)^2} + \dots$$

poles $p_{1,2} = \pm i\Omega$ contribute significantly to the desired representation. Hence,

$$(8.33)$$

From the preceding analysis, at random vibration of the system with a small clearance, the frequency of collisions is high. As the clearance decreases, the probability of the occurrence of collisions of the body with the bumper (or the solid stop in the case of momentary interaction) increases. Hence, the design of vibration protection systems requires a proper increase in the rattle space Δ that must be allowed around the equipment, for example

$$|\Delta| \gg \sigma_0$$

where σ_0^2 is the variance of the process calculated for the linear system. However,

due to occasional disturbances, the system may achieve the stops and be involved in the consequent persistent motion with collisions.

Engineering analysis of such regimes involves the estimation of the conditions minimising the duration of such events. To obtain such estimates the method proposed by Kolovsky [79] is used in [17].

Let the SDOF system be subjected to the normal ergodic centred random excitation $\xi(t)$ and then the equation of motion takes the form

$$\ddot{x} + \Omega^2 x + 2b\dot{x} + \lambda \Phi(x) = \xi(t) - g(x, \dot{x}) \quad (8.34)$$

where $g(x, \dot{x})$ is the damping force at the interaction, and $\Phi(x)$ is the threshold force

with threshold Δ for which condition (8.33) is satisfied.

Imagine that, due to the occasional disturbance, a vibro-impact regime develops in the system. If the levels of excitation and damping are low, mainly inertial and elastic forces govern the motion. Hence, such a regime may be realised exclusively in accordance with the resonant law described by the solution $x_0(t)$ of the corresponding conservative problem. From Sect. 4.9, for such intensive interactions the solutions may be written in the form

$$x_0(t) \approx -J_\lambda \chi(t - \tau_\lambda)$$

where J_λ is the impulse of interaction, and $\tau_\lambda = \tau_\lambda(J_\lambda)$ is half of the interaction time. The frequency, ω , of the PGF is not defined initially. Nevertheless, it is known that the above frequency belongs to the intervals of the natural frequencies, $\Omega < \omega < 2\Omega$, for the system with clearance ($\lambda \rightarrow \infty$) and $\omega > \Omega$ ($\lambda \rightarrow \infty$) for the system with preload.

We will find the condition for sustaining the regime $x_0(t)$ using the random force $\xi(t)$ from reasonings which are physically close to the energy balance principle. Denote by N_1 the power of the force $\xi(t)$ on the motion $x_0(t)$ over the observation

time T_N

$$N_1 = \frac{1}{T_N} \int_0^{T_N} \xi(t) x_0(t) dt \quad (8.35)$$

where the integration is performed in the sense of the theory of random processes. The random value N_1 is centred, hence

$$\langle N_1 \rangle = \left\langle \frac{1}{T_N} \int_0^{T_N} \xi(t) dx_0(t) \right\rangle = \frac{1}{T_N} \int_0^{T_N} \langle \xi(t) \rangle dx_0(t) = 0$$

where the angle brackets denote the statistical averaging. The variance N_1 is expressed through the correlation function $K_\xi(\tau)$ as

$$\langle N_1^2 \rangle = \frac{1}{T_N} \int_0^{T_N} \int_0^{T_N} dt \int_0^{T_N} dx_0(t) x_0(t + \tau) K_\xi(\tau) d\tau \quad (8.36)$$

where the periodicity of $x_0(t)$ is taken into account. It was assumed that the observation time T_N is a multiple of the period $T = 2\pi\omega^{-1}$.

The value N_1 is distributed normally

$$P(N_1) = \frac{1}{\sqrt{2\pi}\sigma_N} \exp\left(-\frac{N_1^2}{2\sigma_N^2}\right)$$

where $\sigma_N^2 = \langle N_1^2 \rangle$ (see formula (8.36)).

The power of the dissipation forces over the time T_N is given by the equation

$$N_2(\omega) = \frac{1}{T_N} \int_0^{T_N} \left\{ 2bx_0(t) + g[x_0(t), \dot{x}_0(t)] \right\} dx_0(t)$$

The regime with impacts will be sustained only if the power of excitation dominates that of the damping.

The probability of the downward jump to the impactless regime is

$$P\{N_1 < N_2(\omega)\} = \int_{N_2(\omega)}^\infty P(N_1) dN_1$$

If the correlation function $K_\xi(\tau)$ decreases, i.e. $\lim_{\tau \rightarrow \infty} K_\xi(\tau) = 0$, then as $T_N \rightarrow \infty$ the

value of σ_N^2 decreases also. As a result of the above reasoning, with the desired probability we can estimate the time T_N^* during which the relationship

$$N_2(\omega) > N_1$$

does not hold, thus, the duration of the process with the impacts may be assessed. Let $\xi(t)$ be white noise and $K_\xi(\tau) = S_0 \delta(\tau)$. Then, from (8.36)

$$\sigma_N^2 = \langle N_1^2 \rangle = \frac{S_0}{T_N} \langle x_0^2(t) \rangle$$

Let $P\{N_1 < N_2(\omega)\} = p_0$. By accounting for the properties of the normal distribution the above probability may be expressed using the Laplace function [139] as

$$N_2(\omega) > n_0(p_0)\sigma_N \quad (8.37)$$

where $n_0(p_0)$ is the number depending on the desired probability: $n_0 = 2$ when $p_0 \approx 0.98$, $n_0 = 4$ when $p_0 \approx 0.99$ etc.

The comparison of two latter expressions yields the desired estimate

$$T_N^* > \frac{N_2^2(\omega)}{n_0^2(p_0)S_0\langle x_0^2(t) \rangle} \quad (8.38)$$

If the correlation function $K_\xi(\tau)$ is not tending to zero as $\tau \rightarrow \infty$, the estimates, naturally, will be different (see Sect. 8.10).

Formula (8.10) may be simplified in the case of momentary impact, $\lambda \rightarrow \infty$. We find $x_0(t) = -J\chi(t)$, $J = -\Delta\chi^{-1}(0)$. Hence, for a symmetric system

$$\langle x_0^2(t) \rangle = J^2 \frac{1}{T} \int_0^T x_0^2 dt = \frac{2\cos^2(\pi\Omega/2\omega)}{\Delta^2\Omega^2} \left[1 - \frac{2\pi\Omega}{\omega} \sin \frac{2\pi\Omega}{\omega} \right]$$

where $T = 2\pi/\omega$ and the above assumption about the duration of observation time is taken into account. Similarly,

$$N_2(\omega) = \frac{b\Delta^2\Omega^2}{\cos^2(\pi\Omega/\omega)} \left[1 - \frac{\omega}{2\pi\Omega} \sin \frac{\omega}{2\pi\Omega} \right] + \frac{\pi}{\Delta^2\Omega^2\omega} \frac{1+R}{1-R} \tan^2 \frac{\omega}{2\pi\Omega}$$

Hence, for the asymmetric system, formula (8.38) takes the form

$$T_N^* > \frac{1}{2} n_0^2(p_0) S_0 \frac{\cos^2 \frac{\pi\Omega}{2} \left(1 - \frac{\omega}{2\pi\Omega} \sin \frac{\omega}{2\pi\Omega} \right)}{\Delta^2\Omega^2 \left\{ b \left[1 - \frac{\omega}{2\pi\Omega} \sin \frac{\omega}{2\pi\Omega} \right] + \frac{\omega}{1-R} \sin^2 \frac{\omega}{2\pi\Omega} \right\}}$$

Similarly, for the symmetric system

$$T_N^* > \frac{1}{2} n_0^2(p_0) S_0 \frac{\sin^2 \frac{\pi\Omega}{2} \left(1 + \frac{\omega}{2\pi\Omega} \sin \frac{\omega}{2\pi\Omega} \right)}{\Delta^2\Omega^2 \left\{ b \left[1 + \frac{\omega}{2\pi\Omega} \sin \frac{\omega}{2\pi\Omega} \right] + \frac{\omega}{1+R} \cos^2 \frac{\omega}{2\pi\Omega} \right\}}$$

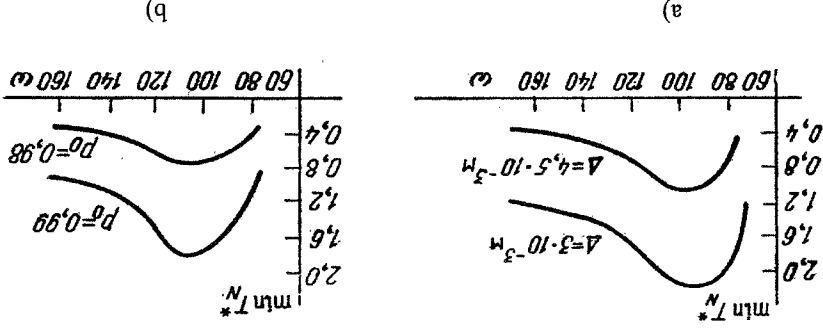


Fig. 8.5

In Fig. 8.5 the examples of the calculation of the limiting duration of the vibro-impact process in a symmetric system are illustrated. In Fig. 8.5a the desired probability is $p_0 = 0.98$ ($n_0(p_0) = 2$); in Fig. 8.5b the value of the clearance is $\Delta = 4.5 \cdot 10^{-3}$ m. The parameters of the system in both figures are: $\Omega = 62.8$ s $^{-1}$; $b = 12.5$ s $^{-1}$; $R = 0.5$; $S_0 = 2$ m 2 s $^{-3}$. The time T_N^* is measured in seconds, and the frequency ω is expressed in s $^{-1}$.
The above method gives the estimates for the duration of the vibro-impact process in the vicinity of the resonant frequencies. The frequency ω of the analysed regimes is not known a priori, because its value depends on the random factors.
The above random regimes and regular deterministic resonances are very similar. Due to this similarity, such processes are said to be *quasi-resonances*.
The estimation of duration of quasi-resonant regimes is often the aim of the

8.7. Parametric Systems in a Quasi-isochronous Approximation

In this section and in the following sections we analyse the strongly nonlinear systems with $\lambda \rightarrow \infty$ under parametric stochastic excitation.
Our analysis will be based on the calculation of the fluctuation correction to the solution of the averaged equation proposed by Stratonovich [164].
The essence of the method [72] may be formulated as follows. Let the stochastic system be given by the equation

$$\ddot{z} = \varepsilon F(z, t) + \varepsilon^2 F_1(z, t), \quad MF = 0, \quad z \in \mathbb{R}^n$$

where the functions F and F_1 represent the random processes satisfying the conditions of strong mixing and a number of additional requirements. The averaged vector moments of the processes F and F_1 are

$$\begin{aligned} F_0(z) &= \lim_{T \rightarrow \infty} \frac{1}{T} \int_{t_0+T}^{t_0+T} MF_1(z, t) dt \\ F^k(z) &= \lim_{T \rightarrow \infty} \frac{1}{T} \int_{t_0+T}^{t_0+T} ds \int_{t_0+T}^{t_0+T} dt M[F^k(z, s) F^k(z, t)] \\ K_0(z) &= \lim_{T \rightarrow \infty} \frac{1}{T} \int_{t_0+T}^{t_0+T} ds \int_{t_0+T}^{t_0+T} dt M[F^k(z, s) F^l(z, t)] \end{aligned} \quad (8.39)$$

where the uniform existence of the limits in t_0 and z is assumed, $F^{(j)}(z, s)$ denotes the j -indexed component of the process.
We expect that on the time interval $0 \leq \varepsilon^2 t < \tau_0$ the process $z(\tau)$, where $\tau = \varepsilon^2 t$, which is the solution of the initial system, as $\varepsilon \rightarrow 0$ may be approached by the solution of the stochastic equation of Ito (compare with Sect.8.5)

$$dz^0(\tau) = B[z^0(\tau)] d\tau + \sigma[z^0(\tau)] dw(\tau) \quad (8.40)$$

where $w(\tau)$ is the n -dimensional Wiener process, $B(z) = K_0(z) + F_0(z)$, and the asterisk denotes transposition.

The drift factor $B(z)$ defines, therefore, the deterministic part of the equation obtained as a result of averaging of the initial system. The fluctuation correction is given by the diffusion factor.

$$0 = \left[x_1 J^0 K - (x, x)^z \Phi - x \partial_z \Phi \right] z^2 + (x, x)^0 \Phi + x \left((t)^1 \zeta^3 + 1 \right) z \zeta + x$$

satisfying the estimation (8.29). Under the assumption that the system is close to

$$f = \{(\phi - 1)\chi_2 \mathcal{U} + (\phi - 1)\chi_2 \mathcal{V} + (\phi - 1)\chi_2 \mathcal{W}\} \mathcal{Z} =$$

$$\phi = -4\epsilon J^{-1} \left[2\epsilon b J \chi(t) - (\phi - t) \chi^0 K^{\mathcal{E}} - (\phi - t) \chi J \Omega^2 + (\phi - t) \chi \Omega^2 \nabla^2 \chi \right] \chi(t) - (\phi - t) \chi \Omega^2 \nabla^2 \chi$$

$$F_{(1)} = -4J\Omega^2 \xi_1(t) g_1(t) (\phi - \phi), \quad F_{(2)} = -4J\Omega^2 \xi_2(t) \chi_2(t) (\phi - \phi)$$

$$(\phi - 1)_L^{-1} \mathcal{G} \mathcal{H} -$$

where, accounting for the definition of the PGF, for the isochronous case

$$0 = \tau_1 v, \quad \pi(\tau_1 \tau_2)^5 S_2 \tau_2 f_1 = 11v$$

$${}^0K^z_l+r(q+\mathfrak{U}_{1-\mathfrak{u}})-={}_{(1)}^0H^{\circ},\,r\mathfrak{U}(\mathfrak{U}\mathfrak{Z})^{\mathfrak{s}}S_z\mathfrak{U}^{\mathfrak{g}}_3={}_{(1)}^0K^{\circ}$$

$$(8.41) \quad \pi^{1/2} \int_0^\infty \left[\pi^{1/2} S^{1/2} \left(\frac{1}{2} \right) + K_0 \right] \left[q + \pi^{1/2} \left(\frac{1}{2} \right) - \pi^{1/2} S^{1/2} \left(\frac{1}{2} \right) \right] = \pi p / \sigma$$

where $\xi_0(t)$ is white noise with the unit intensity. The above equation may be analysed using the methods of the theory of the Markov processes [164,172]. The corresponding FPK equation takes the form

$$\left\{ \left({}_o\mathcal{F} \right)^D \left[\left({}_o\mathcal{F} \right)^{11} v \left({}_o\mathcal{F} \right)^{11} v \frac{z}{1} + \left({}_o\mathcal{F} \right)^B \right] \right\} \frac{{}_o\mathcal{P}}{P} = \left[\left({}_o\mathcal{F} \right)^D \left({}_o\mathcal{F} \right)^{11} v \right] z \frac{\left({}_o\mathcal{P} \right)^P}{z^2 P} \frac{z}{1}$$

The substitution of the values $B(f_0)$, $a_{11}(f_0)$ and integration yields

where

It is seen easily that $\lim_{j \rightarrow 0} P(j) = 0$ for arbitrary γ . Hence, one of the boundary

conditions is always satisfied. Assume that $\gamma < 0$, then $(\gamma - 1) < 0$, hence $\lim_{J \rightarrow \infty} P(J) = 0$ and the second boundary condition is also satisfied. The condition $\gamma > 0$ gives the relationship between the damping and the excitation levels which define the stability of the system on average

The spectral density is taken here at the frequency of the fundamental linear parametric resonance. This reflects the fact that in the assumed approximation at this particular frequency the energy absorption of the system is maximal. The energy is supplied by the fluctuating wideband excitation.

The expression of the probability density $P(j^0)$ contains the arbitrary constant, defined from the normalisation condition on the half-axis $0 < j < \infty$. Using [56]

where $\Gamma(x)$ is the Euler's Γ -function, then

The obtained probability density defines, within the assumed accuracy, the process $J^0(\tau)$. The moments of the process

$$m_p^{(p)} = \int_0^\infty \left[J^0 \right]^p P(J^0) dJ^0 = \alpha^p \frac{\Gamma(-\gamma+p)}{\Gamma(-\gamma)}$$

are defined for $\gamma + p < 0$. This yields the improved stability condition

$$r\Omega\pi^{-1} + b > \frac{3+p}{8}\Omega^2 S_\xi^2(2\Omega)$$

Now we obtain the *mode* of the process $J^0(\tau)$. The maximum of the probability

density $P(J^0)$ yields

$$J_0^0 = \alpha(1-\gamma)^{-1} = \frac{r\Omega\pi^{-1} + b}{\frac{1}{2}K_0} = \frac{r\Omega\pi^{-1} + b - \frac{1}{4}\Omega^2 S_\xi^2(2\Omega)}{\pi}$$

If the system tends to a deterministic system, then $S_\xi \rightarrow 0$ and $P(J^0) \rightarrow \delta(J^0 - J_0^0)$.

The value J_0^0 coincides with the standard value of the impulse, obtained using the averaging method. This may be established from Eq. (8.41) by considering $S_\xi^2(2\Omega) = 0$ and making the right-hand side of the equation zero.

A variety of strongly nonlinear systems may be analysed similarly. It is necessary to note that for this study we have to assume $J \neq 0$. In another case the right-hand sides of equations, which are written in a standard form, become senseless due to the degeneration of the transformation of variables "impulse-phase" supposing a priori the existence of the regime with impacts. For the study of random vibration in the vicinity of the point $J = 0$ it is convenient to apply the methods of the theory of outbursts of random processes; see also [37].

In the previous section we demonstrated the method based on the analysis of quasi-resonant regimes. In the paper by Kovalova¹ a new method allowing similar analysis of the parametric systems is proposed.

8.8. Perturbed Periodic Green Functions

In §7 we found that the PPGF, obtained in a regular way, play a pivotal role in the analysis of the parametric vibration of strongly nonlinear systems. For such systems

¹ Kovalova AS, Analysis of vibro-impact resonances in systems with random parametric excitation, Machine Study, 1983, N2 (in Russian).

under random excitation the concept of periodic motion is, generally speaking, senseless. Nevertheless, under the assumption of small perturbations and the resulting motion being close to periodic for the approximate description of the stationary processes, we can use the integral equations of periodic motion. We introduce a new concept of *disturbed periodic Green function* (DPGF) for the system with random parametric excitation. Our analysis is based on the method of *multi-scale decomposition* [124, 154] which was modified for the stochastic systems in [112].

Consider the equation of the perturbed motion of a linear SDOF system in the form

$$(8.42) \quad \ddot{x} + 2\varepsilon^2 b\dot{x} + \Omega^2(1 + \varepsilon\xi(t))x = Q_0(t)$$

Here $\xi(t)$ is a stationary random process with zero mean value and spectral density

$$S_\xi(\omega), Q_0(t) = Q_0(t+T) \text{ is the external force.}$$

Similarly to the deterministic case (§3), the DPGF of the system with random parametric excitation will be defined using the steady-state response of a system to the T -periodic excitation

$$Q_0(t, t_0) = \delta^T(t - t_0) = \sum_{k=-\infty}^{\infty} \delta(t - t_0 + kT)$$

where $\delta(t)$ is the Dirac δ -function. Denoting the DPGF as $\kappa_T(t, t_0)$, for the asymptotically stable system we find

$$(8.43) \quad \kappa_T(t, t_0) = \int_{t_0}^t h(t, s) \delta^T(s - t_0) ds$$

where $h(t, s)$ is the Green function of a linear system.

If the system is unstable, then [148]

$$\kappa_T(t, t_0) = -\int_{t_0}^t h(t, s) \delta^T(s - t_0) ds$$

The Green function $h(t, s)$ may be obtained as the solution of the homogeneous equation

$$(8.44) \quad \ddot{x} + 2\varepsilon^2 b\dot{x} + \Omega^2[1 + \varepsilon\xi(t)]x = 0$$

with the initial conditions $x(t_0) = 0, \dot{x}(t_0) = 1$.

The method of multi-scale decompositions is an effective method for the solution of equations with random coefficients [112]. We are looking for a solution of equation (8.44) in the form

$$(8.45) \quad h(t, \tau) = x_0(t, \tau) + \varepsilon x_1(t, \tau) + \varepsilon^2 x_2(t, \tau) + \dots, \tau = \varepsilon^2 t$$

It follows from above that

$$\frac{\partial h}{\partial t} = \frac{\partial x_0}{\partial t} + \varepsilon \frac{\partial x_1}{\partial t} + \varepsilon^2 \frac{\partial x_2}{\partial t} + \dots$$

The initial conditions will be taken in the form

$$x_0(t, \tau) = C_1(\tau)e^{i\Omega\tau} + C_2(\tau)e^{-i\Omega\tau}$$

$$\frac{\partial^2 h}{\partial t^2} = \frac{\partial^2 x_0}{\partial t^2} + \varepsilon \frac{\partial^2 x_1}{\partial t^2} + \varepsilon^2 \left(\frac{\partial^2 x_2}{\partial t^2} + 2 \frac{\partial^2 x_0}{\partial t^2} \right) + \dots$$

where $\tau_0 = \varepsilon^2 t_0$.

The dependences $C_{1,2}(\tau)$ may be defined from the condition of regularity of decomposition (8.45). Introducing (8.45) into (8.42) and equalising the coefficients of the like orders of ε yields the system of recurrent equations. In the first approximation we find

$$\frac{\partial^2 x_1}{\partial t^2} + \Omega^2 x_1 = -\Omega^2 \xi(t) x_0$$

Hence,

$$x_1 = -\Omega \int_{t_0}^t \sin \Omega(t-s) \left[C_1(\tau) e^{i\Omega s} + C_2(\tau) e^{-i\Omega s} \right] \xi(s) ds \quad (8.46)$$

and

$$x_1(t_0, \tau_0) = 0, \quad \left[\partial x_1 / \partial t \right]_{t=t_0, \tau=\tau_0} = 0, \quad \tau_0 = \varepsilon^2 t_0$$

The equation of the second approximation is used for the definition of the functions

$$C_{1,2}(\tau)$$

$$\frac{\partial^2 x_2}{\partial t^2} + \Omega^2 x_2 = F_1(t, \tau)$$

$$F_1(t, \tau) = -2i\Omega \left(e^{i\Omega t} \frac{\partial}{\partial C_2} - e^{-i\Omega t} \frac{\partial}{\partial C_1} \right) - \Omega^2 \xi(t) x_1 - 2bi\Omega (C_1 e^{i\Omega t} - C_2 e^{-i\Omega t})$$

Satisfying the condition [112]

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_T^0 \langle F(t, \tau) \rangle \exp(\pm i\Omega t) dt = 0$$

is required for the regularity of decomposition (8.45) and for the absence of secular components in the term x_2 .

The latter condition yields equations

$$\frac{dC_1}{d\tau} + 2bC_1 = -\frac{1}{2i\Omega} g(\tau, -i\Omega)$$

$$\frac{dC_2}{d\tau} + 2bC_2 = \frac{1}{2i\Omega} g(\tau, i\Omega)$$

where

$$g(\tau, \pm i\Omega) = \Omega^2 \lim_{T \rightarrow \infty} \frac{1}{T} \int_T^0 \langle x_1(t, \tau) \xi(t) \rangle \exp(\pm i\Omega t) dt$$

and integration is carried out with respect to the fast time t only.

$$\text{Taking into account expression (8.46), for } x_1, \text{ after rearrangements, we find}$$

$$g(\tau, -i\Omega) = -\frac{\Omega C_1}{2i} [L_\xi(0) - L_\xi(2i\Omega)]$$

where

$$L_\xi(p) = \int_0^\infty K_\xi(t) e^{-pt} dt$$

so the spectral density is $S_\xi(\omega) = 2 \operatorname{Re} L_\xi(i\omega)$.

The substitution of the representation obtained into the equation for the function

$$C_{1,2}(\tau) \text{ yields}$$

$$\frac{dC_1}{d\tau} + \gamma C_1 + i\delta C_1 = 0, \quad \frac{dC_2}{d\tau} + \gamma C_2 - i\delta C_2 = 0 \quad (8.47)$$

where we denote

$$\gamma = b - \frac{1}{8}\Omega^2 [S_\xi(2\Omega) - S_\xi(0)], \quad \delta = \frac{1}{4}\Omega^2 \operatorname{Im} L_\xi(i\omega)$$

The initial conditions for Eq. (8.47) are as follows

$$C_1(\tau_0) = (2i\Omega)^{-1} e^{-i\Omega t_0}, \quad C_2(\tau_0) = -(2i\Omega)^{-1} e^{-i\Omega t_0}$$

The desired solutions take the form

$$C_1(\tau) = (2i\Omega)^{-1} \exp[-(\gamma + i\delta)(\tau - \tau_0) - i\Omega t_0]$$

$$C_2(\tau) = -(2i\Omega)^{-1} \exp[-(\gamma - i\delta)(\tau - \tau_0) + i\Omega t_0]$$

The above equations are stable if

$$\gamma > 0, \quad b > \frac{1}{8}\Omega^2 [S_\xi(2\Omega) - S_\xi(0)] \quad (8.48)$$

System (8.47) is stationary, hence the solution of problem (8.44) defining the Green

function h may be represented in a form of the decomposition

$$h(t, t_0, \varepsilon) = h_0(t - t_0, \varepsilon) + \varepsilon h_1(t, t_0, \varepsilon) + \dots$$

where

$$h_0(t, \varepsilon) = C_1(\varepsilon^2 t) \exp(i\Omega t) + C_2(\varepsilon^2 t) \exp(-i\Omega t)$$

Therefore, the DPGF of the initial system (8.42) may be obtained as a result of substitution of this decomposition into (8.43),

$$\kappa_T(t, t_0, \varepsilon) = \chi_{0T}(t - t_0, \varepsilon) + \varepsilon \dots$$

The fundamental term of the decomposition

$$\chi_{0T}(t, \varepsilon) = \int_{-\infty}^{\infty} h_0(t - s, \varepsilon) \delta_T(s) ds$$

may be given (as a regular PGF) in the form of the Fourier series

$$\chi_{0T}(t, \varepsilon) = T^{-1} \sum_{k=-\infty}^{\infty} H(ki\omega, \varepsilon) e^{k i \omega t} \quad (8.49)$$

where the coefficients $H(k\omega, \varepsilon)$ are obtained in terms of the *disturbed dynamic compliance operator* (compare with §1.2)

$$H(p, \varepsilon) = \int_0^\infty h_0(t, \varepsilon) e^{-pt} dt$$

It is seen that $\chi_{0T}(t) = \chi_{0T}(t+T)$ and, therefore, the steady-state response of system (8.42) to the arbitrary T -periodic excitation $\mathcal{Q}_0(t)$ is

$$x(t, \varepsilon) = \int_T^\infty h_0(t-s, \varepsilon) \mathcal{Q}_0(s) ds + \varepsilon \dots$$

where stability condition (8.48) holds.

So, $x = x_0(t, \varepsilon) + \varepsilon \dots$ and the steady-state motion may be treated as a periodic process

modulated by a small random vibration.

The function $\chi_{0T}(t, \varepsilon)$ may be written for $0 \leq t < T$ in a closed form. In accordance

with the obtained representations for $G_{1,2}(\tau)$ and the definition of the operator

$H(p, \varepsilon)$ the coefficients of series (8.49) take the form

$$H(k\omega, \varepsilon) = \left(1 - \frac{\varepsilon^2 \delta^2}{\Omega} \right) \left[\left(\Omega - \varepsilon^2 \delta^2 \right)^2 - k^2 \omega^2 + 2k\omega \varepsilon^2 \gamma + \varepsilon^4 \gamma^2 \right]^{-1} \quad (8.50)$$

In accordance with (2.12) for $0 \leq t < T$, within the accuracy of small terms,

$$\chi_{0T}(t) = \frac{e^{-\gamma t}}{\Omega_1} \frac{\sin \Omega_1 t + e^{-\gamma_1 T} \sin \Omega_1(T-t)}{1 + e^{-2\gamma_1 T} - 2e^{-\gamma_1 T} \cos \Omega_1 T}$$

where

$$\gamma_1 = 2\varepsilon^2 \gamma, \quad \Omega_1 = \sqrt{(\Omega - \varepsilon^2 \delta^2)^2 - \gamma_1^2}$$

The DPGF for the symmetric systems and for the parametric systems with deterministic periodic or quasi-periodic excitation may be obtained similarly.

8.9. Application of Perturbed Periodic Green Functions to the Analysis of a Vibro-impact System

The latter results in combination with the calculations of §5.7 allow the analysis of a variety of strongly nonlinear systems under random parametric and force excitation. For the sake of simplicity, we consider the momentary impact interaction. Consider the equation of a vibro-impact system in the form

$$\ddot{x} + \Omega^2 [1 + \varepsilon \xi(t)] x + \varepsilon^2 [2bx + \Phi_2(x, \dot{x})] + \Phi_0(x, \dot{x}) = \mathcal{Q}_0(t + \varphi) \quad (8.51)$$

where Φ_0 and Φ_2 are the elastic and dissipative components of the generalised function describing the force of the impact interaction (see Sect. 5.6);

$\mathcal{Q}_0(t + \varphi) = \mathcal{Q}_0(t + T + \varphi)$; the process $\xi(t)$ is defined in Sect. 8.8; φ is the unknown

phase of the impact; the asymmetric system is considered for definiteness.

If the excitation is given by a random function, it is not feasible to consider periodic motions in such a system. Assume, however, that disturbances causing the deviation of the process from the periodic regime are small enough. Therefore, the qualitative character of the steady-state process is not changed. Assume, for example, that the stationary regime is close to the T -periodic regime with a single impact over the period. Coinciding the time zero reference with impact at t , by analogy with (7.23), we write

$$x(t, \varepsilon) = f(t + \varphi, \varepsilon) - J\kappa_T(t, 0, \varepsilon) + \varepsilon \dots \quad (8.52)$$

Here we use the DPGF of the linear part of the system and, $f(t, \varepsilon)$ is the forced steady-state motion in the absence of impacts

$$f(t, \varepsilon) = \int_T^\infty \kappa_T(t-s, \varepsilon) \mathcal{Q}_0(s) ds$$

Since the DPGF is given by the asymptotic decomposition

$$\kappa_T(t, s, \varepsilon) = \kappa_{0T}(t-s, \varepsilon) + \varepsilon \dots$$

representation (8.52) may be rewritten as

$$x(t, \varepsilon) = f_0(t + \varphi, \varepsilon) - J\kappa_{0T}(t, \varepsilon) + \varepsilon \dots = x_{0T}(t, \varepsilon) + \varepsilon \dots$$

and, within the assumed accuracy,

$$f_0(t) = \int_T^\infty \chi_{0T}(t-s, \varepsilon) \mathcal{Q}_0(s) ds$$

(an expression for the DPGF χ_{0T} may be found in the previous section).

Formula (5.45) yields the values of the impulse and phase defining the fundamental part of the desired decomposition $\chi_{0T}(t, \varepsilon)$ in the case of a sine excitation.

A similar representation may be obtained for the regime which is close to the subperiodic process of the period lT or even for a more complicated regime. In order to obtain relationships, which are more suitable for interpretation, we will

assume that the periodic process $\chi_{0T}(t, \varepsilon)$ is resonant, i.e.

$$\chi_{0T}(t, \varepsilon) \approx -J^0 \chi(t), \quad J^0 = -\Delta \chi^{-1}(0)$$

where $\chi(t)$ is the PGF of the system. We will define the existence conditions of such regimes based on the energy balance principle (see Sect. 5.10).

Due to the fact that the imaginary part of the *disturbed dynamic stiffness* $\text{Im } H^{-1}(im) = 2\varepsilon^2 \gamma \omega$ (see (8.50)), energy balance equation (5.47) takes the form

$$\left(J^0 \right) \frac{1-R}{2(1+R)} + \int_T^\infty [2\varepsilon^2 \gamma x_{0T} - \mathcal{Q}_0(t + \varphi)] dx_{0T} = 0$$

where the parameter γ is involved in (8.47), thus the influence of the random disturbances is reduced to *negative friction*.

Assume that $\mathcal{Q}_0(t) = \mathcal{Q}_0 \cos \omega t$, $\mathcal{Q}_0 = O(\varepsilon^2)$, and $1 - R = O(\varepsilon^2)$. In accordance with the calculations of §5.7 the above equation takes the form

$$J_0 \frac{1-R}{2(1+R)} + \frac{4}{1} J_0 \frac{\sin^2(\Omega T/2)}{\gamma} (T - \Omega^{-1} \sin \Omega T) + \frac{\omega \mathcal{Q}_0 \sin \varphi}{\Omega^2 - \omega^2} = 0$$

and the existence condition the of the resonant regimes ($|\sin \varphi| \leq 1$) takes the form

$$\frac{\omega \mathcal{Q}_0}{\Omega^2 - \omega^2} \geq J_0 \left[\frac{1-R}{2(1+R)} + \frac{4}{1} \frac{\sin^2(\Omega T/2)}{\gamma} (T - \Omega^{-1} \sin \Omega T) \right]$$

In accordance with Sect. 8.8 (see (8.48)) for the stable system

$$\gamma = b - \frac{1}{8} \Omega^{-2} [S_\xi^2(2\Omega) - S_\xi^2(0)] > 0$$

Hence, it may happen that in the absence of the parametric excitation, $\xi(t)$, a vibro-

impact process is not developing, but in its presence ($S_\xi^2(2\Omega) > S_\xi^2(0)$) the steady-state

vibro-impact regime does exist. Note, if $\xi(t)$ is white noise, the influence of the

random excitation within this approximation is negligible because $S_\xi^2(\omega) = \text{const}$. The

duration of the existence of the vibro-impact processes may be assessed in accordance

with Sect. 8.6. The obtained results are valid if the resonant conditions hold. The intensive vibration

are sustained due to the small excitation balancing the small damping.

The proposed method may be generalised for other cases of parametric excitation,

containing deterministic components, for example. Taking into consideration the

nonmomentary impacts, the energy balance equation using singularisation must be

written for the motion $\chi_{0T}(t) \approx -J_0^0 \chi(t - \tau_2)$ (see §4).

8.10. Narrowband Excitation

Up to now, most attention was paid to the wideband types of random excitation. This was not only because of the existence of the well-developed methods of analysis. Such systems turn out to be the most interesting from a physical point of view.

Nevertheless, the dynamic design of a variety of structures and machines requires an analysis of strongly nonlinear systems subjected to the excitation which is represented by single or multiple harmonics with random amplitudes and phases.

Here, we do not have an opportunity to give a detailed analysis of such systems. We will analyse the quasi-resonant regimes arising due to the narrowband excitation in the

system with momentary interaction (compare with Sect. 8.6). So, in equation (8.34) let $\lambda \rightarrow \infty$, the losses of energy at impact are accounted for by the restitution ratio R , and the excitation level is such that condition (8.33) holds.

Moreover, the external force $\xi(t)$ satisfies the conditions of Sect. 8.4 and takes the form

$$\xi(t) = \xi_1 \sin \nu t + \xi_2 \cos \nu t$$

where ν is frequency, ξ_1, ξ_2 are random values with the same variance equal $\langle \xi^2 \rangle$, where $\xi^*(t)$ is the envelope of the random process [169]. Under the assumption that the phase of the process $\xi(t)$ is distributed symmetrically, for the

correlation function $K_\xi(\tau)$ we find [169] that

$$K_\xi(\tau) = \langle \xi^*(t) \xi^*(t+\tau) \rangle \cos \nu \tau$$

Substitute this expression into (8.36) and consider the asymmetric system, when

$$\dot{x}_0 = -J \chi(t) = -JT^{-1} \sum_{k=1}^{\infty} \frac{\Omega^2 - (k\omega)^2}{k\omega \sin(k\omega t)}$$

where $T = 2\pi\omega^{-1}$, and ω is the resonant frequency. For the time of observation $T_n = nT$, $n = 1, 2, \dots$ taking into account $J = -\Delta \chi^{-1}(0)$, we find

$$\sigma_N^2 = \langle N_1^2 \rangle = \frac{4\pi^2 n^4 T^2 \chi^2(0) \int_{nT}^{\infty} \sum_{k=1}^{\infty} \frac{\Omega^2 - (k\omega)^2}{k \sin(k\omega t)} dt \int_{nT}^{\infty} \sum_{k=1}^{\infty} \frac{\Omega^2 - (k\omega)^2}{k \sin(k\omega t)} dt}{\cos \nu \tau d\tau}$$

For $\nu \neq \omega$ we have $\sigma_N^2 \rightarrow 0$ as $n \rightarrow \infty$ and, therefore, the steady-state vibro-impact regime at the resonant frequency $\nu = \omega$ is impossible. Under this assumption the system may be analysed similarly to the case of wideband excitation (see Sect. 8.6).

The latter equation under the condition $\nu = \omega$ yields

$$\sigma_N^2 = \frac{4\pi^2 \chi^2(0) (\Omega^2 - \omega^2)^2}{\langle \xi^2 \rangle \omega^4 \Delta^2}$$

The quantity $\sigma_N^2 \neq 0$ and, therefore, with a particular probability the steady-state resonance in the system is possible. Relationship (8.37) yields the condition when the

system will be free of such a resonance. With allowance made for the quantity $N_2(\omega)$, calculated in Sect. 8.6, we find¹

$$n_0(p_0) \sqrt{\langle \xi^2 \rangle \omega^2} \left[\frac{\pi(\omega^2 - \Omega^2)}{2} \right]$$

$$> - \left\{ \frac{2b\Delta\Omega}{\sin 2(\pi\Omega/\omega)} \left[1 - \frac{\omega}{2\pi\Omega} \sin \frac{\omega}{2\pi\Omega} \right] + \Delta\Omega \frac{\omega}{\pi} \frac{1+R}{1-R} \tan \frac{\omega}{2\pi\Omega} \right\}$$

A similar condition may be written for the symmetric system (see Sect. 9.13).

¹ Note that for resonant regimes $\omega > \Omega$.

Chapter 3 Multiple-Degree-of-Freedom Systems

§9 Forced Vibration in Multiple-Degree-of-Freedom Systems

9.1. Preliminary Considerations

In this paragraph we consider the analytical methods for the investigation of forced vibration in specific strongly nonlinear multiple-degree-of-freedom systems¹. The reduction of a threshold strong nonlinearity to a singular nonlinearity, which defines momentary interaction, allows application of the method of integral equations of periodic vibration (see §5.7). Consequently, the exact or approximate solutions (derived, for example, using the energy balance method) which describe the desired *periodic* regimes may be obtained.

It is evident that vibration in the multiple-degree-of-freedom system is much more complicated compared to that in SDOF system. Hence, effective analysis may be provided only for the simplest forms of resonant regimes. It is worth noting, however, that these regimes play a fundamental role in various problems of mechanics. Analysis is carried out mainly using the "operator language" of §1, allowing the general method of analysis for the systems of arbitrary order. Assuming the periodicity of the external excitation, the descriptive operator equations may be obtained using the PGF (see §2).

The majority of conservative strongly nonlinear systems can not be integrated in quadratures and corresponding equations of motion can not be represented in a standard form. Therefore, the averaging method is ineffective here. However, for particular systems the averaging method in combination with the *single-frequency method* [117] yields useful formulae (see sections 9.11 and 9.12). We consider that the strong threshold nonlinearity is the single nonlinearity which is represented in the system. The corresponding generalisations in the case of small nonlinear nonconservative forces, in particular forces of nonlinear friction, may be easily done.

The present material is based, in general, on references [10,17,89,90,93,94].

¹ Understanding this chapter requires basic knowledge of the theory of linear vibration in multiple-degree-of-freedom systems. The reader may consult, for example, Rao SS, Mechanical Vibrations, Addison-Wesley, 1995, or Thomson WT, Theory of Vibrations with Applications Chapman & Hall, 1988.

9.2. Integro-differential Equation of Periodic Regimes in System of Two Strongly Interacting Linear Subsystems

Consider two uncoupled stationary mechanical subsystems, which may contain both lumped and distributed elements (compare with Sect. 1.9). The equations of motion for these subsystems yield dynamic compliance operators, relating, as usual, the forces and corresponding dynamic responses. Assume that $N^{(I)}$ points of the subsystem $\mathbf{A}^{(I)}$, whose motion is described using the generalised coordinates $x_j^{(I)}$, $j = 1, \dots, N^{(I)}$, are subjected to the action of the forces $\mathcal{G}_j^{(I)}(t)$. Suppose that we observe the time

history of coordinate $x_k^{(I)}$. With the use of (1.13) we find

$$x_k^{(I)}(t) = L_{(I)}^{rk}(d) \mathcal{G}_k^{(I)}(d; t)$$

where the operator function $\mathcal{G}_k^{(I)}(d; t)$ is defined in accordance with Sect. 1.9 by the

system of operators $L_{(I)}^{hj}(d)$ and forces $\mathcal{G}_j^{(I)}(t)$. Similarly, for the second subsystem $\mathbf{A}^{(II)}$ we find $N^{(II)}$ points with coordinates $x_j^{(II)}$, $j = 1, \dots, N^{(II)}$, where the forces

$\mathcal{G}_j^{(II)}(t)$ are applied. This defines the system $L_{(II)}^{hj}(d)$, the operator function $\mathcal{G}_k^{(II)}(d; t)$ and the relationship

$$x_k^{(II)}(t) = L_{(II)}^{rk}(d) \mathcal{G}_k^{(II)}(d; t)$$

Let the above subsystems $\mathbf{A}^{(I)}$ and $\mathbf{A}^{(II)}$ be installed so that when $x_k = x_k^{(II)} - x_k^{(I)} = \Delta$ the single-dimensional interaction starts (see Fig. 9.1). The force of interaction contains the elastic and dissipative components $\lambda \Phi_2(x_k)$ and $\Phi_2(x_k, \dot{x}_k)$ and the interaction starts when the relative coordinate $x_k \geq \Delta$. We

suppose that the function $\Phi(x_k)$ belongs to the class of threshold nonlinearities $\{\Phi\}_\Delta$ as described in §4.

Our problem is the definition of the entire field of displacements $\{x_j^{(I)}, x_j^{(II)}\}$ for all points of the system $\{\mathbf{A}^{(I)}, \mathbf{A}^{(II)}\}$.

The interaction forces may be treated as the external forces applied to the points $x_k^{(I)}$ and $x_k^{(II)}$, hence

The case of the distributed forces is considered similarly.

$$\begin{aligned} x_k^{(I)} &= L_{(I)}^{rk}(d) \left[\mathcal{G}_k^{(I)}(d; t) + \lambda \Phi(x_k) + \Phi_2(x_k, \dot{x}_k) \right] \\ x_k^{(II)} &= L_{(II)}^{rk}(d) \left[\mathcal{G}_k^{(II)}(d; t) - \lambda \Phi(x_k) - \Phi_2(x_k, \dot{x}_k) \right] \end{aligned} \quad (9.1)$$

For the relative coordinate x_k using (9.1), we find the following operator equation

$$x_k = L_{(II)}^{rk}(d) \mathcal{G}_k^{(II)}(d; t) - L_{(I)}^{rk}(d) \mathcal{G}_k^{(I)}(d; t) - \lambda \Phi(x_k) + \Phi_2(x_k, \dot{x}_k) \quad (9.2)$$

where $L_{(I)}^{rk}(d) = L_{(I)}^{rk}(d) + L_{(II)}^{rk}(d)$. If the function $x_k(t)$ is defined from equation (9.2), then using (9.1) we find the functions $\{x_k^{(I)}(t), x_k^{(II)}(t)\}$ and using the operator relationships for other coordinates we find the entire field of displacement of the system $\{\mathbf{A}^{(I)}, \mathbf{A}^{(II)}\}$.

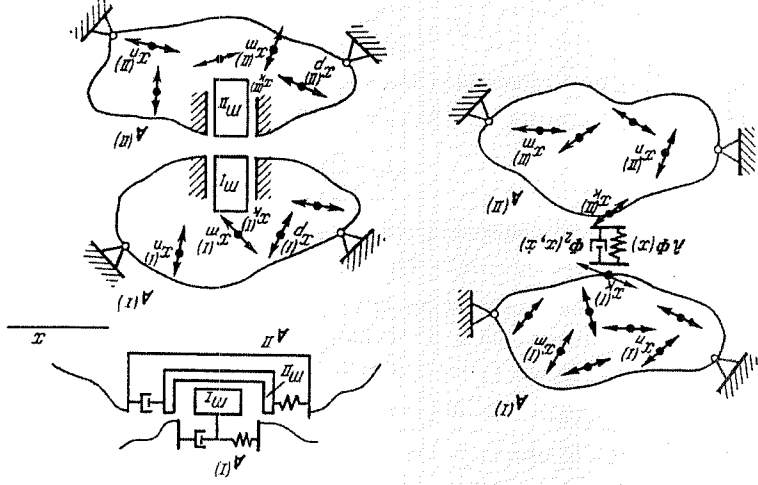


Fig. 9.1

The operator Eq. (9.2) may be generalised for the cases of the interactions of multiple subsystems or a few interactions of two subsystems, etc. The corresponding relationships will be similar. It is important to note that the problem of the definition of the entire field of displacements may be reduced to the problem of the definition of a single, or, maybe, a few relative coordinates. Then for the considered systems

$$\mathcal{G}_{(I)}^{rk}(d; t+T) = \mathcal{G}_{(I)}^{rk}(d; t), \quad \mathcal{G}_{(II)}^{rk}(d; t+T) = \mathcal{G}_{(II)}^{rk}(d; t)$$

Let all forces applied to the subsystems $\mathbf{A}^{(I)}$ and $\mathbf{A}^{(II)}$ have the period $T = 2\pi\omega^{-1}$.

The PGF $\chi(t)$ for the operator $L(p)$ may be introduced as (see (2.4))

$$\chi(t) = T^{-1} L_k e_k(t), \quad L_k \equiv L(ki\omega)$$

where summation takes place along the repeated indices.

Then for the T -periodic regime $x_k^k(t)$ we find the integro-differential equation which

follows from (2.2)

$$x_k^k(t) = \int_0^T \chi_{(II)}^k(t-s) g_{(I)}^k(p; s) ds - \int_0^T \chi_{(I)}^k(t-s) g_{(I)}^k(p; s) ds - \int_0^T \chi_{(II)}^k(t-s) g_{(I)}^k(p; s) ds$$

where the dynamic compliance operators and the PGF are related as

$$\chi_{(II)}^k(t) \Leftrightarrow L_{(II)}^k(p), \quad \chi_{(I)}^k(t) \Leftrightarrow L_{(I)}^k(p), \quad \chi_{(II)}^k(t) \Leftrightarrow L_{(II)}^k(p)$$

The first two addends in Eq. (9.3) yield the periodic function

$$f_k(t) = \int_0^T \chi_{(II)}^k(t-s) g_{(I)}^k(p; s) ds - \int_0^T \chi_{(I)}^k(t-s) g_{(I)}^k(p; s) ds$$

describing the linear motion x_k neglecting the impact interactions.

Eq. (9.3), therefore, takes the form of the following integro-differential equation

$$(9.4) \quad x(t) = f(t) - \int_0^T \chi(t-s) \{ \lambda \Phi [x(s)] + \Phi_2 [x(s), \dot{x}(s)] \} ds$$

where, for simplicity, the indices which depend on the number k are omitted.

The above equation will be the fundamental object of our study. It may be generalised to the case of multi-dimensional interaction (the PGF matrix is to be considered) and to the case of symmetric systems (Fig. 9.1b) (the symmetric PGF is to be considered).

Such generalisations will be considered later.

Rozenwasser [149, 150] studied equation (9.4) as applied to the problems of the theory of vibration. Numerous mathematical problems associated with the application of this equation are beyond the scope of the present monograph (see [86, 194]).

It is important to note that Eq. (9.4) may be obtained without differential equations of motion. For example, the PGF $\chi(t)$ may be constructed using the measured dynamic compliance operator $L(p)$, when the complex dynamic compliance $L(i\omega)$ is given by

its modulus and argument in the limited frequency band (see Sect. 2.8).

The corresponding calculations may be simplified significantly if we consider an interaction of the single subsystem $\mathbf{A}$ with a buffer. The fundamental relationships are to be formulated for the absolute coordinate of the subsystem $\mathbf{A}$.

9.3. Newtonian interaction

In the interacting system $\{ \mathbf{A}_{(I)}, \mathbf{A}_{(II)} \}$ let the impact be due to the momentary

interaction of lumped pointwise bodies with masses m_I and m_{II} in accordance with

the Newton's hypothesis (see Fig. 9.1b)

$$(9.5) \quad x(t_a) = \Delta, \quad R\dot{x}(t_a - 0) = -\dot{x}(t_a + 0)$$

where x is the relative coordinate, t_a is the α th moment of impact, and $R \in [0, 1]$ is the restitution ratio. The impact is considered to be one-dimensional. Using the reasoning of Sect. 4.4, 5.6 and 5.7 we find that the force of the impact interaction $\Phi_1(x, \dot{x})$ is given by a singular generalised function of the coordinate. In the time domain

$$(9.6) \quad \Phi_1[x(t), \dot{x}(t)] \Big|_{t=t_a} = J\delta(t-t_a)$$

where $\delta(t)$ is the Dirac δ -function, we find

$$J = \int_{t_a-0}^{t_a+0} \Phi_1[x(t), \dot{x}(t)] dt$$

the impulse of impact, represented as a sum of impulses in the loading and restitution phases $J = J_- + J_+$, where $J_- = m\dot{x}_-$, $J_+ = m\dot{x}_+$, $\dot{x}_\pm = \dot{x}(t_a \pm 0)$, we have

$$m = \frac{m_I m_{II}}{m_I + m_{II}}$$

the effective mass of the interacting points. From the second relationship of (9.5)

$$(9.7) \quad J = m(1+R)\dot{x}_- \geq 0$$

The appearance of the effective mass m may be explained using the mechanical sense of Newton's hypothesis. Since the impact is considered to be momentary, in the vicinity of the time instant t_a the bodies with the masses m_I and m_{II} are subjected to the action of the impact force $\Phi_1(x, \dot{x})$ only. In other words, the equations of motion for $t_a - 0 < t < t_a + 0$ take the form

$$m_I \ddot{x}_{(I)} = \Phi_1(x, \dot{x}), \quad m_{II} \ddot{x}_{(II)} = -\Phi_1(x, \dot{x})$$

For the relative coordinate $x = x_{(II)}^k - x_{(I)}^k$, using the effective mass, we find

$$m\ddot{x} = -\Phi_1(x, \dot{x})$$

and after simple manipulations we obtain (9.7).

We assume that lumped bodies are involved in the impact, hence the singular force of the impact may be substituted into Eq. (9.4)

$$(9.8) \quad x(t) = f(t) - \int_0^T \chi(t-s) \Phi_1[x(s), \dot{x}(s)] ds$$

because in accordance with Sect. 1.8 in this case $L(p) = O(p^{-2})$ and the PGF $\chi(t)$ is continuous. Solutions to this equation are continuous functions possessing, in the instant of impact, a first-order jump, which is reciprocal to the effective mass, i.e. $\chi(+0) - \chi(-0) = m^{-1}$ (see Sect. 2.3).

The form of Eq. (9.8) (when $m = 1$) is similar to (5.40). Using (9.6) for ν interactions over the period of motion we find the analogue of relationship (5.41) in the form

$$x(t) = f(t) - \sum_{k=1}^K J_k \chi(t - \phi_k)$$

which represents the T -periodic motion exactly. The system for the definition of the unknown parameters J_k and ϕ_k takes the form (5.42). The geometrical conditions, as usual, are $x(t) \leq \Delta$, $J_k \geq 0$.

The obtained 2ν -parametric representation is more general than (5.42). The PGF $\chi(t)$ corresponds to the arbitrary form of the operator $L(p) = O(p^{-2})$. Hence, for Newtonian interaction, the unknown field of displacements is defined by the PGF $\chi(t)$ (which may be obtained from the particular solution of the corresponding linear problem) and 2ν parameters define by the system of transcendental equations (5.42). The analogous representations may be obtained for the symmetric systems.

Multiple-impact processes are not studied here. Consider, in detail, the case when $\nu = 1$, $f(t) = f_1 \cos(\omega t + \phi)$, where the phase is unknown and the instance of impact is situated in the time zero reference. The magnitude f_1 is defined from the solution of the linear problem using the system of operators $L_{(1)}^{(k)}(p)$ and $L_{(1)}^{(k)}(p)$ (see Sect. 1.9). The function $f(t)$ may be sinusoidal only if all forces that are applied to the system $A_{(1)}$ and $A_{(1)}$ are sinusoidal, or, if, due to the filtering feature of linear systems, the higher-order harmonics are filtered in the transformation from the point of application to the contact point. In this case, following the manipulations of Sect. 5.7 the solution may be represented as

$$x(t) = f_1 \cos(\omega t + \phi) - J(t) \chi(t) \quad (9.9)$$

where impulse and phase are defined in accordance with (5.45)

$$J_{1,2} = \frac{-\Delta \chi(0) \pm \sqrt{f_1^2 \chi_2^2(0) - \frac{\omega^2}{1} \left[\chi_-(0) + \frac{(1+R)m}{1} \right]_2^2 (\Delta^2 - f_1^2)}}{\Delta + J_{1,2} \chi(0)} \quad (9.10)$$

$$\sin \phi_{1,2} = -\frac{J_{1,2} \omega f_1}{\left[\chi_-(0) + \frac{(1+R)m}{1} \right]_2^2} \cos \phi_{1,2} = \frac{f_1}{\Delta + J_{1,2} \chi(0)}$$

The fundamental difference between these formulae and (5.45) follows. Here the PGF $\chi(t)$ is given by the general form operator $L(p)$. It even may be given experimentally. Knowledge of the relative coordinate $x(t)$ gives us the full information about the

entire field of displacements of the system $\{A_{(1)}, A_{(1)}\}$. In a general case the corresponding relationships may be written using the system of operators $L_{(1)}^{(k)}(p)$, $L_{(1)}^{(k)}(p)$ and also representations similar to (9.1). The necessary manipulations are carried out for a number of particular cases.

Regimes (9.9) and (9.10) must be analysed with respect to stability and the realisation of the conditions $x \leq \Delta$, $J \geq 0$. The second problem, in general, may be solved only numerically. The analysis of stability may be conducted using the Rozenwasser's method [149] resulting in the appearance of the equations in variations containing the generalised functions. Such a scheme of analysis as applied to the considered systems leads to tedious calculations. Thus, it is reasonable to use again the energy balance condition (7.25) (see the discussion of section 7.6).

With reference to our previous analysis, which is based on the energy balance equations, we conclude that the second equality of (9.10) may be treated as the balance of nonconservative forces on the steady-state periodic motion with the parameters $J_{1,2}$ and $\phi_{1,2}$. The phase and impulse are related in the third equation from (9.10) [17].

For an unstable regime

$$\frac{d}{dt} \left\{ \frac{J}{\omega f_1} \left[\chi_-(0) + \frac{(1+R)m}{1} \right] + \sin \phi \right\}_{J=J_{1,2}} > 0$$

i.e. in the unstable stationary solution the excitation forces dominate over the stabilising dissipation forces. Differentiating using the dependence $\phi(J)$, which is given by the third equation of (9.10), yields the instability condition

$$J > -\frac{\Delta \chi(0)}{\chi_2^2(0) + \frac{\omega^2}{1} \left[\chi_-(0) + \frac{(1+R)m}{1} \right]_2^2} \quad (9.11)$$

This indicates instability of the regimes corresponding to the negative sign before the radical in the first formula of (9.10). Note that the quantity $\chi_-(0)$ is related uniquely to the dissipation in the linear part of the system defined by the value $\text{Im } L(i\omega)$,

$$\chi(t) = T^{-1} k i \omega L_k e_k(t)$$

When $\text{Im } L(i\omega) = 0$ we have $\chi_-(0) = -1/(2m)$ and the square parentheses in (9.10) and (9.11) give the well-known multiplier from Carnot's theorem (Sect. 5.6),

$$\left[\chi_-(0) + \frac{(1+R)m}{1} \right] = \frac{2m(1+R)}{1-R}$$

Satisfying the condition, which is reciprocal to (9.11), is necessary but not sufficient for stability of the periodic regimes (see [14]). In spite of this, for a number of important practical cases, the positive sign before the radical in the first equality (9.10) corresponds to the regimes which are actually stable.

The combination of formulae (9.9) and (9.10) with that obtained in §1 for representations for operators of linear systems and the corresponding PGF $\chi(t)$ (see §2), yield the solution to the problem of the definition of single-impact regimes for a very wide class of vibro-impact systems. Similar expressions may be obtained for the description of subperiodic regimes of period lT , $l = 1, 2, \dots$

It is worth noting that in comparison with the widespread stitching method (point mapping) [5, 75, 126], the class of linear systems, which are involved in the modelling of the impact interaction, may be widened significantly here. The corresponding PGF may be defined only using particular solutions of corresponding linear problems. Knowledge of the general integral is not required at all. This fact became decisive while considering various types of linear systems involved in the interaction, such as that described by linear integro-differential equations of the theory of viscoelasticity, equations with deviating arguments, etc. Representations (9.9) and (9.10) may be based on the experimentally obtained dynamic compliances.

In spite of the apparent compactness of the above expressions, the substitution of the PGF $\chi(t)$ of specific systems yields lengthy expressions. Significant simplification may be achieved while considering an important class of motions, namely resonant regimes. The class of models describing the impact interaction may be widened in this case. Non-Newton models described by the force $\lambda\Phi(x) + \Phi_2(x, x)$ in integral equation (9.4) for $\lambda < \infty$ also may be analysed.

9.4. Principle of Energy Balance

For further analysis a generalisation of the principle of energy balance (5.47) for systems with multiple-degree-of-freedom is required.

Let us consider the scleronomous mechanical system with n degrees of freedom and full dissipation (Sect. 1.3) described by the equation of motion

$$A\ddot{x} + C(x) = g(x, \dot{x}, t) \quad (9.12)$$

where $x \in \mathbb{R}^n$ is a vector of generalised coordinates; the kinetic energy $\mathcal{K}$ is defined by the quadratic form with the symmetric positively definite matrix A :

$$\mathcal{K} = \frac{1}{2}(A\dot{x}, \dot{x})$$

the conservative forces are defined by the potential $\Pi(x)$, as follows

$$C(x) = \text{grad} \Pi(x)$$

the vector $g \in \mathbb{R}^n$ defines nonconservative forces; it is assumed that

$g(x, \dot{x}, t + T) = g(x, \dot{x}, t)$, where T is period. Let $x(t)$ be a T_1 -periodic solution of the equation of motion. Similarly to Sect. 5.10, let us substitute the above solution into (9.12), multiply the right-side and left-side parts of the obtained equation by $\dot{x}(t)$ and integrate the result over the period T_1 . The left-hand side yields null because

$$\int_{T_1}^0 [(A\dot{x}, \dot{x}) + (C(x), \dot{x})] dt = \int_{M_0}^0 (A\dot{x}, dx) + (C(x), dx) \Big|_{M_0}^0 \equiv 0$$

where M_0 is the point with coordinates $x_1(0), \dots, x_n(0)$, and, therefore,

nonconservative forces are balanced on the periodic motion

$$\int_{T_1}^0 (g(x, \dot{x}, t), \dot{x}(t)) dt = 0 \quad (9.13)$$

which reflects the principle of energy balance.

Assume that a periodic solution of the equation of motion is known in the form $x_0(t, \phi)$, where ϕ is undefined parameter. The principle of energy balance (9.13) produces an equation for the definition of the unknown parameter ϕ .

For systems of a more general structure it is convenient to formulate the principle of energy balance based on the operator form of the equations of motion. If the operator of dynamic compliance $L(p)$ for the observed coordinate $x(t)$ of the system is known

and

$$x(t) = L(p)[g(x, \dot{x}, t)] \quad (9.14)$$

where g is a T -periodic function, then the operator equation

$$\int_{T_1}^0 \{x(t) - g[x(t), \dot{x}(t), t]\} \dot{x}(t) dt = 0 \quad (9.15)$$

where T_1 is the period of the solution $x(t)$, corresponds to Eq. (9.13).

Different forms of expression of the principle of energy balance may be written for the systems given by the operator equations.

The above relationships are convenient for analysis of the resonance regimes in the strongly nonlinear systems. For such systems it is easy to derive the two-parametric family of the periodic solutions to a conservative problem, which may be used as follows.

Let system (9.12) take the form

$$A\ddot{x} + C(x) = \varepsilon[P(t) + g_0(x, \dot{x})] \quad (9.16)$$

where $P(t + T) = P(t)$ is the force of excitation, and $g_0(x, \dot{x})$ are the friction forces. Assume that at $\varepsilon = 0$ the T_0 -periodic solution of the generative system is known in the form $x_0(A, \phi; t)$, where A and ϕ are the two integrals of motion. Here $A = A(\omega_0)$, where ω_0 is the frequency of free vibration belonging to the subset

$\{\omega_0\}$ of the set of positive numbers¹

We suppose by analogy with Sect. 5.10 that resonant regimes arising in system (9.16) are close in structure to free vibration. Thus, work done by nonconservative forces must be balanced in these motions and the corresponding resonances may be analysed as above.

Let the excitation frequency ω be a multiple of one of the natural frequencies of the corresponding conservative system, $\omega_0 = \omega q^{-1}$, where q is an integer. The resonant subperiodic regime $1:q$ may be approached as

$$x(t) \approx x_0 \left[A(\omega q^{-1}), \varphi, t \right]$$

where the function x_0 is periodic with the period $T_1 = qT$, and the unknown phase of the process is defined from the condition of balance of work done by the excitation and damping on this motion, that is:

$$E(\varphi) \equiv \int_0^{qT} \left\{ \left[P(t), \dot{x}_0(\varphi, t) \right] + \left(g_0, \dot{x}_0(\varphi, t) \right) \right\} dt = 0$$

where for the sake of simplicity some of the arguments are not indicated here.

The condition of existence of such resonant regimes, therefore, is reduced to the condition of resolvability of the transcendental equation $E(\varphi) = 0$. The number of pairs of solutions of the above equation yields the number of pairs of resonant regimes (compare with Sect. 5.12).

The consideration of the combined regimes $p:q$ or superperiodic regimes $p:1$ requires some modifications to the above analysis (see Sect. 5.14).

Note that the solutions analysed within the above-mentioned scheme are of single-frequency form; in such systems regimes of a different nature may exist. Let us study a problem of analysis of resonant regimes in the systems, defined by the operator equations in the form of (9.14). Let the equation of motion take the form

$$x(t) = L(p, \varepsilon) \left\{ \varepsilon [P(t) + g_0(x, \dot{x})] + C_0(x) \right\}$$

where $P(t+T) = P(t)$, $C_0(x) = -dT/dx$ and $L(p, 0) = L_0(p)$, so at $\text{Im } L_0(i\omega) = 0$ and $\varepsilon = 0$ we have the conservative system

$$x(t) = L_0(p) C_0(x)$$

If $x_0(A, \varphi, t)$ is a T_0 -periodic solution of the above equation and $A = A(\omega)$, then for the study of the resonant regimes $1:q$ in the case of $\omega_0 = \omega q^{-1}$ we have approximately

$$x(t) \approx x_0 \left[A(\omega q^{-1}), \varphi, t \right]$$

The unknown phase of the process may be obtained from the equation of energy

¹For the conservative systems analysed in this monograph much more complex regimes may exist [6].

balance

$$E(\varphi) \equiv \int_0^{qT} \left\{ \left[L^{-1}(p, \varepsilon) x_0(\varphi, t) \right] \left[x_0(\varphi, t) - \varepsilon [P(t) + g_0(x, \dot{x})] x_0(\varphi, t) \right] \right\} dt = 0$$

hence the condition of solvability of the resonant processes may be reduced to the condition of solvability of the transcendental equation $E(\varphi) = 0$.

Quite similarly resonance regimes may be calculated for other coordinates of the system. The desired two-parameter family of solutions may be obtained by means of the procedure of singularisation.

9.5. Singularisation

Consider the conservative strongly nonlinear system, given by the operator equation (9.17)

$$x(t) = -L_0(p) [\lambda \Phi(x)]$$

where $L_0(p) = O(1/p^2)$ and $\text{Im } L_0(i\omega) = 0$. The force $\Phi(x)$ belongs to the class of threshold nonlinearities $\{\Phi\}_\Delta$.

Similarly to Sections 4.8 and 4.9 we can apply the schemes of singularisation and improved singularisation for the calculation of the periodic regimes described by the above equation. Corresponding manipulations are quite similar to those in Sections 4.8 and 4.9.

For large λ in Eq. (9.17) we consider that as $\lambda \rightarrow \infty$ the threshold nonlinearity may be substituted by the impact nonlinearity, $\lambda \Phi(x) \rightarrow \Phi_0(x, \dot{x})$, where the singular function $\Phi_0(x, \dot{x})$ is defined in §4 (compare with (9.6))

$$\Phi_0[x(t), \dot{x}(t)]|_{t=t_a} = J\delta(t - t_a)$$

where t_a is the moment of impact, $J = 2m\dot{x}(t_a - 0) \geq 0$ is the impulse of the impact, and m is the effective mass reduced to coordinate x .

The above transformation is based on the mean-value theorem. Assume, for simplicity, that during one period of vibration T_0 a single impact takes place and, of course, in system (9.17) periodic regimes are possible. The above solutions may be defined from integral equation (9.4) when $J = \Phi_2 = 0$

$$x(t) = - \int_{t_0}^0 \lambda \Phi[x(s)] ds$$

The expression in the braces differs from zero only if $x \geq \Delta$. In this zone the body with coordinate x is subjected to the large potential force $\lambda \Phi(x)$ and to the relatively small forces related to the elasticity of the system. For large λ and impulses we can neglect these small forces and, therefore, for $x \geq \Delta$ the coordinate x may be described

approximately¹ by the second-order equation

$$m\ddot{x} + \lambda\Phi(x) = 0, \quad x(0) = \Delta, \quad \dot{x}(0) = V_0$$

where for definiteness the beginning of the interaction coincides with the zero time reference, and V_0 denotes the preimpact velocity. In accordance with §4 the interaction time t_2 is small. The mean-value theorem yields

$$x(t) = -\int_{t_2}^t \lambda\Phi[x(s)] \{ \lambda\Phi[x(s)] \} ds = -J_{\lambda} x[t - \Theta(t)] \quad (9.18)$$

where $0 < \Theta < t_2$ and the impulse of interaction is

$$J_{\lambda} = \int_{t_2}^{t_1} \lambda\Phi[x(s)] ds$$

As $\lambda \rightarrow \infty$ (similar to Sect. 4.8) $t_2 \rightarrow 0$ and $J_{\lambda} \rightarrow J = 2mV_0$. Therefore, Eq. (9.18) in the singular approach yields

$$x(t) = -J\chi(t) \quad (9.19)$$

By virtue of the autonomy of the system, solution (9.19) defines the two-parametric family $x(t) = -J\chi(t - \varphi)$, where φ is the arbitrary phase of the impact. The above limit in the solution corresponds to the transform

$$\lambda\Phi[x(t)] \rightarrow \Phi_0[x(t), x(t)] = J\delta_{T_0}(t)$$

where $\delta_{T_0}(t)$ is a T_0 -periodic δ -function.

Representation (9.19) may be found readily by a straight substitution of the impact nonlinearity into the above integral equation (compare with (9.9)). The above manipulations demonstrated the correlation of "preliminary nonlinearities" with the limit singular force.

From the definition of a PGF, with provision for $L_0(p) = O(p^{-2})$, we find that the solution $x(t)$ is represented by a continuous function, and the derivative possesses first-order discontinuities of magnitude m^{-1} . In order to obtain the dependence $J(\omega_0)$, we will use the condition $x(0) = \Delta$. Then the definition for the PGF yields

$$J(\omega_0) = -\Delta \left[T_0^{-1} \sum_{k=-\infty}^{\infty} L(ki\omega) \right]^{-1}$$

Since the initial time instant is arbitrary, the second condition of impact $J = 2m\dot{x}_-(0)$ is satisfied automatically, because $\dot{x}_-(0) = -J\chi_-(0)$, $\chi_-(0) = -1/(2m)$.

Similarly, it may be established that, for example, in the case of the elastic impact, representation (9.9) with $R = 1$ defines the corresponding solutions of the problems of

¹ The substitution of strong threshold nonlinearities by the singular forces may be carried out for arbitrary J_{λ} . This assumption will be used below.

$m \neq 1$

and for the characteristic $\lambda\Phi(x) = \lambda(x - \Delta)^{\alpha} \eta(x - \Delta)$ in accordance with (4.52) for

$$\lambda\Phi(x) = \lambda(x - \Delta) \eta(x - \Delta), \quad \tau_{\lambda} = \frac{1}{2} \pi(m/\lambda)^{1/2}$$

piecewise linear characteristics

zone, similar to the SDOF systems (see §4). In the case of large impulses, for the where the time $\tau_{\lambda} = \tau_{\lambda}(J_{\lambda})$ is defined from the equation of motion in the contact

$$x(t) = -J_{\lambda} \chi(t - \tau_{\lambda}) \quad (9.20)$$

Then, approximately,

$$\text{are the same, } \Theta(t) \approx \tau_{\lambda} = \frac{2}{t_2}.$$

interaction is fully elastic, the duration of the load and restitution phases of the impact In notation (9.18) the function $\Theta(t)$ belongs to the time interval $[0, t_2]$. Since the reflected in the obtained solutions.

The above results help us to obtain improved singular approximations (Sect. 4.9) for the conservative problems. At the same time the specifics of the initial nonlinearity are forced vibration of the system with the threshold nonlinearities from the class $\{\Phi\}_{\Delta}^1$.

where relationships (4.53) and (4.54) hold.

Equation (9.20) may be obtained using the more rigorous manipulations of Sect. 4.9. The series in the form (4.36) for the solution $x(t)$ may be calculated easily.

The transcendental equation defining the impulse J_{λ} follows from the condition

$$x(0) = \Delta,$$

$$\Delta = -J_{\lambda} \chi[\tau_{\lambda}(J_{\lambda})]$$

where, since damping is absent, $\chi(-t) = \chi(t)$. The above equation, in a general case (as for the SDOF systems), may be solved only numerically. In order to simplify the calculations in (9.20) it is reasonable to consider that $J_{\lambda} = J$, where J is the impulse of the impact taken from the solution of the problem with momentary interaction.

¹ The analysis of the models of contact damping producing Newton's hypothesis as a limiting case is beyond the scope of the present monograph. Some reasoning may be found in [11].

Therefore, for the construction of the solutions in the improved singular approximation for large λ and impact impulses the nonlinear characteristic $\lambda\Phi(x)$ may be substituted using the characteristic $\Phi_0[x(t), \dot{x}(t)] = J_\lambda \delta_{T_0}^2 [t - \tau_\lambda(J_\lambda)]$, where the interaction time $t_\lambda(J)$ is defined by the elastic component $\lambda\Phi(x)$.

It is worth noting, that the above singularisation is feasible only if the system possesses well-pronounced filtering features, i.e. if the operator of the dynamic compliance takes the form $L(p) = O(p^{-k})$, $k \geq 2$. When $k = 1$, the PGF is discontinuous and limiting solutions are senseless. The plane section is massless and the hypothesis of momentary impact is not applicable (see §1.2).

If the motion $x(t)$ is defined using the parameter $\tau_\lambda(J)$, the motion of other points of the system may be obtained. Thus, if $L_{xy}(p)$ is the transient dynamic compliance operator from point x into the point y , and if $\lambda\Phi(x)$ is the single nonlinearity represented in the system, we find

$$y = -L_{xy}(p)[\lambda\Phi(x)] \\ \text{The substitution } \lambda\Phi(x) \rightarrow J_\lambda \delta_{T_0}^2 [t - \tau_\lambda(J_\lambda)] \text{ yields} \\ y(t) = -J_\lambda \chi_{xy} [t - \tau_\lambda(J_\lambda)]$$

where the PGF is

$$\chi_{xy}(t) = T^{-1} L_{xy} e^k(t)$$

Typically, the transient dynamic compliance operators demonstrate better filtering features as compared with the local dynamic compliances. If this statement is valid for the operator $L_{xy}(p)$, then the function $y(t)$ is smoother than $x(t)$. In other words, for the systems which possess the filtering features, the strong nonlinear interaction is filtered from the point of interaction to the point of observation.

9.6. Interaction of Two Systems with Lumped Parameters

Let the interacting subsystems $A^{(I)}$ and $A^{(II)}$ be described by the equations of motion

$$A_1 \ddot{x}^{(I)} + C_1 \dot{x}^{(I)} + \lambda F_1(x) = \varepsilon [P^{(I)}(t) - B_1 \dot{x}^{(I)} - \Phi_{21}(x, \dot{x})] \\ A_2 \ddot{x}^{(II)} + C_2 \dot{x}^{(II)} - \lambda F_2(x) = \varepsilon [P^{(II)}(t) - B_2 \dot{x}^{(II)} - \Phi_{22}(x, \dot{x})] \quad (9.21)$$

where the matrices $A_j, C_j, B_j, j = 1, 2$ possess the features mentioned in Sect. 1.3:

the dimensions of the matrices are $[N_1 \times N_1]$ and $[N_2 \times N_2]$, respectively; the vectors

$x^{(I)} \in \mathbb{R}^{N_1}$ and $x^{(II)} \in \mathbb{R}^{N_2}$ define the states of the subsystems $A^{(I)}$ and $A^{(II)}$; the forces $F_j(x) = (\Phi(x), 0, \dots, 0)$, $\Phi_j(x, \dot{x}) = (\Phi_2(x, \dot{x}), 0, \dots, 0)$ have all null components with the exception of the first component and describe the interaction of the subsystems. For definiteness $x = x^{(I)} - x^{(II)}$, hence the interaction starts at $x = \Delta$;

the external forces $P^{(I)}(t)$ and $P^{(II)}(t)$ are supposed to be T -periodic.

We want to define periodic regimes of system (9.21) under the assumption that a single impact takes place over the period of motion.

In the case when all the components of the vectors $P^{(I)}(t)$ and $P^{(II)}(t)$ are sinusoidal functions and the impact interaction may be described by Newton's hypothesis, the problem possesses the exact solution which is represented by (9.9) and (9.10). If an interaction is nonmomentary we can apply singularisation and then find the existence conditions of resonant regimes. In both cases, however, knowledge of the dynamic compliance operators of the subsystems $A^{(I)}$ and $A^{(II)}$ along with the corresponding PGF is required.

Consider the problem of the definition of resonant regimes. For this purpose, we have to find firstly the solution of the conservative problem (9.21) when $\varepsilon = 0$.

When $\varepsilon = \lambda = 0$ Eq. (9.21), in accordance with (1.8), define components of the dynamic compliance matrix operators $L^{(I)}(p)$ and $L^{(II)}(p)$ in the form of eigenform decompositions

$$L^{(I)}(p) = \sum_{N_1} \frac{\Theta_{(I)}^{(I)} \Theta_{mj}^{(I)}}{p^2 + \Omega_{mj}^2}, \quad L^{(II)}(p) = \sum_{N_2} \frac{\Theta_{(II)}^{(II)} \Theta_{mj}^{(II)}}{p^2 + \Omega_{mj}^2}$$

where $\{\Omega_{js}\}$ ($s = 1, 2$) are the eigenfrequencies of the subsystems $A^{(I)}$ and $A^{(II)}$, and $\Theta_{kj}^{(I)}$ and $\Theta_{kj}^{(II)}$ are the normalised coefficients of the j th eigenform.

In accordance with the above and (2.19) the components of the matrix PGF for $0 \leq t < T_0$ take the form

$$\chi_{km}^{(I)}(t) = \sum_{N_1} \frac{2\Omega_{j1}}{\cos[\Omega_{j1}(t - 1/2 T_0)]} \sin(1/2 \Omega_{j1} T_0) \\ \chi_{km}^{(II)}(t) = \sum_{N_2} \frac{2\Omega_{j2}}{\cos[\Omega_{j2}(t - 1/2 T_0)]} \sin(1/2 \Omega_{j2} T_0)$$

where T_0 is the period of free vibration.

For the coordinate x from (9.4) we have the integral equation

$$x(t) = -\lambda \int_0^t \chi(t-s) \Phi[x(s)] ds, \quad \chi(t) = \chi_{(I)}^{\Pi}(t) + \chi_{(II)}^{\Pi}(t)$$

In accordance with (9.20) we find from the above

$$x(t) = -J_{\lambda} \chi(t - \tau_{\lambda}), \quad \Delta = -J_{\lambda} \chi(t - \tau_{\lambda}), \quad \Delta = -J_{\lambda} \chi(t - \tau_{\lambda}) \quad (9.22)$$

As $\lambda \rightarrow \infty$, the impulse J_{λ} may be defined as

$$J = -\frac{\Delta}{\chi(0)} = -\Delta \left[\sum_{j=1}^{N_1} \frac{2\Omega_{j1}}{\Omega_{j1}T_0} \cot \frac{2\Omega_{j1}}{2\Omega_{j2}} + \sum_{j=1}^{N_2} \frac{2\Omega_{j2}}{\Omega_{j2}T_0} \cot \frac{2\Omega_{j2}}{2\Omega_{j1}} \right]^{-1} \quad (9.23)$$

Using (9.20) we can construct the representation for $\tau_{\lambda}(J_{\lambda}) \approx \tau_{\lambda}(J)$ and, thus, define (9.22) completely.

Now we will find the components of the vectors $\mathbf{x}^{(I)}$ and $\mathbf{x}^{(II)}$. By definition, $x = x_{(I)}^I - x_{(II)}^I$. Due to this, and taking into account the results of the latter section, it may be found that

$$x_{(I)}^I = -J_{\lambda} \chi_{(I)}^I(t - \tau_{\lambda}), \quad x_{(II)}^I = J_{\lambda} \chi_{(II)}^I(t - \tau_{\lambda})$$

where the value of J_{λ} is already defined. Similarly, for each coordinate $x_{(I)}^j, j = 1, \dots, N_1$ and $x_{(II)}^n, n = 1, \dots, N_2$ we find that

$$x_{(I)}^j = -J_{\lambda} \chi_{(I)}^j(t - \tau_{\lambda}), \quad x_{(II)}^n = J_{\lambda} \chi_{(II)}^n(t - \tau_{\lambda}) \quad (9.24)$$

Similarly, corresponding representations may be obtained for the case of few interactions taking place in different points of the systems.

Now, when the entire field of displacements (9.24) of the system is defined, we can write the balance of work done by the nonconservative forces. Let the external forces be given by

$$\mathbf{P}^{(I)}(t) = \mathbf{P}_{(I)}^k \cos(k\omega t + \phi_{(I)}^k)$$

$$\mathbf{P}^{(II)}(t) = \mathbf{P}_{(II)}^k \cos(k\omega t + \phi_{(II)}^k)$$

where the summation is carried out for the dummy indices k over positive integers. Let the value of the period of the excitation force, T , belongs to the set of periods of free vibration T_0 of the corresponding conservative system. Therefore, in order to obtain the PGF $\chi_{(I,II)}^{km}(t)$, we put $T_0 = T$, compose representations (9.24) which approximates the resonant regimes, and write Eq. (9.13). Let the beginning of the interaction take place at time $t = 0$. Introducing the unknown phase ϕ into the input excitation yields, for the work done by the external forces,

$$E_1(\phi) = \varepsilon \int_0^T \left[\left(\mathbf{P}_{(I)}^{(I)}(t) + \phi, \mathbf{x}_{(I)}^{(I)}(t) \right) + \left(\mathbf{P}_{(II)}^{(II)}(t) + \phi, \mathbf{x}_{(II)}^{(II)}(t) \right) \right] dt$$

$$= -\varepsilon \sum_{N_1}^{j=1} \int_0^T P_{(I)}^{j_1} \cos(k\omega t + \phi + \phi_{(I)}^{j_1}) \chi_{(I)}^{j_1}(t - \tau_{\lambda}) dt$$

$$+ \varepsilon \sum_{N_2}^{j=1} \int_0^T P_{(II)}^{j_2} \cos(k\omega t + \phi + \phi_{(II)}^{j_2}) \chi_{(II)}^{j_2}(t - \tau_{\lambda}) dt$$

Substitution of the representations for the PGF and taking the integrals yields, finally

$$E_1(\phi) = -\varepsilon J_{\lambda} \left[\sum_{N_1}^{j=1} \sum_{M_1}^{i=1} P_{(I)}^{j_1 i_1} \Theta_{(I)}^{j_1 i_1} \Theta_{(I)}^{i_1 j_1} \cos(k\omega \sin(\phi_{(I)}^k) + \phi + \tau_{\lambda}) - \sum_{N_2}^{j=1} \sum_{M_2}^{i=1} P_{(II)}^{j_2 i_2} \Theta_{(II)}^{j_2 i_2} \Theta_{(II)}^{i_2 j_2} \cos(k\omega \sin(\phi_{(II)}^k) + \phi + \tau_{\lambda}) \right] \quad (9.25)$$

where $P_{(I)}^{j_1 i_1}$ is the j -indexed component of the vector $\mathbf{P}_{(I)}^k$. If, for example, all excitations are sinusoidal and in-phase functions, $\mathbf{P}_{(I,II)}(t) = \mathbf{P}_{(I,II)}^1 \cos \omega t$, then (9.25) may be simplified,

$$E_1(\phi) = -\varepsilon J_{\lambda} \left[\sum_{N_1}^{j=1} \sum_{M_1}^{i=1} P_{(I)}^{j_1 i_1} \Theta_{(I)}^{j_1 i_1} \Theta_{(I)}^{i_1 j_1} \Omega_{(I)}^{j_1 i_1} - \sum_{N_2}^{j=1} \sum_{M_2}^{i=1} P_{(II)}^{j_2 i_2} \Theta_{(II)}^{j_2 i_2} \Theta_{(II)}^{i_2 j_2} \Omega_{(II)}^{j_2 i_2} \right] \omega \sin(\phi + \tau_{\lambda}) \quad (9.26)$$

We will find the work done by the dissipation forces $E_2 = -\varepsilon \int_0^T \left[\left(\mathbf{B}_1 \mathbf{x}_{(I)}^{(I)}(t), \mathbf{x}_{(I)}^{(I)}(t) \right) + \left(\mathbf{B}_2 \mathbf{x}_{(II)}^{(II)}(t), \mathbf{x}_{(II)}^{(II)}(t) \right) \right] dt - \varepsilon E_c$ where $\varepsilon E_c = \varepsilon \int_0^T \Phi_2(x, \dot{x}) dt$ are the energy losses in the contact zone. For Newton's interaction, by Carnot's theorem we find that

$$\varepsilon E_c = J_2 \frac{2m(1+R)}{\varepsilon(1-R)}$$

where, from (9.21), the effective mass is

$$m = \frac{a_{11}^{(I)} a_{11}^{(II)}}{a_{11}^{(I)} + a_{11}^{(II)}}$$

and $a_{ij}^{(I,II)}$ are the components of matrices $\mathbf{A}_{1,2}$. Denoting the components of the matrix $\mathbf{B}_{1,2}$ as $b_{(I,II)}^{jk}$ and integrating yields

$$E_2 = -\varepsilon E_c$$

$$-j_2^2 \left\{ \sum_{N_1} b_{jk(i)}^k \frac{4 \sin(\frac{1}{2} \Omega_{j1} T) \sin(\frac{1}{2} \Omega_{j1} T)}{\Theta_{j1(i)}^i \Theta_{j1(ii)}^{ii} \Theta_{j1(ij)}^{ij}} \times \left[\frac{\sin(\frac{1}{2} T |\Omega_{j1} - \Omega_{q1}|)}{\sin(\frac{1}{2} T |\Omega_{j1} + \Omega_{q1}|)} \right] \right.$$

$$\left. + \sum_{N_2} b_{jk(ii)}^k \frac{4 \sin(\frac{1}{2} \Omega_{j2} T) \sin(\frac{1}{2} \Omega_{j2} T)}{\Theta_{j2(ii)}^{ii} \Theta_{j2(ii)}^{ii} \Theta_{j2(ii)}^{ij}} \times \left[\frac{\sin(\frac{1}{2} T |\Omega_{j2} - \Omega_{q2}|)}{\sin(\frac{1}{2} T |\Omega_{j2} + \Omega_{q2}|)} \right] \right\}$$

(9.27) This formula may be significantly simplified if the matrices $\mathbf{B}_{1,2}$ are diagonal. With provision for the latter calculations we find the energy balance equation

$$E_1(\varphi) + E_2 = 0 \quad (9.28)$$

where $E_1(\varphi)$ is given by (9.25) or (9.26), and E_2 by (9.27). The solvability condition of this equation corresponds to the existence condition of the resonant regimes. The number of its roots equals the number of these regimes. If $E_1(\varphi)$ is given by (9.26) we denote the expression in the parentheses by W_1 and find the existence condition of resonant regimes of the period T in the form

$$\left| \frac{E_b + E_c}{J_\lambda W_1 \omega} \right| \leq 1 \quad (9.29)$$

where εE_b denotes the work done by the viscous friction. A similar condition may be obtained readily by analysing the subperiodic resonances (compare with Sect. 5.1). The analysis of resonant regimes is completed.

Consider briefly the problem of deriving the exact solutions in the case of momentary impact, where the energy dissipation at the contact is accounted for using the restitution ratio.

Here we can use formulae (9.9) and (9.10), which hold under the assumption that all forces applied to the system are sinusoidal. The PGF $\chi(t)$ involved in these formulae corresponds to the operator $L(p) = L_{(I)}^{11}(p) + L_{(II)}^{11}(p)$, depending, in turn, on the components of the matrices of linear damping $\mathbf{B}_{1,2}$ (see Sect. 1.5).

If these components may be treated as small numbers, we can use expressions in the form of (2.17) to derive the PGF.

As a result, for Eq. (9.9) we have the PGF¹

$$\chi(t) = \chi_{(I)}^{11}(t) + \chi_{(II)}^{11}(t)$$

¹Here, for simplicity, the operators and PGF are denoted using the same symbols as in the conservative case.

$$\chi(t) = \sum_{N_1}^{j_1} \frac{\gamma_{j_1}}{1 + e^{-2T_{j_1} \Omega_{j_1} T} - 2e^{-T_{j_1} \Omega_{j_1} T} \cos \gamma_{j_1} T} \left[\Theta_{j_1(i)}^{11} \right]_2 e^{-T_{j_1} \Omega_{j_1} T} \sin \gamma_{j_1} T (T - t) + \sum_{N_2}^{j_2} \frac{\gamma_{j_2}}{1 + e^{-2T_{j_2} \Omega_{j_2} T} - 2e^{-T_{j_2} \Omega_{j_2} T} \cos \gamma_{j_2} T} \left[\Theta_{j_2(ii)}^{11} \right]_2 e^{-T_{j_2} \Omega_{j_2} T} \sin \gamma_{j_2} T (T - t)$$

where $\gamma_{jq} = \Omega_{jq} \sqrt{1 - \Gamma_{jq}^2}$, $q = 1, 2$ and the numbers Γ_{jq} , in accordance with Sect. 1.5, are defined by the components of the matrices $\mathbf{B}_{1,2}$

$$\Gamma_{j1} = \frac{1}{2} \rho_{(I)}^{jj} \Omega_{j1}^{-1}, \quad \rho_{(I)}^{kj} = \sum_{m=1}^{N_1} b_{(I)}^{km} \Theta_{mj}^{(I)}$$

and similarly for the subsystem $\mathbf{A}_{(II)}$.

In formulae (9.9) and (9.10) the magnitude f_1 is also involved. Under the assumption that all excitations are sinusoidal functions, i.e.

$$\mathbf{P}_{(I)}(t) = \mathbf{P}_{(I)}^1 \cos(\omega t + \phi_{(I)}^1), \quad \mathbf{P}_{(II)}(t) = \mathbf{P}_{(II)}^1 \cos(\omega t + \phi_{(II)}^1)$$

we find for the displacements $x_{(I)}^1$ and $x_{(II)}^1$ in the absence of interactions

$$x_{(I)}^1(t) \equiv \varepsilon \sum_{N_1}^{k_1} \left| L_{(I)}^{k_1(i)}(i\omega) \right| p_{(I)}^{k_1(i)} \cos(\omega t + \phi_{(I)}^1 + \psi_{(I)}^{k_1(i)})$$

$$x_{(II)}^1(t) \equiv \varepsilon \sum_{N_2}^{k_2} \left| L_{(II)}^{k_2(ii)}(i\omega) \right| p_{(II)}^{k_2(ii)} \cos(\omega t + \phi_{(II)}^1 + \psi_{(II)}^{k_2(ii)})$$

where $p_{(I,II)}^{k_1}$ are the components of the amplitude vectors $P_{(I,II)}^1$,

$\psi_{(I,II)}^{k_1} = \arg \left(L_{(I,II)}^{k_1(i)}(i\omega) \right)$. Since the unknown phase ϕ is involved in the expression

$$f_1(t) = f_{(I)}^1(t) - f_{(II)}^1(t), \quad \text{the latter may be expressed as}$$

$$f_1(t + \phi) = f_1 \cos(\omega t + \phi)$$

where f_1 is defined using elementary trigonometric manipulations from two previous formulae:

$$f_1 = \varepsilon \left\{ \sum_{N_1}^{k_1} \left| L_{(I)}^{k_1(i)}(i\omega) \right| p_{(I)}^{k_1(i)} \cos(\phi_{(I)}^1 + \psi_{(I)}^{k_1(i)}) - \sum_{N_2}^{k_2} \left| L_{(II)}^{k_2(ii)}(i\omega) \right| p_{(II)}^{k_2(ii)} \cos(\phi_{(II)}^1 + \psi_{(II)}^{k_2(ii)}) \right\}$$

$$+ \left\{ \sum_{N_2}^{j_2} \left| L_{(II)}^{j_2(ii)}(i\omega) \right| p_{(II)}^{j_2(ii)} \sin(\phi_{(II)}^1 + \psi_{(II)}^{j_2(ii)}) - \sum_{N_1}^{j_1} \left| L_{(I)}^{j_1(i)}(i\omega) \right| p_{(I)}^{j_1(i)} \sin(\phi_{(I)}^1 + \psi_{(I)}^{j_1(i)}) \right\}^{1/2}$$

Hence, all the values which are involved in (9.9) and (9.10) are prepared and the desired field of deflections is defined by the equalities

$$x_{(I)}^k(t) = f_{(I)}^k \cos(\omega t + \phi_{(I)}^k) - J \chi_{(I)}^k(t), \quad k = 1, 2, \dots, N_1$$

$$x_{(II)}^j(t) = f_{(II)}^j \cos(\omega t + \phi_{(II)}^j) - J \chi_{(II)}^j(t), \quad j = 1, 2, \dots, N_2$$

where the PGF $\chi_{(I,II)}^k(t)$ and the values of $f_{(I,II)}^k$ and $\phi_{(I,II)}^k$ are derived from the given system $L_{(I,II)}^k(p)$, as above. The impulse and phase are given by (9.10).

Note that in spite of the fact that system (9.30) was obtained using "exact" relationships, the system of dynamic compliance operators was taken here in accordance with Sect. 1.5 under the assumption of small damping. The exact solutions, of course, may be derived for systems with arbitrary damping but it will not be as simple to derive the corresponding representations for the PGF in a closed form. It is worth noting that there exist hypotheses of damping such that formula (1.8) representing the eigenform decomposition turns out to be an exact relationship (see (1.12)).

The above reasoning may be illustrated graphically in a general case by plotting the dependencies $J(\omega)$ characterising the intensity of the interaction and, therefore, the magnitude of the vibration. Consider the case of momentary impact as $\lambda \rightarrow \infty$. Expression (9.23) gives the family of backbone curves

$$J = J_0(\omega) = -\Delta \left\{ \sum_{k=1}^{N_1+N_2} A_k \cot \frac{\pi \Omega_k}{\omega} \right\}^{-1} \quad (9.31)$$

where A_k are the coefficients derived in (9.23), and the eigenfrequencies are sorted in ascending order (it is assumed that there are no multiple eigenfrequencies). The behaviour of the graph of function (9.31) in the resonant zone may be predicted from the following simple reasoning. Note that the resonant frequencies of a nonlinear system, which corresponds to the normal vibration in the linear system with the frequencies $\Omega_1, \dots, \Omega_{N_1+N_2}$, satisfy the system of inequalities $\omega > \Omega_1, \dots, \omega > \Omega_{N_1+N_2}$. It follows from a simple fact. The imposition of the additional restraints (impact interactions, for example) on the linear system oscillating in the k th mode with the frequency Ω_k results in the decrease of the time of motion of the representing point in $2(N_1 + N_2)$ -dimensioned phase space. Hence, the period of vibration may only be decreased.

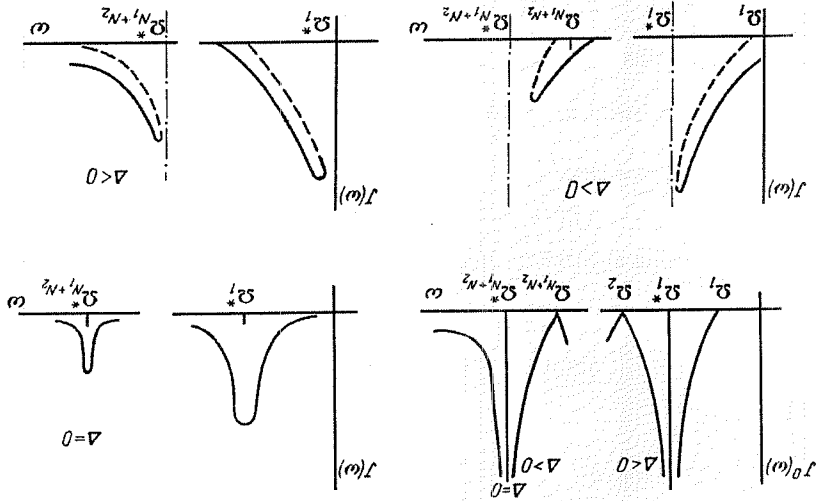
From the above figure, the most intensive regimes are possible in lower modes of vibration. The increase in the frequency results in the growth of the term E_2 , which is involved in the energy balance equation (9.28) defining the energy losses.

The frequencies of the infinite impulses Ω_k^* are related to the natural frequencies of the linear system as $\Omega_1 < \Omega_1^* < \Omega_2 < \dots < \Omega_{N_1+N_2}^* < \infty$. As $\omega \rightarrow \infty$, the impulse $J_0(\omega) \rightarrow 0$. With provision for the condition $J > 0$, the backbone curves take the form as sketched in Fig. 9.2a, where the increasing branches correspond to the case $\Delta > 0$, decreasing to the case $\Delta < 0$, and vertical lines to the case $\Delta = 0$.

Therefore, (9.31) will be considered for $\omega > \Omega_1$. It can be seen readily that for interval $\Omega_1 \leq \omega < \infty$ the function $J_0(\omega)$ possesses exactly $N_1 + N_2$ asymptotes $J_0(\Omega_k^*) = \infty$ which are defined using the equation

$$\sum_{k=1}^{N_1+N_2} A_k \cot \frac{\pi \Omega_k}{\omega} = 0, \quad \Omega_k^* = \omega$$

Fig. 9.2



Note that the above study is qualitative and defines only the general tendencies in the dynamic behaviour of the considered system which is involved in the single-frequency regime. For analysis of particular systems a more accurate analysis of stability is required. The condition $x(t) \leq \Delta$, in general, must be verified numerically. Numerous physical and numerical experiments demonstrated that in the vicinity of the frequencies $\omega = \Omega_k^*$ there exist stable single-impact regimes satisfying the condition $x(t) \leq \Delta$. The behaviour of the system in the vicinity of the frequencies $\omega = \Omega_k$ may be much more complicated. In particular, vibration with multiple impacts over the period of motion (chatter) is possible. For large $\lambda > \infty$ the shapes of the curves change insignificantly and the system may be analysed qualitatively, similarly to the considered limiting case (see also [17]). In the same manner, we can study subperiodic vibration and systems with symmetric nonlinearities (see Sect. 9.13). The above regimes are of the simplest and most well defined form. More complex motions must be analysed numerically.

9.7. Interaction of Rod Systems

Now we will consider the problem of deriving the field of displacements when the interacting subsystems $\mathbf{A}^{(I)}$ and $\mathbf{A}^{(II)}$ possess distributed parameters (thin rods) and the excitation forces are distributed along all plain sections of the systems [90]. We use the results of Sections 1.8, 2.2 and 2.6. In order to demonstrate the procedure for obtaining the exact solutions in the singular case $\lambda \rightarrow \infty$ we assume that pointwise masses are involved in the interaction¹.

Consider two thin rods $\mathbf{A}^{(I)}$ and $\mathbf{A}^{(II)}$ which are installed with the clearance $\Delta \geq 0$ or preload $\Delta < 0$. The above rods are subjected to the action of the T -periodic external forces $f_{j0}(x_j, t) \equiv f_{j0}(x_j, t + T)$, where x_j are the coordinates of the plane sections of the rods $\mathbf{A}^{(I)}$ and $\mathbf{A}^{(II)}$, $j = 1, 2$. Assume that displacements of the plane sections x_j under the applied forces are described by the functions $u_j(x_j, t)$, $j = 1, 2$ and possible that $l_{2j} = \infty$, i.e. the rods may be half-infinite. We assume also that the ends $x_j = l_{1j}$ of the interacting rods carry the lumped masses m_j . Similarly to (1.9), the

¹Otherwise the hypothesis of momentary interaction leads to the appearance of singular solutions.

equations of motion of the system $\{\mathbf{A}^{(I)}, \mathbf{A}^{(II)}\}$ may be written as two equations in

$$\mathcal{L}^{(q)}_j \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x_j} \right) u_j(x_j, t) = \left\{ (-1)^j \Phi_1(u, u) + m_j \frac{\partial^2 u_j(x_j, t)}{\partial t^2} \right\} \delta(x_j - l_{1j}) \quad (9.32)$$

where $\mathcal{L}^{(q)}_j(p, p_1)$ is a polynomial of power $q > 1$ with coefficients which are independent of time t (see Sect. 1.8), $u(t) = u_2(l_{12}, t) - u_1(l_{11}, t)$ is the relative coordinate, and $\Phi_1(u, u)$ is the force of impact interaction, defined above.

Since we are looking for the steady-state regimes, we do not take into consideration zero initial conditions. The boundary conditions are considered to be zero. This may be obtained by introducing the corresponding correction to the external excitation $f_{0j}(x_j, t)$.

The integro-differential equations of the periodic motion and the features of nonlinearity $\Phi_1(u, u)$ yield the solutions of period T .

From Sect. 2.2, those T -periodic regimes which satisfy (9.32) satisfy also the equations in the form

$$u_j(x_j, t) = n_{0j}(x_j, t) - (-1)^j \int_{l_{1j}}^0 \int_T^t \chi_j(x_j, y_j, t - s) \Phi(u, u) \delta(y_j - l_{1j}) dy_j ds \quad (9.33)$$

where the PGF of the distributed systems is

$$\chi_j(x_j, y_j, t) = T^{-1} L_{H_j}(x_j, y_j) e^{ik\omega}(t)$$

and $L_{H_j}(x_j, y_j) \equiv L_j(x_j, y_j, ik\omega)$ are the transient dynamic compliances from the point y_j into the point x_j which are taken at the frequency $k\omega$. The function

$$n_{0j}(x_j, t) = \int_T^t \int_{l_{1j}}^{l_{2j}} \chi_j(x_j, y_j, t - s) f_{0j}(y_j, s) dy_j ds$$

represents the periodic solution of the linear problem.

From (9.33) for the single-impact regimes we find the two-parametric representation (9.34) where the impulse and the phase of impact are involved in the consistency conditions

$$J = m(1 + R)u_-(\phi) \geq 0, \quad u(\phi) = \Delta$$

the reduced mass is $m = \frac{m_1 m_2}{m_1 + m_2}$ and $u(t) = u_2(l_{12}, t) - u_1(l_{11}, t)$. We put $x_j = l_{1j}$ and in (9.34) subtract the second relationship from the first. By doing so we find that

$$u(t) = u_0(t) - J\chi^0(t - \phi)$$

where $u_0(t) = u_{02}(l_{12}, t) - u_{01}(l_{11}, t)$ is the relative motion of the interacting bodies in the absence of an impact. The PGF is $\chi_0(t) = \chi_1(l_{11}, l_{11}, t) + \chi_2(l_{12}, l_{12}, t)$. This representation, within the accuracy of the notation, is analogous to that in Sect. 9.3. If, for example, $u_0(t) = u_{01} \cos \omega t$, then the impulse and the phase from this relationship may be obtained from formula (9.10) by the substitutions $f_1 \rightarrow u_{01}$, $\chi(0) \rightarrow \chi^0(0)$

and $\chi^-(0) \rightarrow \chi^-(0)$.

By virtue of Sect. 1.8 the Fourier coefficients for the PGF $\chi^0(t)$ are decreasing faster than k^{-2} , hence (as is usual in a singular approximation) the obtained solutions are given by the continuous functions (see Sect. 2.3).

With known impulse and phase from (9.34) at $x_j = l_j$ we find the representations for the motion of the interacting masses

$$u_j(l_j, t) = u_{0j}(l_j, t) - (-1)^j J\chi_j(l_j, l_j, t - \phi), \quad j = 1, 2$$

and formula (9.34) represents the desired entire field of displacements.

Solutions (9.34), (9.35) and (9.10) give, in a general form, the representations for the motion of the rod vibro-impact systems.

The structure of the PGF $\chi^0(t)$, which is involved in the above solutions, is defined in Sections 1.8 and 2.6. In accordance with (1.12) and (2.17) we find that

$$\chi_j(x_j, y_j, t) = \sum_{n=1}^{\infty} \Theta_{nj}(x_j) \Theta_{nj}(y_j) \chi_0^n(t) \quad (9.36)$$

where $\Theta_{nj}(x_j)$ are the eigenforms of the rod systems corresponding to the eigenfrequencies Ω_{nj} , and the functions

$$\chi_0^n(t) = \frac{e^{-\Gamma_{nj}\Omega_{nj}t} \left\{ \sin(\gamma_{nj}t) + e^{-\Gamma_{nj}\Omega_{nj}T} \sin[\gamma_{nj}(T-t)] \right\}}{\gamma_{nj} \left[1 + e^{-2\Gamma_{nj}\Omega_{nj}T} - 2e^{-\Gamma_{nj}\Omega_{nj}T} \cos(\gamma_{nj}T) \right]}, \quad 0 \leq t < T \quad (9.37)$$

were involved in the previous analysis. The values $\gamma_{nj} = \Omega_{nj} \sqrt{1 - \Gamma_{nj}^2}$ depend on the assumed theory of energy dissipation (Sect. 1.8), $\Gamma_{nj}^2 < 1$. With provision for (9.36) we find

$$\chi_0(t) = \sum_{n=1}^{\infty} \sum_{j=1}^2 \Theta_{nj}^2(l_j) \chi_0^n(t) \quad (9.38)$$

By accounting only the relevant modes of vibration, we obtain a structure of the PGF

$$\left[\frac{\partial^2}{\partial t^2} - c_j^2 \frac{\partial^2}{\partial x_j^2} - 2b_j \frac{\partial}{\partial x_j} \right] u_j(x_j, t) = \delta_{1j} f_{01} \sin(\omega t + \phi) \delta(x_1 - x_{11}) - \left\{ (-1)^j \Phi_1(u, n) + m_j \frac{\partial^2 u_j(x_j, t)}{\partial t^2} \right\} \delta(x_j - l_j)$$

The equation of motion takes the form
Gravity is neglected.

displacements x_j , $j = 1, 2$ will be measured from the position of static equilibrium. the force $f_{01} \sin(\omega t + \phi)$ applied to the plane section x_{11} of the upper rod. The vibration which is accompanied by collisions of the attached masses m_1 and m_2 under system in Fig. 9.3. Two viscoelastic rods $A^{(I)}$ and $A^{(II)}$ are involved in periodic Now we will derive the relationships defining the motion of the vibro-impact rod instead of (9.38) the full Fourier series may be used.

finite form when the eigenform decomposition may be summed [17]. In this case, exist methods for deriving the complex dynamic compliances $L_j(x_j, y_j, i\omega)$ in a lumped bodies may be carried out in accordance with the methods of Sect. 1.8. There is a lot of reference data concerning the calculation of the operators of the distributed systems. The calculation of the operators of rod systems with attached The methods of calculation of the eigenforms $\Theta_{nj}(x_j)$ are well-developed [9]. In [30]

preceding section.

$\chi_0(t)$ to be similar to that of the system with lumped parameters, as shown in the

where c_j are the velocities of the elastic waves, b_j are the coefficients of the viscous damping, and δ_{nj} is Kronecker symbol.

The standard boundary conditions for the stationary edges are $u_j(0, t) = 0$. The interacting masses are attached to the free ends. This is reflected in the above equation of motion by means of the corresponding addends. We want to find the terms involved in representation (9.34) for the T -periodic fields of deflections $u_j(x_j, t)$.

Let the momentary impact takes place at the time instant $t = 0$, the phase ϕ is unknown. The PGF $\chi_j(x_j, l_j, t)$ is involved in formula (9.34).

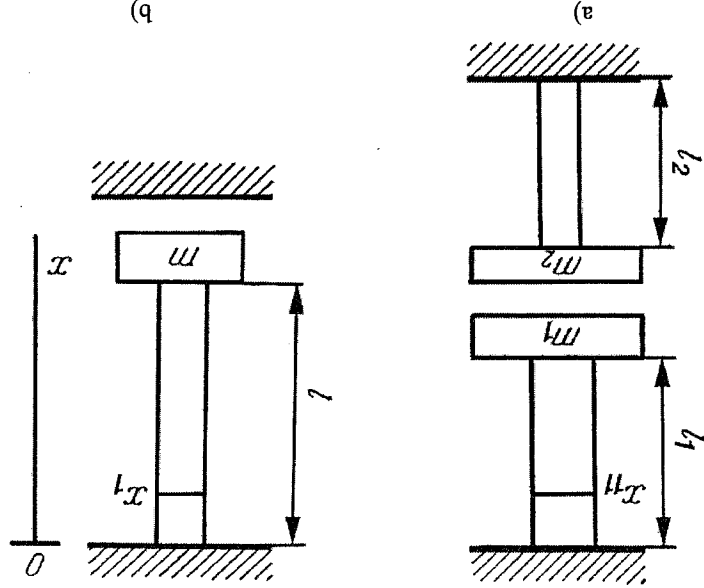


Fig. 9.3

By making use of the known solutions given in [9] we find that the PGF may be defined with the help of (9.36) and (9.37) as

$$\chi_j(x_j, l_j, t) = \sum_{k=1}^{\infty} \frac{2\beta_{kj} \sin(l_j \beta_{kj} x_j) \sin \beta_{kj} \exp(-\Gamma_{kj} \Omega_{kj} t)}{l_j \gamma_{kj} (2\beta_{kj} + \sin 2\beta_{kj})} \times \left[\frac{\sin(\gamma_{kj} t) + \sin[\gamma_{kj}(T-t)] \exp(-\Gamma_{kj} \Omega_{kj} T)}{1 + \exp(-2\Gamma_{kj} \Omega_{kj} T) - 2 \exp(-\Gamma_{kj} \Omega_{kj} T) \cos(\gamma_{kj} T)} \right]$$

where $t \in [0, T]$, β_{kj} are the roots of the transcendental equation $\beta_j \tan \beta_j = l_j m_j^{-1}$ which is well-known in the theory of vibration [9], $\Gamma_{kj} \Omega_{kj} = b_j \beta_{kj} l_j^{-2}$, $\gamma_{kj} = \Omega_{kj} \sqrt{1 - \Gamma_{kj}^2}$, $\Omega_{kj}^2 = c_j^2 \beta_{kj}^2 l_j^{-2}$. We assume that the damping is small, i.e. $\Gamma_{kj}^2 < 1$.

To derive the solution $u_{0j}(x_j, t)$ of the linear problem we note, that $u_{02} \equiv 0$ and

$$u_{01}(x_1, t) = f_{01} |L_1(x_1, x_{11}, i\omega)| \sin(\omega t + \phi) = \int_0^T \int_{l_1} \chi_1(x_1, y, t-s) f_{01} \sin(\omega s + \phi') \delta(y - x_{11}) dy ds$$

Here the phase $\phi = \phi' + \arg L_1$ is still unknown. Using the relationships from [9] we find that

$$|L_1(x_1, x_{11}, i\omega)| = \left[\sum_{k=1}^{\infty} D_k \cos \phi_{0k} \right]^2 + \left(\sum_{k=1}^{\infty} D_k \sin \phi_{0k} \right)^2 \Bigg]^{1/2} \\ \tan \phi_{0k} = \frac{2\Gamma_{k1} \Omega_{k1} \omega}{\Omega_{k1}^2 - \omega^2} \\ D_k = 4\beta_{k1} \sin \frac{l_1}{\beta_{k1} x_{11}} \sin \frac{l_1}{\beta_{k1} x_1} \times \left[l_1 (2\beta_{k1} + \sin 2\beta_{k1}) \sqrt{(\Omega_{k1}^2 - \omega^2)^2 + 4\Gamma_{k1}^2 \Omega_{k1}^2 \omega^2} \right]^{-1}$$

In order to obtain the desired solutions we need to find the values of the phase and impulse. The functions χ_j and u_{0j} which are involved in formulae (9.34), (9.35) and (9.10) are defined, and the desired representations may be constructed easily. The energy instability condition, as applied to the obtained solutions, indicates that a smaller value of impact impulse corresponds to an unstable regime.

It is worth noting that the obtained representations are tedious and difficult to interpret. They become useful only in combination with preliminary analysis carried out using more simple approximate methods, such as equivalent linearisation. In reference [17] the approach required for such preliminary analysis is developed and the basic physical characteristics typical for the considered systems are revealed.

As an example, we consider the vibration of a rod carrying a lumped mass and interacting with a solid obstacle, as shown in Fig. 9.3b. We will obtain another representation for the PGF $\chi_j(x_j, l_j, t)$ defining a solution of the above problem.

The equation of motion is similar to that considered above. For a definition of the solution of type (9.34), (9.35) and (9.10) we want to find the PGF $\chi(x, l, t)$ which is defined by a dynamic compliance of the linear system. The above dynamic compliance may be written in the finite form, as distinguished from the eigenform decomposition which was used above.

The dynamic compliance from the point l into the point x is the complex response of a linear system to the standard excitation $\exp(i\omega t)$. The equation of motion of the linear part of the system without attached mass is

$$\left[\frac{\partial^2}{\partial t^2} - c^2 \frac{\partial^2}{\partial x^2} - b \frac{\partial^2}{\partial t \partial x^2} \right] u(x, t) = 0$$

The external excitation may be accounted for in the boundary conditions,

$$u(0, t) = 0, \quad ES \left(1 + b c^{-2} \frac{\partial}{\partial t} \right) \left[\frac{\partial u(x, t)}{\partial x} \right]_{x=l} = \exp(i\omega t)$$

where E is the elasticity modulus, and S is the cross-section area of the rod. The steady-state response is thought of to be in the form

$$u(x, t) = L_0(x, l, i\omega) e^{i\omega t}$$

where L_0 is the dynamic compliance of the rod without an attached body.

Substitute this expression into the equation of motion and boundary conditions and we find the boundary problem for the ordinary differential equation

$$-\omega^2 L_0(x, l, i\omega) - (c^2 + i\omega b) \frac{d^2 L_0(x, l, i\omega)}{dx^2} = 0$$

$$L_0(x, l, i\omega) = 0, \quad ES \left(1 + i\omega c^{-2} \omega \right) \left[\frac{dL_0(x, l, i\omega)}{dx} \right]_{x=l} = 1$$

The solution takes the form

$$L_0(x, l, i\omega) = - \frac{\alpha(i\omega) c^2 \sinh[\alpha(i\omega)x]}{\alpha(i\omega) c^2 \cosh[\alpha(i\omega)l]}$$

$$\text{where } \alpha(i\omega) = i\omega(c^2 + i\omega b)^{-1/2}.$$

The expression for the dynamic compliance of the system with the attached body $L(x, l, i\omega)$ is based on the expression of dynamic compliance $L_0(x, l, i\omega)$ (see Sect. 1.8),

$$L(x, l, i\omega) = \frac{L_0(x, l, i\omega)}{L_0(x, l, i\omega) \sinh[\alpha(i\omega)x]} = \frac{1 - m\omega^2 L_0(x, l, i\omega)}{c^2 \cosh[\alpha(i\omega)l] - m\omega^2 c^2 L_0(x, l, i\omega) \sinh[\alpha(i\omega)x]}$$

$$\text{where } C_0(i\omega) = -\alpha(i\omega) c^2 (ES\omega^2)^{-1}.$$

The expressions for the desired PGF in terms of a full Fourier series is now found

$$\chi(x, l, t) = T^{-1} L^k(x, l) e^k(t), \quad L^k(x, l) \equiv L(x, l, k i\omega)$$

Similar expressions may be obtained in a different way. The attached mass may be accounted in the boundary conditions and the response of the system to the standard excitation $\delta^T(t) = T^{-1} \sum \exp(ik\omega t)$ may be calculated directly.

Further solution of the problem in Fig. 9.3b follows the same scenario. The solutions obtained in terms of the eigenform decomposition or the Fourier series are equivalent. The choice is entirely up to the researcher's taste.

The results of this section indicate the advantages of the developed approach. Knowledge of a particular solution of the linear problem defines fully the motion of the nonlinear system. Similar motions may be obtained for systems with the dynamic compliances represented in Sect. 1.10.

The systems defined by the integro-differential equations of the theory of hereditary viscoelasticity or by the equations with the deviating argument in the nonlinear approach were not studied before. Now, as $\lambda \rightarrow \infty$, we can find the corresponding exact solutions. The derivation scheme is quite similar to that represented above. The approximate approach to such problems is given in the next section and also in

9.8. Resonant Regimes in Systems with Arbitrary Dynamic Compliance Operators

sections 9.11 and 9.12. The obtained results and the energy balance principle yield more simple approximate solutions describing resonant regimes. In the next section we apply the proposed procedure to systems defined by linear operators in a general form.

Knowledge of the operators of interacting systems allows the derivation of the exact solutions in the singular approach. However, this is sometimes illusive. The above exact solutions may be constructed easily but they are too cumbersome for interpretation and analysis. In this section we refer to the approximate analysis of resonant regimes in the strongly nonlinear systems $\{A^{(I)}, A^{(II)}\}$ (see Sects. 9.4 and 9.6).

Assume that the operator $L(p)$ at the arbitrary point of the system is measured or calculated theoretically and all forces applied to the system are reduced to this point. Let the displacement of this point be described by the function $x(t)$. In accordance with (9.2) and (9.14) the operator equation takes the form

$$(9.39) \quad x(t) = L(p) \{ p(t) - [\lambda \Phi(x) + \Phi_2(x, x)] \}$$

The term in the parentheses represents, as usual, the force of the contact interaction, $p(t+T) = p(t)$ which is the T -periodic external force, $L(p) = O(p^{-2})$ as $p \rightarrow \infty$. The operator $L(p)$ is given in a finite form, or complex dynamic compliance is defined using corresponding real and imaginary parts, i.e. $L(i\omega) = Q_1(i\omega) + iQ_2(i\omega)$. Using the energy balance principle in the form (9.15) and representation (9.20) we can derive the existence conditions of single-impact resonant regimes in system (9.39), where Q_2 , p and $1-R$ are all small values.

Let the imaginary part of the dynamic compliance be zero, $Q_2(i\omega) = 0$. We put $P(t) = \Phi_2(x, x) = 0$ and derive the solution $x_0(t)$ of the conservative problem in the singular approximation (9.20),

$$(9.40) \quad x_0(t) = -J_\lambda \chi(t - \tau_\lambda), \quad \chi(t) = T^{-1} Q_1 e^k(t)$$

where the interaction time $t_\lambda = 2\tau_\lambda$ may be defined experimentally or using (4.52). The impulse of the interaction may be found from the impact condition $x_0(0) = \Delta$,

$$\Delta = -J_\lambda \chi[\tau_\lambda(J_\lambda)]$$

or $J_\lambda \approx J$ from the solution as $\lambda \rightarrow \infty$, where $J = -\Delta/\chi(0)$.

The approximate solution of (9.40) for small damping and external excitation levels describes the single-frequency resonant regimes. We will find the existence conditions of subperiodic resonances of arbitrary order l under the external force $P(t) = P_1 \cos(\omega t + \varphi)$. The period of the regime (9.40) and the period of the PGF $\chi(t)$ is $T_1 = 2\pi\omega^{-1}$.

The work done by the linear viscous damping on motion (9.40) is

$$E_2 = \int_0^{T_1} [L^{-1}(p)x_0(t)] \dot{x}_0(t) dt$$

The period of the motion is $T_1 = 2\pi\omega^{-1}$, hence, $\omega_1 = l\omega^{-1}$, $L_k = L(ik\omega)$ and

$$x_0(t) = -J_\lambda (T_1)^{-1} \bar{Q}_k e_k(t - \tau_\lambda)$$

$$\dot{x}_0(t) = -J_\lambda (T_1)^{-1} ik\omega_1 \bar{Q}_k e_k(t - \tau_\lambda)$$

The integration yields the series

$$E_2 = -(2\pi^2)^{-1} k\omega_2^2 J_\lambda^2 [\text{Im } L_k^{-1}] \bar{Q}_k^2 \geq 0$$

where the symbol $\text{Im } L_k^{-1}$ denotes the imaginary part of dynamic stiffness

$$\text{Im } L^{-1}(ik\omega_1) = -\bar{Q}_2^2(ik\omega_1) [\bar{Q}_2^2(ik\omega_1) + \bar{Q}_2^2(ik\omega_1)]^{-1}$$

The work done by the contact dissipation, as usual, is calculated in accordance with the assumed hypothesis of impact. If we assume the restitution ratio hypothesis, then

$$E_c = J_\lambda^2 \frac{2m(1+R)}{1-R}$$

where the value of the reduced mass m is calculated using the operator $L(p)$ (see Sects. 1.8 and 2.3),

$$m^{-1} = \lim_{p \rightarrow \infty} p^2 L(p)$$

The work done by the external source

$$E_1 = \int_0^{T_1} P(t) \dot{x}_0(t) dt$$

in this particular case, when $P(t)$ is a sinusoidal force, takes the form

$$E_1 = P_1 \omega_\lambda |Q_1(i\omega)| \sin(\varphi + \tau_\lambda)$$

Finally, the energy balance equation (9.15)

$$E_1 = E_2 + E_c$$

¹ The convergence of series (9.40) for the PGF $\chi(t)$ may be improved if the complex dynamic compliance $L(i\omega)$ is given experimentally in a finite frequency span.

takes the form

$$J_\lambda \left\{ \frac{2m(1+R)}{1-R} - (2\pi^2)^{-1} k\omega_2^2 [\text{Im } L_k^{-1}] \bar{Q}_k^2 \right\} = P_1 \omega |Q_1(i\omega)| \sin(\varphi + \tau_\lambda) \quad (9.41)$$

This equation gives two values for the unknown phase $0 \leq \varphi + \tau_\lambda \leq \pi$ and the existence condition of resonant regimes (9.40) in the form

$$P_1 \omega |Q_1(i\omega)| \geq J_\lambda \left\{ \frac{2m(1+R)}{1-R} - (2\pi^2)^{-1} k\omega_2^2 [\text{Im } L_k^{-1}] \bar{Q}_k^2 \right\} \quad (9.42)$$

Similarly, the energy balance equation may be written in the case of polyharmonic excitation for different models of contact damping or additional nonlinearities. The full description of the entire field of deflections may be obtained using the system of dynamic compliances operators (see above).

The above calculations are feasible for operators of linear systems given both in analytical and experimental form. As a matter of fact, explicit knowledge of the external force $P(t)$ is not required. Thus, if the motion is given by the relationship which is slightly different from (9.39)

$$x(t) = f(t) - L(p)[\lambda\Phi(x) + \Phi_2(x, \dot{x})]$$

then the work done by the external force involved in the energy balance equation takes the form

$$E_1 = \int_0^{T_1} [L^{-1}(p)f(t)] \dot{x}_0(t) dt \quad (9.44)$$

9.9. Quasi-resonant Regimes

A full analogy between the structure of resonant regime (9.40) in the multiple-degree-of-freedom system and the SDOF system allows the generalisation of the method of

section 8.6 for the multi-dimensional case. For example, assuming that in Eq. (9.39) the external force represents a stationary normal ergodic random process with zero mean value, the duration of a quasi-resonant regime with interactions may be assessed using formula (8.38). The only difference between the present case and that of Sect. 8.6 is the more complicated representation of the PGF $\chi(t)$, for example in the form of series (9.40).

Assume that condition (8.33) holds and the quasi-resonant regime

$$x_0(t) \approx -J_\lambda \chi(t - \tau_\lambda)$$

is sustained by a random "white noise" process. Using the calculations of the previous section and formula (8.38) we find for the duration of the process $(N_2(\omega) > N_1)$ as

$$\lambda \rightarrow \infty$$

$$T_N > n_0^2(p_0) S_0 \frac{k^2 \omega^2 \bar{Q}_{1k}^2}{\Delta^2} \left[\frac{\chi^2(0)}{1-R} - \frac{2\pi}{k\omega^2} \left(\text{Im } L_k^{-1} \right) \bar{Q}_{1k}^2 \right] \quad (9.45)$$

where, similarly to Sect. 8.6, the number $n_0(p_0)$ depends on the predefined probability of a backward jump of the vibration with interactions. For example, $n_0(p_0) = 2$ at $p_0 = 0.98$, etc.

The above formula gives an estimate of the time of existence of single-frequency regimes. In such systems, regimes of that kind are the most intensive and affect mostly the strength properties and the vibration activity of the structures involved. If the system is subjected to the narrowband random excitation $\xi(t) = \xi_1 \sin vt + \xi_2 \cos vt$, as mentioned in Sect. 8.10, the condition of absence of vibro-impact processes with the frequency $\omega = v$ takes the form

$$\omega \bar{Q}_{1k} n_0(p_0) \sqrt{\left\langle \frac{1}{2} \xi_2^*(t) \right\rangle} > - \frac{\Delta}{1-R} \left[\frac{\chi(0)}{2m(1+R)} - \frac{2\pi}{k\omega^2} \left(\text{Im } L_k^{-1} \right) \bar{Q}_{1k}^2 \right] \quad (9.46)$$

where $L_k \equiv L(ik\omega)$, $\bar{Q}_{1k} = \text{Re } L_k$ and the period of the PGF is $2\pi\omega^{-1}$.

Similarly to the case of wideband excitation, a sufficient condition for the feasibility of the assumed approximation $x_0(t) \approx -J\chi(t - \tau_k)$ may be formulated in the form of (8.33), $|\Delta| \gg \sigma_0$, where σ_0^2 is the variance of the process without interactions.

9.10. Analysis of Multi-dimensional Systems using Markov Processes

In general, multi-dimensional systems which are excited by a δ -correlated random process may be analysed using the FPK equations [27,37,73]. In particular cases, the above equations allow exact solutions and the probability densities of the joint $2n$ -dimensional distribution of the phase coordinates may be obtained. The methods, which are required for the analysis of vibro-impact systems, will be represented now. General methods of the theory of random processes of multi-dimensional systems may be found in references [27,37,73,147].

Let the vibration of the multi-dimensional system be defined by the system of differential equations

$$\ddot{x}_k + 2b\dot{x}_k + \Phi_k(x_1, \dots, x_n) = \xi_{n0}(t), \quad k = 1, \dots, n \quad (9.47)$$

where b is the damping coefficient which is assumed to take the same value for all coordinates of the system, $\xi_{k0}(t)$ are the Gaussian white noise sources of equal

$$\sum_{k=1}^n \left\{ H, P \right\}_k - 2b \frac{\partial}{\partial} (y_k P) - \frac{1}{2} S \frac{\partial^2 P}{\partial y_k^2} = 0$$

Here

$$\left\{ H, P \right\}_k = \frac{\partial H}{\partial P} \frac{\partial P}{\partial y_k} - \frac{\partial y_k}{\partial x_k} \frac{\partial x_k}{\partial y_k} \quad \text{are the Poisson brackets for } k\text{th mass,}$$

$$H = H(x_1, \dots, x_n, y_1, \dots, y_n) = \frac{1}{2} \sum_{k=1}^n y_k^2 + U(x_1, \dots, x_n) \quad \text{is the Hamilton function.}$$

Direct verification indicates that the desired stationary density [73]

$$P(x_1, \dots, x_n, y_1, \dots, y_n) = C \exp \left\{ - \frac{4b}{S} H(x_1, \dots, x_n, y_1, \dots, y_n) \right\} \quad (9.48)$$

takes the form of a *canonical Gibbs distribution*. In accordance with the type of Hamilton function, the distributions of the velocities and displacements are independent, and besides, the velocity distribution is normal. The constant C which is involved in (9.48) may be found from the normalisation condition

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} P(x, y) dx dy = 1$$

where the integration is provided with respect to all $2n$ -dimension phase space.

Integration of (9.48) with respect to coordinates yields the *Maxwell distribution* for velocities

$$P(y_1, \dots, y_n) = \text{const} \times \exp \left\{ - \frac{2b}{S} \sum_{k=1}^n y_k^2 \right\} \quad (9.49)$$

At the same time integration with respect to the velocities yields the *Boltzmann distribution* for coordinates

$$P(x_1, \dots, x_n) = \text{const} \times \exp \left\{ - \frac{4b}{S} U(x_1, \dots, x_n) \right\} \quad (9.50)$$

All three types of distributions are well known in statistical physics. The above distributions are especially useful as applied to the analysis of oscillatory systems with regular structure.

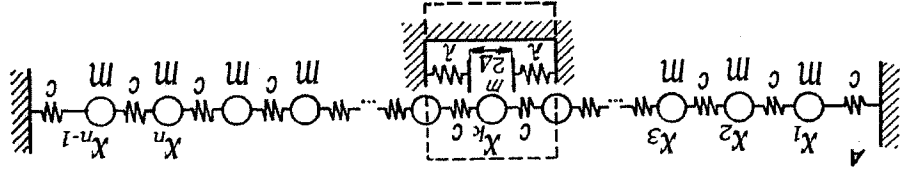
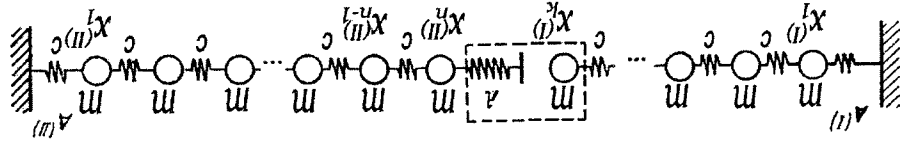


Fig. 9.4

Consider the dynamic model in Fig. 9.4a. Two chains consisting of pointwise bodies with masses m and stiffnesses c are interacting by means of a buffer, represented as the strongly nonlinear spring with characteristics $\lambda\Phi(x_{(II)}^n - x_{(I)}^k)$, $\Phi(x) \in \{\Phi\}_\Delta$. The

coordinates $x_j^{(I)}$, $j = 1, \dots, n$, $x_j^{(II)}$, $j = 1, \dots, n$ define the deflections of j th bodies in

the chains $\mathbf{A}^{(I)}$ and $\mathbf{A}^{(II)}$. We assume that the above chains are situated in a

homogeneous viscous liquid, hence the friction forces applied to each body are proportional to its absolute velocities. We assume also that each mass is subjected to wideband stationary random excitation (due to the intensive circulation of the liquid, for example). Neglecting the energy dissipation at the interaction we can apply the above methods to the analysis of this particular system. The dynamic model with symmetric buffers in Fig. 9.4b is described similarly. The above models are discrete models of rod systems with the linear operator in the form

$$\mathcal{L}_2 \left(\frac{\partial}{\partial t}, \frac{\partial x}{\partial t} \right) = a_1 \frac{\partial^2}{\partial t^2} - a_2 \frac{\partial^2 x}{\partial t^2} + a_3 \frac{\partial}{\partial t}$$

We will consider in detail the symmetric system in Fig. 9.4b. The Lagrangian

$$L(x_1, \dots, x_n, \dot{x}_1, \dots, \dot{x}_n)$$

$$= \frac{1}{2} m \sum_{j=1}^n \dot{x}_j^2 - \frac{1}{2} c \left[x_1^2 + x_n^2 + \sum_{j=2}^n (x_j - x_{j-1})^2 \right] - \lambda \int_{x_k}^{x_k} \Phi^*(x_k - \Delta) dx_k$$

where $\Phi^*(x_k - \Delta)$ is the elastic characteristic of the buffer, yields the equation of motion in the form

The analysis of distributed models requires the application of very complex mathematical tools which are beyond the scope of the present monograph.

$$\begin{aligned} \ddot{x}_1 + c(2x_1 - x_2) + 2bx_1 &= \xi_{01}(t) \\ \ddot{x}_j + c(2x_j - x_{j-1} - x_{j+1}) + 2bx_j &= \xi_{0j}(t), \quad j \neq 1, k, n \\ \ddot{x}_k + c(2x_k - x_{k-1} - x_{k+1}) + 2bx_k + \lambda\Phi^*(x_k - \Delta) &= \xi_{0k}(t), \quad k \neq 1, n \\ \ddot{x}_n + c(2x_n - x_{n-1}) + 2bx_n &= \xi_{0n}(t) \end{aligned} \quad (9.51)$$

For definiteness we assume that the interacting body is not situated on the edge of the chain and all parameters of the system are reduced to the unit mass $m = 1$. System (9.51) is a particular case of (9.47). The corresponding Hamiltonian which is written in the form

$$H(x_1, \dots, x_n, y_1, \dots, y_n) = \frac{1}{2} \sum_{j=1}^n y_j^2 + c \left(\sum_{j=1}^n x_j^2 - \sum_{j=1}^n x_j x_{j+1} \right) + \lambda U^*(x_k)$$

defines the type of the canonical Gibbs distribution (9.48). The above distribution satisfies zero boundary conditions as $x_j, y_j \rightarrow \pm\infty$.

Distributions (9.49) and (9.50) may be obtained similarly. Distribution (9.49) corresponds to our problem exactly and in formula (9.50) we have to put

$$U(x_1, \dots, x_n) = c \left(\sum_{j=1}^n x_j^2 - \sum_{j=1}^n x_j x_{j+1} \right) + \lambda U^*(x_k)$$

For large λ , an appearance of the coordinate x_k above the threshold level Δ is unlikely. Hence, it is reasonable to consider the problem as $\lambda \rightarrow \infty$. The Gibbs distribution

$$P(x_1, \dots, x_n, y_1, \dots, y_n) = C \exp \left\{ -\frac{4b}{S} \left[\frac{1}{2} \sum_{j=1}^n y_j^2 + c \left(\sum_{j=1}^n x_j^2 - \sum_{j=1}^n x_j x_{j+1} \right) \right] \right\} \quad (9.52)$$

satisfying the boundary conditions of elastic impact

$$\begin{aligned} P(x_1, \dots, x_{k-1}, \Delta, x_{k+1}, \dots, x_n, y_1, \dots, y_{k-1}, -U, y_{k+1}, \dots, y_n) \\ \equiv P(x_1, \dots, x_{k-1}, -\Delta, x_{k+1}, \dots, x_n, y_1, \dots, y_{k-1}, U, y_{k+1}, \dots, y_n) \\ \equiv P(x_1, \dots, x_{k-1}, -\Delta, x_{k+1}, \dots, x_n, y_1, \dots, y_{k-1}, -U, y_{k+1}, \dots, y_n) \end{aligned}$$

and zero conditions with respect to other variables at infinity, corresponds to this limiting case. The constant C may be defined from the normalising condition

$$\int P(x_1, \dots, x_n, y_1, \dots, y_n) dx dy = 1$$

where the integration domain (xy) is defined as

$$-\infty < x_j < \infty \quad (j \neq k); \quad -\Delta \leq x_k \leq \Delta; \quad -\infty < y_j < \infty$$

Probability density (9.52) contains full information about the process $\{x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n\}$. Using this formula we can define the moments of a process, the ordinary probability densities and other characteristics defining the dynamics of the above system.

Consider some examples.

The Boltzmann distribution for generalised coordinates takes the form

$$(9.53) \quad P(x_1, \dots, x_n) = C_1 \exp \left[-\frac{4bc}{S} \sum_n x_j^2 - \sum_{n=1}^j x_j x_{j+1} \right], \quad |x_k| \leq \Delta$$

To normalise this expression, the constant C_1 must be defined from the condition

$$\int_{(x)} P(x_1, \dots, x_n) dx = 1$$

The desired integral is taken over the domain $(x): -\infty < x_j < \infty, j \neq k; |x_k| \leq \Delta$. In accordance with this, distribution (9.53) has to be integrated $n-1$ times over the infinite limits and once over the domain $-\Delta \leq x_k \leq \Delta$. In accordance with the tables in reference [56] we find

$$\int_{-\infty}^{\infty} \exp(-px^2 + qx) dx = \exp \left(\frac{q^2}{4p} \right) \sqrt{\frac{p}{\pi}}$$

First, integrate $k-1$ times with respect to variables $x_1, \dots, x_{k-1}$. As a result we find the

joint density

$$P(x_k, x_{k+1}, \dots, x_n) = C_1 \left(\frac{a}{\pi} \right)^{(k-1)/2} \exp \left[-a(1 - \alpha_{k-1}) x_k^2 \right] \exp \left[-a \left(\sum_n x_j^2 - \sum_{n=1}^j x_j x_{j+1} \right) \right]$$

(9.54)

where $|x_k| \leq \Delta$, $a = 4bcS^{-1}$. The numbers α_j are calculated recurrently using the

$$\text{formula } \alpha_j = \frac{1}{2} (1 - \alpha_{j-1}), \quad \alpha_0 = 0.$$

As a result of $n-k$ integrations with respect to the variables $x_{k+1}, \dots, x_n$ we find the

single-dimensioned density

$$P(x_k) = C_1 \left(\frac{a}{\pi} \right)^{(n-1)/2} \prod_{k=2}^{n-1} (1 - \alpha_j)^{1/2} \prod_{n-k-1}^{n-2} (1 - \alpha_j)^{1/2} \exp \left[-a(1 - \alpha_{k-1} - \alpha_{n-k-1}) x_k^2 \right]$$

(9.55)

From the definition of the numbers α_j , it may be proven by induction that $\alpha_j < 1/2$

for all j . Hence, $(1 - \alpha_{k-1} - \alpha_{n-k}) > 0$.

Under the assumption $|x_k| \leq \Delta$ formula (9.55) shows that the single-dimensioned distribution $P(x_k)$ is a truncated normal shape. Hence, the single-dimensioned characteristics of the process $x_k(t)$ may be calculated similarly to the case of the SDOF system (see Sect. 8.2).

The integration of (9.55) over the domain $x_k \in [-\Delta, \Delta]$ and accounting for the normalisation condition yields

$$C_1^{-1} = \left(\frac{a}{\pi} \right)^{n/2} \frac{2\sqrt{\Delta} \sqrt{2a(1 - \alpha_{k-1} - \alpha_{n-k})}}{\sqrt{(1 - \alpha_{k-1} - \alpha_{n-k}) \prod_{j=0}^{k-1} (1 - \alpha_j)^{1/2} \prod_{n-k-1}^{n-2} (1 - \alpha_j)^{1/2}}} \frac{4bc}{S} a = \frac{S}{4bc}$$

Substitution of the obtained expression into expressions (9.53) and (9.55) fully defines the desired probability densities. Similarly, we can define the joint density of the arbitrary configuration $\{x_{j1}(t), x_{j2}(t), \dots, x_{jn}(t)\}$, $j_n \leq n$. Consider now the Maxwell distribution (9.49). Denoting the corresponding normalisation constant as C_2 we find that

$$\int_{(y)} P(y_1, \dots, y_n) dy = C_2 \int_{(y)} \exp \left\{ -\frac{S}{2b} \sum_n y_j^2 \right\} dy = \left(\frac{2b}{\pi S} \right)^{n/2}$$

and, finally,

$$C_2 = (2b/\pi S)^{n/2}$$

The calculated values C_1 and C_2 allow definition of the normalisation constant C in the Gibbs distribution (9.52), $C = C_1 C_2$.

It may be seen readily that the single-dimensioned density $P(y_k)$ is of the same form as in the SDOF system. Hence, the moments of the random process $y_k(t)$ may be calculated similarly to Sect. 8.2. By virtue of the independence of the random processes $x_j(t)$ and $y_j(t)$ we can make a similar conclusion with respect to the two-dimensional density $P(x_k, y_k)$ of the process $\{x_k, y_k\}$.

The problem in Fig. 9.4a may be analysed similarly. It is worth noting, that for simplicity of the solution, it is feasible to consider instead of the absolute coordinates $x_{(I)}^k$ and $x_{(II)}^k$ another set, namely the relative coordinates $x = x_{(II)}^k - x_{(I)}^k \leq \Delta$ in combination with $x_{(II)}^k$ or $x_{(I)}^k$.

Now we will analyse the strongly nonlinear multiple-degree-of-freedom systems using

the averaging method in the deterministic case.

9.11. Systems with Relaxation

As was indicated above, the exact solutions as $\lambda \rightarrow \infty$ or the approximate resonant solutions may be obtained if the dynamic compliance operators of the linear parts of the interacting subsystems are known. The single requirement imposed on these operators is the existence of a well-pronounced filtering feature. In some particular cases, however, it is possible to carry out an analysis of the above system using averaging. The corresponding solutions are more comprehensive than those obtained using the energy balance principle and more observable than those obtained by exact methods.

In a very common case, the conservative, strongly nonlinear and multiple-degree-of-freedom systems are nonintegrable in quadratures. Thus, reduction of the corresponding equations to a standard form is problematic for quasi-conservative systems. The averaging method, therefore, may be applied effectively only for restricted class of problems. In this section we mention an example of such a system, and possible generalizations are discussed further.

In Fig. 9.5a the model of a viscoelastic system is comprised of a massless rod with a tip unit mass. The rod is produced of polymer-wise material where the stress follows the strain in accordance with the Boltzmann-Volterra principle, see Sect. 1.10. The above lumped mass is involved in an interaction with a buffer and the elastic force of the impact interaction is given by the characteristics $\lambda\Phi(x)$, $\Phi(x) \in \{\Phi\}^\Delta$. Energy losses due to the interaction are described by the dissipative function $\varepsilon\Phi_2(x, \dot{x})$.

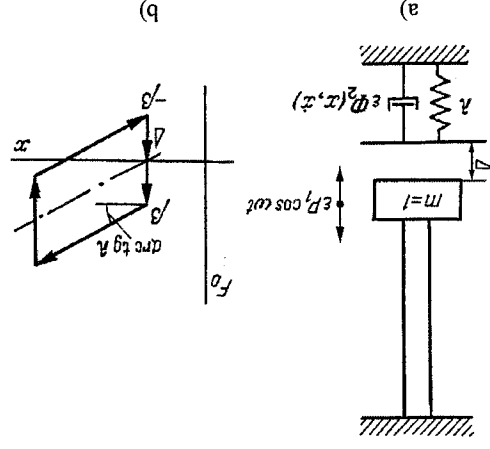


Fig. 9.5

Suppose that the vibration is due to the force $P(t) = \varepsilon P_1 \cos \omega t$ applied to the lumped body. The equation of motion of the above body might be in the form

$$\ddot{x} + \Omega^2 x + \lambda \Phi(x) = \varepsilon \left[\Phi_2(x, \dot{x}) + P_1 \cos \omega t + \int_{-\infty}^{\infty} \Gamma^*(t-s)x(s)ds \right]$$

where $\Gamma^*(t)$ is the *relaxation kernel*, defined by the properties of the rod material. The infinite lower limit in the integral added corresponds to the stationary process which is analysed. Assume for definiteness

$$\Phi(x) = (x - \Delta)\eta(x - \Delta), \quad \varepsilon\Phi_2(x, \dot{x}) = -(\beta \operatorname{sgn} \dot{x})\eta(x - \Delta)$$

After changing the integration limits we find

$$\ddot{x} + \Omega^2 x + \lambda(x - \Delta)\eta(x - \Delta) = \varepsilon \left[P_1 \cos \omega t + \int_0^{\infty} \Gamma^*(s)x(t-s)ds \right] - (\beta \operatorname{sgn} \dot{x})\eta(x - \Delta) \quad (9.56)$$

where $\lambda = O(\varepsilon^{-1/2})$, and because the interaction is fast, the order of the magnitude of the dry friction force is unity (see Sect. 5.8).

Hence, in this problem the force of the buffer reaction (Fig. 9.5b) and the stress-strain characteristic are of *hysteresis* type [63, 140].

The averaging method is now applied formally to obtain the approximate solution to Eq. (9.56) as described in §5. The mathematical basis for this method is given, for example, in [45, 120]. When $\varepsilon = 0$ a solution of the conservative problem may be taken in the form of the improved singular approximation

$$x_0(t) = -J_\lambda \chi(t - \tau_\lambda - \phi), \quad \phi = \text{const}$$

where, due to the fact that the generative system is a SDOF system,

$$\chi(t) = T_0^{-1} L_k e^{k t}, \quad -L_k \equiv (k^2 \omega_0^2 - \Omega^2)^{-1}, \quad T_0 = 2\pi \omega_0^{-1}$$

and T_0 is the period of free vibration. In the representation for $x_0(t)$ the value $\tau_\lambda = \pi/(2\sqrt{\lambda})$ is half of the interaction time. The dependence of the impulse on the frequency of the motion is calculated using (4.46)

$$J_\lambda(\omega_0) = -\frac{\chi(0; \omega_0) + 1/4 \pi \lambda^{-1/2}}{\Delta}$$

The obtained system is analysed in accordance with the scheme of Sect. 5.8. For this purpose, put the new variables $J \equiv J_\lambda$ and ϕ in (9.50). The system takes the standard form

¹ Note that the considered representation is valid only for large impulses of interaction.

Taking into account the equalities $\tilde{\Gamma}_{nc}^{*} = \tilde{\Gamma}_{ns}^{*}$ and $\tilde{\Gamma}_{nc}^{*} = -\tilde{\Gamma}_{ns}^{*}$, yields

$$I_3 = T^{-2} J |L_n|^2 n \omega \tilde{\Gamma}_{ns}^{*} \text{ and } I_4 = T^{-2} J |L_n|^2 n \omega \tilde{\Gamma}_{nc}^{*}$$

The above series may be summed exactly for particular kernels or approximately for the small relaxation times. Usually, the above series are converging fast enough, typically better then k^{-4} . Thus, the residual of the above series may be estimated as $O(k^{-3})$. By making use of the values I_3 and I_4 we can keep only the first terms in

corresponding decompositions

$$I_3 = 2T^{-2} J \left[\frac{\omega \tilde{\Gamma}_{ns}^{*}(\omega)}{2\tilde{\Gamma}_{nc}^{*}(\omega)} + \dots \right], \quad I_4 = T^{-2} J \left[\frac{\Omega^4}{2\tilde{\Gamma}_{nc}^{*}(\omega)} + \frac{\Omega^2 - \omega^2}{2\tilde{\Gamma}_{nc}^{*}(\omega)} + \dots \right]$$

The above calculations allow averaging of the system represented in standard form

$$J = -\frac{8\beta}{T} [\chi(\tau_2) - \chi(0)] - 4\epsilon \left[2T^{-2} J_0 \frac{\omega \tilde{\Gamma}_{ns}^{*}(\omega)}{\omega^2 P \sin[\omega(\varphi + \tau_2)]} + \frac{2\pi(\Omega^2 - \omega^2)}{2\tilde{\Gamma}_{nc}^{*}(\omega)} + \dots \right]$$

$\phi =$

$$\left\{ \frac{2\epsilon(J_0)^{-1} \omega P_1 \cos[\omega(\varphi + \tau_2)]}{\pi(\Omega^2 - \omega^2)} + 4\epsilon T^{-2} J_0 \left[\frac{\Omega^4}{2\tilde{\Gamma}_{nc}^{*}(\omega)} + \frac{\Omega^2 - \omega^2}{2\tilde{\Gamma}_{nc}^{*}(\omega)} \right] + \frac{\omega}{J - J_0} \omega_0(J_0) + \dots \right\}$$

The sine transformation of the relaxation kernel characterises the dissipative properties of the material; it is involved in the first averaged equation. The cosine transformation characterises the elastic properties and introduces correction to the second averaged equation. In the above system we denote

$$J_0 = J(\omega_0) \Big|_{\omega_0 = \omega} = -\frac{\chi(0, \omega) + i^{1/4} \pi \chi^{1/2}}{\Delta}$$

and further analysis (within the accuracy of quantitative corrections) repeats the results of Sections 5.6 and 5.8. Equating the right-hand sides of the averaged system to zero we find the stationary values for the impulse and phase. The regime with the greater impulse is stable and the existence conditions of resonant regimes are

$$\left| \frac{4\pi(\omega^2 - \Omega^2)}{\beta} \left\{ \frac{T}{\beta} [\chi(\tau_2) - \chi(0)] + \frac{\epsilon J_0 \omega \tilde{\Gamma}_{ns}^{*}(\omega)}{J_2(\Omega^2 - \omega^2)} \right\} \right| \leq 1$$

The obtained dependencies $J(\omega)$ may be represented graphically similarly as in §5.

By making use of the method of Sect. 5.9, we can analyse systems with piecewise

where in accordance with the fundamental resonance condition $\theta(J) = \omega - \omega_0(J)$, and

the dependence $\omega_0(J)$ was obtained as a result of inversion of the function $J_2(\omega_0)$

(see Sect. 5.8). The obtained system is subjected then to averaging with respect to

explicit time $0 \leq t \leq T$. The terms corresponding to excitation and damping at contact

were calculated in section 5.8. In order to perform averaging of the integral terms we

have to evaluate the following integrals

$$I_3 = T^{-1} \int_0^T \int_T^\infty \Gamma^*(s) [-J\chi(t - \tau_2 - \phi - s)] \chi(t - \tau_2 - \phi) dt ds$$

$$I_4 = T^{-1} \int_0^T \int_T^\infty \Gamma^*(s) [-J\chi(t - \tau_2 - \phi - s)] \chi(t - \tau_2 - \phi) dt ds$$

For the integral I_3 taking into account the definition for the PGF and

$e_k(t) = \exp(ik\omega)$, we find that

$$I_3 = -T^{-1} \int_T^\infty dt T^{-1} ik \omega L_k e_k(t - \tau_2 - \phi) T^{-1} L_n e_n(t - \tau_2 - \phi) \int_0^T \Gamma^*(s) e_n(-s) ds$$

The latter integral is the Fourier image of the relaxation kernel, $\tilde{\Gamma}^*(-i\omega)$. We denote

it as $\tilde{\Gamma}_{nc}^{*}$ and integrate with respect to time t .

By accounting for the periodicity of the regime we find

$$I_3 = -T^{-3} J L_k L_n i \omega k \tilde{\Gamma}_{nc}^{*} \int_T^\infty e_k(t - \tau_2 - \phi) e_n(t - \tau_2 - \phi) dt = -T^2 J L_k L_n i \omega k \tilde{\Gamma}_{nc}^{*} \delta_{k,-n}$$

where $\delta_{k,-n}$ is the Kronecker symbol. Finally, we obtain the series

$$I_3 = T^{-2} J |L_n|^2 i n \omega \tilde{\Gamma}_{nc}^{*}$$

where the summation is performed for all integers n and $L_n = (\Omega^2 - n^2 \omega^2)^{-1}$. Similar

calculations yield

$$I_4 = -T^{-2} J |L_n|^2 i n \omega \tilde{\Gamma}_{nc}^{*}$$

The value of $\tilde{\Gamma}_{nc}^{*}$ may be expressed in terms of sine and cosine Fourier

transformations of relaxation kernels, as

$$\tilde{\Gamma}_{nc}^{*} = \int_0^\infty \Gamma(s) e^{-in\omega s} ds = \int_0^\infty \Gamma(s) \cos(n\omega s) ds - i \int_0^\infty \Gamma(s) \sin(n\omega s) ds \equiv \tilde{\Gamma}_{nc}^{*} - i \tilde{\Gamma}_{ns}^{*}$$

elastic characteristics of buffer $\Phi(x) = \lambda(x - \Delta)^a \eta(x - \Delta)$, $a > 1$, or for different systems described using second-order integro-differential equations. It is essential for the solution that as $\varepsilon \rightarrow 0$ the operator of the linear part of the system is $(p^2 + \Omega^2)^{-1}$. The single-frequency vibration in the systems possessing this feature may be analysed using the following method.

9.12. Single-frequency Vibration in Systems Given by the Operator Equation

In the previous section we represented the scheme of the solution of a strongly nonlinear integro-differential equation. This scheme is based on the fact that when $\varepsilon = 0$ the degenerated system possesses a single-degree-of-freedom. The corresponding equation of motion may be written in the symbolic operator form

$$(p^2 + \Omega^2)x(t) + \lambda\Phi(x) = \varepsilon\{\Phi_2(x, px) + F_1 \cos \omega t + F_1^*x(t)\}$$

In this section we consider the possibility of generalising this equation by introducing additional variables [14].

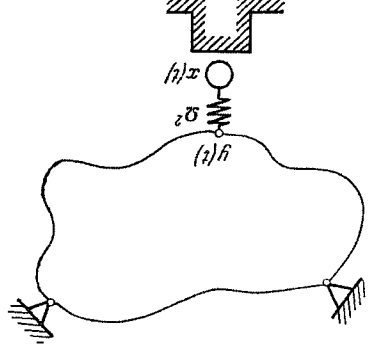


Fig. 9.6

Let the multiple-degree-of-freedom system in Fig. 9.6 embodies the oscillator which is *lightly coupled* with the other degrees of freedom. Denote by $x(t)$ the displacement of the unit mass of the oscillator involved in the interaction with the buffer and $y(t)$ the displacements of the suspension point.

The exact solutions (as $\lambda \rightarrow \infty$) for similar systems are not represented here. They may be done easily using the method described in this paragraph.

The motion of such a system may be described by the following operator equations

$$y(t) = \varepsilon\{L(p)\Omega^2x + F_1(p; t, x, y)\} + \varepsilon^2 \dots$$

$$(p^2 + \Omega^2)x(t) + \lambda\Phi(x) = \Omega^2y + \varepsilon\{\Phi_2(x, px) + F_2(p; t, x, y)\} + \varepsilon^2 \dots$$

where $L(p) = O(p^{-2})$ as $p \rightarrow \infty$ is the dynamic compliance operator at the point y , and the operator functions F_1 and F_2 are T -periodic in time. For simplicity, we consider the above operator functions to be analytic functions of spatial variables. The motion for all points of the system may be found using dynamic compliance operators. The above equations will be studied using the *single-frequency method* [117]. For this purpose we substitute the first of the above equations into the second. Making allowance for the analyticity of the operator functions F_1 and F_2 yields

$$(p^2 + \Omega^2)x(t) + \lambda\Phi(x) = \varepsilon g(p; t; x) + \varepsilon^2 \dots$$

where we denote

$$g(p; t; x) = \Omega^2 L(p)x + \Omega^2 F_1(p; t; x, 0) + F_2(p; t; x, 0) + \Phi_2(x, px)$$

Then the function $x(t)$ may be defined using the averaging method. The transformation of the phase coordinates

$$x = -J\lambda\chi[\psi, \omega_0(J\lambda), \tau\lambda(J\lambda)], \quad px = -J\lambda\omega_0(J\lambda)\chi[\psi, \omega_0(J\lambda), \tau\lambda(J\lambda)]$$

yields system which is almost similar to (5.23)

$$pJ\lambda = -4\varepsilon\omega_0(J\lambda)g(p; t; -J\lambda\chi)\chi\psi$$

$$p\psi = \omega_0(J\lambda)[1 + 4\varepsilon J\lambda^{-1}g(p; t; -J\lambda\chi)] + J\lambda\chi\omega_0 + J\lambda\chi\tau\lambda(J\lambda)$$

$$p\tau = 1$$

An example of the analysis of such relationships was represented in the previous section. The coordinate $y(t)$ and other coordinates may be defined quite simply.

The considered system may contain the lag elements and even more complex structures defined using operator functions. It is worth noting that the methods of analysis based on the integro-differential equations of the periodic vibration (see § 10) are much more efficient for the analysis of forced and parametric vibration. The averaging method, however, allows detailed analysis of the stability of the obtained solutions and also allows us to analyse the autonomous systems given, for example, in the form

$$\ddot{x} + \Omega^2x + \lambda\Phi(x) = \varepsilon g(x, \dot{x}, y) \\ \dot{y} = \varepsilon V(x, \dot{x}, y), \quad y, V \in \mathbb{R}^n$$

where the vector y specifies the influence of the additional degrees of freedom, drive and another factors. Detailed analysis of such systems is beyond the scope of this monograph (see [117, 190]). The solution obtained in this paragraph obviously may be used in the iteration

procedures of the analysis of more complex motions [57]. Progress in the study of strongly nonlinear systems may be achieved using the methods of analysis of multiple-frequency solutions.

We have mainly considered the analysis of asymmetric systems. In this section we make an attempt to apply the results obtained to the analysis of the symmetric, strongly nonlinear threshold-type system.

9.13. Analysis of Symmetric Systems

the systems with symmetric nonlinearities.

Consider the model in Fig. 9.7¹, and reduce all forces applied to different points of the system to the point of interaction (see Sect. 9.2). The motion of the above point is then defined by the operator equation, which takes the form of (9.2)

$$x = L(p)g(p; t) - L(p)[\lambda\Phi^*(x) + \Phi_2^*(x, \dot{x})]$$

where the force function $\Phi^*(x)$ is assumed to be symmetric, $-\Phi^*(x) = \Phi^*(-x)$, and may be constructed from the nonlinearities $\Phi \in \{\Phi\}^\Delta$ in accordance with Sect. 4.12: $\Phi^*(x) = \Phi(|x| - \Delta)\text{sgn } x$ ($\Delta > 0$). The function $\Phi_2^*(x, \dot{x})$ defines the energy losses at the interaction.

Let the operator function $g(p; t)$ be a T -periodic function of time and the displacement $L(p)g(p; t) = f^*(t)$ be symmetric: $f^*(t + T/2) = -f^*(t)$. Hence, in this system we can expect the existence of symmetric T -periodic regimes, which are

¹ The case of two interacting subsystems may be studied similarly.

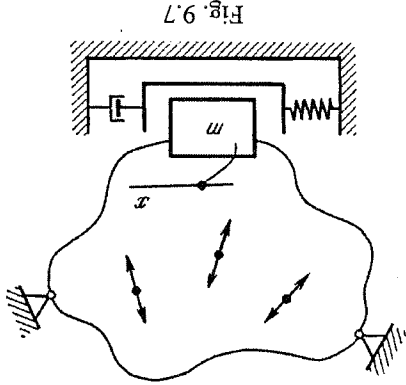


Fig. 9.7

defined similarly to (9.4)

$$x(t) = f^*(t) - \int_{-T/2}^0 \chi^*(t-s)\{\lambda\Phi^*[x(s), \dot{x}(s)]\}ds$$

where in accordance with the results of §2

$$\chi^*(t) = 2T^{-1}L_{2k+1}e^{2k+1}(t)$$

is a symmetric PGF, and

$$f^*(t) = \int_{-T/2}^0 \chi^*(t-s)g(p; s)ds$$

is the solution of the corresponding linear problem.

Hence, the fundamental integro-differential equation of symmetric regimes appears to be analogous to that of the asymmetric case. It means that the results of §2 may be formally extended to the symmetric case with the trivial substitution $\chi \rightarrow \chi^*$. For example, this substitution may transform representations (9.9) and (9.10) describing the single-impact regimes into the representation of symmetric double-impact regimes (single impact over half of the period of the motion).

The energy balance equation for symmetric regimes may be obtained after the integration over half-period. The representations for the solutions in the improved singular approximation take the form $x_0(t) = -J_\lambda \chi^*[t - \tau_\lambda(J_\lambda)]$. This allows, in particular, the application of effective schemes for the calculation of symmetric resonant and quasi-resonant regimes, etc.

Consider now a few examples of practical calculations.

First, we analyse in the deterministic approach an example from Sect. 9.1 (see Fig. 9.4b). Suppose, for simplicity, that the sinusoidal force $P_1 \cos \omega t$ is applied to the interacting body and damping in the system is described by the system of viscous dampers, as shown in Fig. 9.8. Similarly to (9.51) the equations of motion take the form

$$\begin{aligned} m\ddot{x}_1 + c(2\dot{x}_1 - \dot{x}_2) + b(2\dot{x}_1 - \dot{x}_2) &= 0 \\ m\ddot{x}_j + c(2\dot{x}_j - \dot{x}_{j-1} - \dot{x}_{j+1}) + b(2\dot{x}_j - \dot{x}_{j-1} - \dot{x}_{j+1}) &= 0, \quad j \neq 1, k, n \\ m\ddot{x}_k + c(2\dot{x}_k - \dot{x}_{k-1} - \dot{x}_{k+1}) + b(2\dot{x}_k - \dot{x}_{k-1} - \dot{x}_{k+1}) + \Phi_1^*(x_k, \dot{x}_k) &= P_1 \cos \omega t \\ m\ddot{x}_n + c(2\dot{x}_n - \dot{x}_{n-1}) + b(2\dot{x}_n - \dot{x}_{n-1}) &= 0 \end{aligned}$$

where $\Phi_1^*(x_k, \dot{x}_k)$ denotes the force of the impact interaction comprising, as usual, the elastic and dissipative components. Without loss of generality it is possible to consider $m = 1$ in all equations.

$$\Gamma_j = \frac{1}{2} \beta_{jj} \Omega_j^{-1} = \frac{1}{2} \Omega_j^{-1} \sum_{n=1}^b b_{jq} \Theta_{jq} = \begin{cases} \Omega_1^{-1} b(2\Theta_{11} - \Theta_{12}) \\ \Omega_2^{-1} b(2\Theta_{22} - \Theta_{21} - \Theta_{23}) \\ \vdots \\ \Omega_n^{-1} b(2\Theta_{nn} - \Theta_{n,n-1} - \Theta_{n,n+1}) \end{cases}$$

Taking into account the above calculations and formulae (2.3) and (2.18) the system of a periodic PGF may be represented in terms of the Fourier series containing the odd harmonics

$$\chi_{jq}^*(t) = 2T^{-1} L_{jq}^{(2k+1)} e^{2k+1} (t)$$

where $L_{jq}^{(2k+1)} \equiv L_{jq} [i(2k+1)\omega]$ or in a finite form for $0 \leq t < T/2$

$$\chi_{jq}^*(t) = \sum_{n=1}^v \frac{2}{\pi} \sin \frac{\pi j v}{2} \sin \frac{\pi q v}{2} \sin \frac{\pi n v}{2} \chi_{jq}^*(t)$$

$$\chi_v^*(t) = \frac{e^{-\Gamma_v \Omega_v t} \left[\sin \gamma_v t + e^{-\frac{1}{2} \Gamma_v \Omega_v T} \sin \gamma_v \left(t - \frac{T}{2} \right) \right]}{\gamma_v \left[1 + e^{\Gamma_v \Omega_v T} + 2e^{-\frac{1}{2} \Gamma_v \Omega_v T} \cos \gamma_v T \right]}$$

where we denote $\gamma_v = \Omega_v \sqrt{1 - \Gamma_v^2}$ and assume that $|\Gamma_v| < 1$.

Representation (9.9) defines the fundamental regimes of period T and takes the form $x_k(t) = f_k^* \cos(\omega t + \phi) - J \chi_{kk}^*(t)$

where the amplitude of the "linear vibration" is

$$f_k^* = |L_{kk}(i\omega)| R_1 = \sum_{j=1}^v \frac{\Theta_{kj}^2 R_1}{\sqrt{(\Omega_j^2 - \omega^2)^2 + 4\Gamma_j^2 \Omega_j^2 \omega^2}}$$

the impulse J and the phase ϕ are defined using relationships (9.10). In the undamped case the dependence $J = J^0(\omega)$ is given by the formula

$$J^0(\omega) = -\Delta \left[\chi_{kk}^*(0) \right]_{-1}^{-1} = \Delta \left[\sum_{j=1}^v \frac{\sin^2 \frac{\pi j v}{2} \tan \frac{\pi \Omega_j}{2} \frac{(n+1)\Omega_j}{n+1}}{\pi \Omega_j} \right]_{-1}^{-1}$$

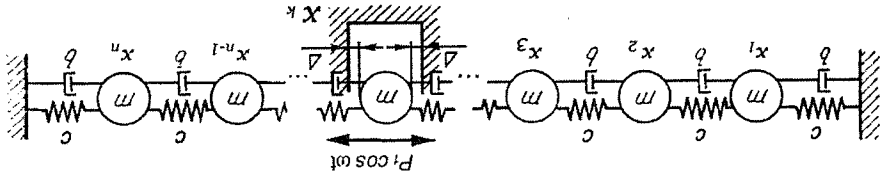


Fig. 9.8

As shown above, the representation of the fundamental and subperiodic regimes with single interaction over the half-period of motion requires analytical expressions for the symmetric PGF $\chi_{jq}^*(t)$ of the corresponding problem. The above PGFs are fully defined by the dynamic compliance operators of the linear part of the system. Hence, the solution starts from the constructing the desired PGFs. For this purpose we apply the results of section 1.5 and well-known features of linear chains [82,108,121]. The eigenfrequencies of the linear system are expressed as

$$\Omega_j = 2\sqrt{c} \sin \frac{\pi j}{2(n+1)}, \quad j = 1, \dots, n$$

The matrix of normalised coefficients of the eigenforms is given by

$$[\Theta_{jq}] = \sqrt{\frac{2}{n+1}} \sin \frac{\pi j q}{2}, \quad j, q = 1, \dots, n$$

The linear equation of the considered system may be written in matrix form, as follows

$$A\ddot{x} + B\dot{x} + Cx = P(t)$$

where $A \equiv I$, and B and C are banded matrices

$$B = \begin{bmatrix} 0 & & & \\ & 2b & & \\ & & \ddots & \\ -b & 2b & & \\ & & & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & & & \\ & 2c & & \\ & & \ddots & \\ -c & 2c & & \\ & & & 0 \end{bmatrix}$$

As a result, decomposition (1.8), obtained under the assumption of the viscous friction being small, takes the form

$$L_{jq}(p) = \sum_{v=1}^n \frac{2}{\pi} \sin \frac{\pi j v}{2} \sin \frac{\pi q v}{2} \frac{1}{p^2 + 2\Gamma_v \Omega_v p + \Omega_v^2}$$

where, in accordance with the features of the matrix B ,

Regimes of higher order may be analysed similarly. In the context of the above problem, it is appropriate to discuss the problem of the filtration of impacts. In Sect. 1.6 it was assumed that a wide class of linear systems possesses the feature of the low-pass filter. The higher harmonics of excitation are

$$f_j^* = |L_{hj}(i\omega)| R.$$

with the phase lags $\phi_j = \varphi - \arg L_{hk}(i\omega) + \arg L_{hj}(i\omega)$ and the magnitudes

$$x_j(t) = f_j^* \cos(\omega t + \phi_j) - J \chi_{hj}^*(t)$$

obtained easily

After the parameters J and φ are defined, the desired field of displacements may be The corresponding analysis was carried out in the resonant frequency band $\omega > \Omega_1$.

$$\sum_{n=1}^j [n+1] \Omega_j \left[\sin^2 \frac{\pi n j}{2\omega} \tan \frac{\pi \omega}{2\Omega_j} \right] = 0$$

to the roots of the transcendental equation virtue of the energy condition) regimes. The vertical asymptotes $\omega = \Omega_j^*$ correspond hold. This must be verified numerically. The dashed lines correspond to unstable (by frequencies $\omega = \Omega_j$ (grazing regimes) the geometrical condition $|x(t)| \leq \Delta$ may not function $J(\omega)$ when $b \neq 0$, $R < 1$. It is worth noting that in the vicinity of the Figure 9.9a shows qualitatively the above dependence. Figure 9.9b portrays the

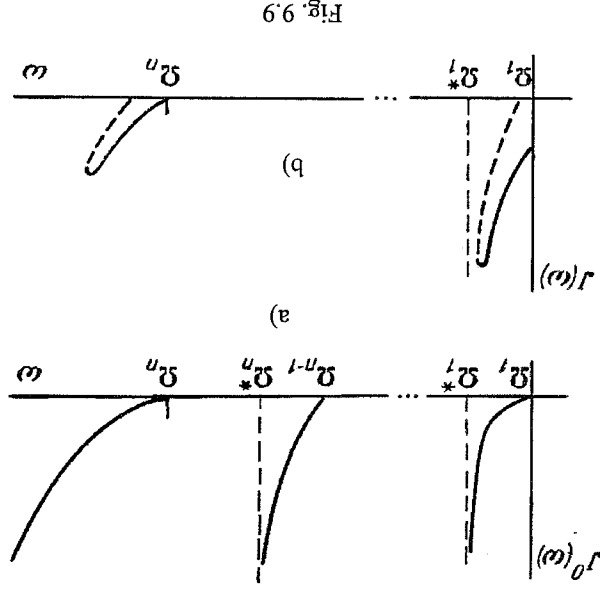


Fig. 9.9

suppressed by a mechanical system as compared with the amplification of low-frequency components. As applied to our problem this feature may be revealed from the following reasoning.

If $L_{hk}(p) = O(p^{-2})$, then the regime of the motion $x_k(t)$ is continuous, but possesses a discontinuous derivative. The corresponding Fourier series contain coefficients $O(k^{-2})$. It may be shown readily that the regimes of motion $x_{k\pm 1}(t)$ for the neighbouring masses are defined by the operators $L_{k,k+1}(p)$ and $L_{k,k-1}(p)$, the coefficients of which are decreasing at the rate of $O(p^{-3})$ as $p \rightarrow \infty$. Speaking generally, the distancing from the interacting point results in a further decrease of the coefficients of the functions $L_{hj}(p)$. As $p \rightarrow \infty$, the coefficients of the series for $L_{hj}(p)$ are $O(p^{-\alpha})$, where $\alpha = 2 + |k - j|$. Hence, the PGF $\chi_{hj}^*(t)$, which is involved in the representation of the motions $x_j(t)$, has $(k - j)$ continuous derivatives. If the body under observation is well distanced from the interacting body, its motion will be practically sinusoidal and only the fundamental harmonic is relevant. This feature of the operators $L_{hj}(p)$ may be verified directly from the observation of their form (the feature of orthogonality of the matrix $[\Theta_{ij}^q]$ is to be considered) or from direct integration of the equations of motion. It is worth noting that if the frequency of excitation is high enough, the fundamental harmonic might be also suppressed. As a result, the well-distanced bodies might be practically stationary!

Therefore, the further smoothing and attenuation of the motion, which is observed as the distance from the point of impact increases, is the most typical manifestation of the impact (or strongly nonlinear interaction) filtration. An example of the evolution of the

motions $x_j(t)$ is shown in Fig. 9.10.

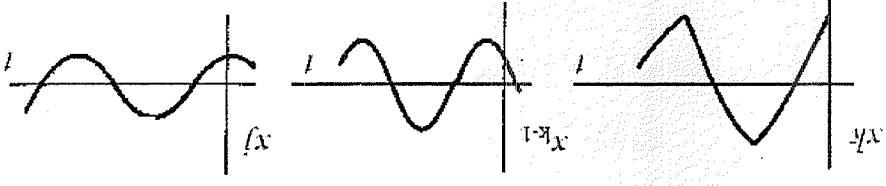


Fig. 9.10

As the second example, consider the quasi-resonant regimes in the symmetric systems (compare with Sect. 9.9). By using the representations for the PGF $\chi^*(t)$ in terms of a

See monograph [121] for details.

Fourier series and after manipulations similar to those in Sect. 8.6, we find for the time of duration of the quasi-resonant process under a wideband excitation

$$T_0 > n_0^2(p_0)S_0 \frac{(2k+1)^2 \omega^2 Q_{2,2k+1}^2}{\Delta^2} \left\{ \frac{\chi^*(0)}{2m(1+R)} - (2k+1)\omega^2 \pi^{-1} \left(\text{Im} L_{2k+1}^{-1} \right) Q_{2,2k+1}^2 \right\}^2$$

The condition of the absence of interactions under the narrowband excitation takes the form (compare with (9.46))

$$\omega Q_{1,n_0}(p_0) \sqrt{\frac{1}{2} \xi^2(t)} > -\frac{\chi^*(0)}{1-R} \left[\frac{2m(1+R)}{1-R} - (2k+1)\omega^2 \pi^{-1} \left(\text{Im} L_{2k+1}^{-1} \right) Q_{2,2k+1}^2 \right]$$

where all notation is similar to that in Sect. 9.9. The corresponding expressions for the power of the external source and the dissipation in the latter two formulae were obtained over half of the period.

As the last example, consider the system in Fig. 9.11. The cantilever beam is produced of viscoelastic material and carries a tip body, which is disposed between two solid obstacles with the clearance 2Δ and whose mass is m . The beam is involved in bending vibration under the T -periodic force $P(t)$ which is applied to the tip body.

Assuming that $P(t+T/2) = -P(t)$, in this system we can expect the appearance of symmetric T -periodic regimes.

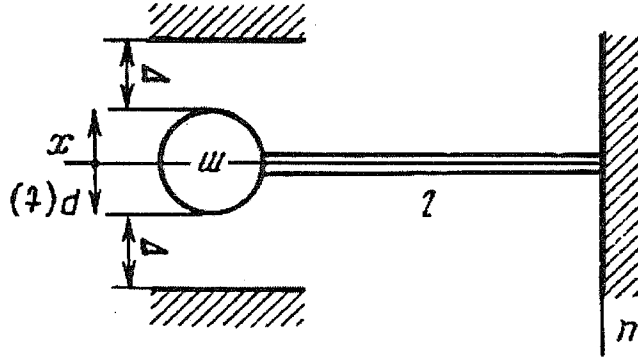


Fig. 9.11

The equation of motion takes the form

$$\left(\frac{\partial^2}{\partial t^2} + c^2 \frac{\partial^4}{\partial x^4} \right) u(x, t) + \left\{ \Phi_0^* [u(l, t)] + m_1 \frac{\partial^2 u(x, t)}{\partial t^2} \right\} \delta(x-l) = \varepsilon D \left(t, u; \frac{\partial}{\partial t}, \frac{\partial}{\partial x} \right)$$

where Φ_0^* is the symmetric force of the elastic impact, symbol εD denotes the nonconservative forces applied to the system, $c^2 = EI\mu^{-1}$, EI is flexural stiffness, μ

is the linear density, and $m_1 = m/\mu$.

When $\varepsilon = 0$ the system is conservative and by analogy with Sect. 9.7 the law of motion of the tip body is given by the relationship

$$u(l, t) = -J \chi^*(l, l; t), \quad J = -\Delta \left[\chi^*(l, l; 0) \right]^{-1}$$

where the symmetric PGF is

$$\chi^*(x, y; t) = 2T^{-1} L_{2k+1}(x, y) e^{2k+1}(t)$$

where $L_{2k+1}(x, y) \equiv L[x, y; i(2k+1)\omega]$ is the dynamic compliance from point y to point x at the frequency $(2k+1)\omega$.

The operator $L(l, l; i\omega)$ may be found in reference [18] as

$$L(l, l; i\omega) = \frac{l^3 B_0[\alpha(\omega)]}{EI \alpha^3(\omega) \{ E_0[\alpha(\omega)] - m_0 \alpha(\omega) B_0[\alpha(\omega)] \}}, \quad m_0 = m_1/(l\mu)$$

where $B_0(\alpha) = \coth \alpha - \sinh \alpha \cos \alpha$, $E_0(\alpha) = 1 + \coth \alpha \cos \alpha$ are the tabulated functions [4], $\alpha(\omega) = l[\mu \omega^2 (EI)]^{-1/4}$. Hence, by making the substitution

$\omega \rightarrow (2k+1)\omega$ and obtaining the series for the symmetric PGF $\chi^*(l, l; t)$ we find the motion $u(l, t)$. In particular, the expression for the impact impulse takes the form

$$J = -\Delta \left[\sum_{k=\pm 1, \pm 3, \dots} \frac{EI \alpha_k^3 [E_0(\alpha_k) - m_0 \alpha_k B_0(\alpha_k)]}{2T^{-1} l^3 B_0(\alpha_k)} \right]^{-1}, \quad \alpha_k \equiv \alpha(k\omega)$$

In order to find the existence conditions of the resonant regimes it is necessary to obtain the energy balance condition for the corresponding motion, as was done above. This problem is analysed in detail in reference [17] using equivalent linearisation. The expression for the dynamic compliance operator from the cited reference allows us to obtain the exact solutions for the above problem using representations (9.9) and (9.10). However, the corresponding expressions obtained as a result of the above manipulations are very tedious and may be used for the quantitative improvement of approximate solutions.

¹ A more general problem was analysed in [8].

§10 Parametric Vibration in the Multiple-Degree-of-Freedom Systems

10.1. Method of Analysis

We continue our study of parametric strongly nonlinear systems. Similarly to §7 we assume that the interaction is momentary and may be described by a singular force of impact. Some of the results obtained below will also be valid for the nonmomentary interaction. In spite of this, consideration of the limiting case as $\lambda \rightarrow \infty$ allows approximations of the desired motion in terms of the PPGF (§3) and performing an analysis of multiple-degrees-of-freedom (MDOF) systems using the scheme of §7. The results obtained in the previous paragraph indicate that (at least from the formal point of view) the periodic regimes which arise in the MDOF systems may be described using the integral (integro-differential) equations which are similar to those of the simplest systems. The only difference is that the corresponding PPGFs are of a more complicated structure. Bearing this in mind, in the present analysis we omit some evident details.

Consider the system described by the following operator equation

$$(10.1) \quad x(t) = L(p) \{ \mu c(t) x(t) - \Phi_1(x, \dot{x}) + \mathcal{E}^p(t) \}$$

where $L(p) = O(p^{-2})$ as $p \rightarrow \infty$, $\mu = O(\varepsilon)$ is modulation level, $c(t)$ and $P(t)$ are T -periodic functions defining the parametric and force excitation, respectively, and $\Phi_1(x, \dot{x})$ is a singular nonlinearly defining both the elastic and dissipative components of the force of impact interaction. We assume also that $L(i\omega) = Q_1(i\omega) + i\varepsilon Q_2(i\omega)$, i.e. the order of dissipation and excitation is the same. In accordance with (3.16) the T -periodic regimes $x(t)$ are defined by the integro-

differential equation

$$(10.2) \quad x(t) = \int_0^T \kappa(t, s; \mu) \{ \mathcal{E}^p(s) - \Phi_1[x(s), \dot{x}(s)] \} ds$$

where the PPGF $\kappa(t, s; \mu)$ is obtained using the method of §3, this is defined by the operator $L(p)$ and the parametric excitation $\mu c(t)$.

For the impact nonlinearity, describing the momentary interaction in the single-impact regime (see Sect. 9.3), we find that

$$(10.3) \quad \Phi_1[x(t), \dot{x}(t)] = J \delta^T(t - \phi)$$

where the impulse J and phase ϕ are subject to the consistency conditions

$$(10.4) \quad x(\phi) = \Delta, \quad J = mx(\phi - 0)(1 + R) \geq 0$$

and for the convenience of calculations the undefined phase may be related to the force or to the parametric excitation (see §7); the moment of impact coincides with the time zero reference. Substituting (10.3) into (10.2) and accounting for (10.4) we find the representation

$$(10.5) \quad \begin{aligned} x(t) &= \mathcal{E}(t) - J \kappa(t, \phi; \mu) \\ \mathcal{E}(t) &= \mathcal{E} \int_0^T \kappa(t, \phi; \mu) P(s) ds \end{aligned}$$

where the unknown parameters J and ϕ are to be defined from the system of

$$(10.6) \quad J = \frac{\mathcal{E}(\phi) - \Delta}{\mathcal{E}(\phi) - \Delta}, \quad J = m(1 + R) [\mathcal{E}(\phi) - J \kappa(t, \phi; \mu)] \Big|_{t=\phi-0}, \quad \kappa(\phi, \phi; \mu)$$

Hence, the problem of the definition of the single-impact periodic regimes is reduced to a construction of representation (10.5), the solution of system (10.6) and verification of the condition $x(t) \leq \Delta$, $J \geq 0$.

The obtained relationships are simple and compact. However, even approximate expressions for the PPGF $\kappa(t, s; \mu)$ are cumbersome; an analysis of Eq. (10.6) in a very common case is possible using numerical calculations only. Nevertheless, the simplifications used in §7 allow for analytical delivery of the important information about the studied process.

The second condition (10.4) may be substituted by the energy balance equation written for motion (10.5) or corresponding approximation. Speaking generally, formulae (10.5) and (10.6) do not assume the smallness of the parameter ε . However, for the purposes of analysis of the resonant regimes at low levels of excitation and dissipation the construction of the PPGF $\kappa(t, s; \mu)$ may be simplified and relatively simply interpretable results may be obtained. The unstable regimes may be separated easily using the instability energy condition (see (9.11) and above).

Similarly to the analysis of the SDOF system, the analysis of resonant regimes may be conducted now in two steps. First, we construct the regimes that are close to the periodic vibration of the generative system under external excitation. Second, by accounting for the higher approximations we obtain the corrections reflecting the specifics of the parametric excitation. Due to the presence of multiple-degrees-of-freedom, the parametric phenomena in the linear part of the system become more complicated. Along with the *simple resonances* the *combination resonances* may arise in the system.

For the simplification of further notation in Sect. 10.2-10.4, we assume that the period of the solution equals that of the excitation. Besides, as usual, we consider the case $\Delta \neq 0$.

Parametric vibration in the MDOF systems was studied, in particular, in the cited monographs by Bolotin [25] and Schmidt [155]. The related references may be found

in [41]. The following results are based on papers [16,92].

10.2. Equation of Energy Balance

Let the parametric system be described by Eq. (10.1) and the force excitation $P(t)$ may be neglected

$$x(t) = L(p)\{\mu c(t)x(t) - \Phi_1(x, \dot{x})\} \quad (10.7)$$

When $\mu = 0$, $R = 1$ and $\text{Im } L(i\omega) = \mathcal{Q}_2(i\omega) = 0$ the system is conservative. We restrict our study to the case of fundamental resonance regimes, and assume that the frequency $\omega = 2\pi T^{-1}$ coincides with one of the natural frequencies of the generative system. Thus allowing a T -periodic solution in the form

$$x_0(t) = -\int_T^0 \chi(t-s)\Phi_1[x(s), \dot{x}(s)]ds = -J\chi(t-\varphi) \quad (10.8)$$

where the period of the PGF $\chi(t)$ is T , the phase of impact φ takes an arbitrary value, and $J = -\Delta/\chi(0)$. The structure of the PGF, $\chi(t)$, is defined by the real part of the operator $L(i\omega)$,

$$\chi(t) = \frac{T}{\mathcal{Q}_{ik}} e^{ik} (t) = \sum_{j=1}^{\infty} \frac{A_j}{\cos[\Omega_j(t - \frac{1}{2}T)]} \frac{\sin(\frac{1}{2}\Omega_j T)}{\Omega_j} \quad (10.9)$$

where the latter equality is written using (2.19). It is valid for $0 \leq t < T$, and the coefficients A_j depend on the coefficients of the eigenforms of the linear system, Ω_j are the natural frequencies, and all eigenforms are involved in summation (10.9).

For small modulations and damping, when $\text{Im } L(i\omega) = \mathcal{Q}_2(i\omega)$ and $(1-R) = O(\varepsilon)$, $\mu = O(\varepsilon)$ we can expect that the form of T -periodic regimes corresponding to the parametric excitation

$$c(t) = \sum_{k=1}^K c_k \cos(k\omega t + \varphi_k)$$

is close to that of (10.8). Now we will relate the unknown phase of the impact to the external excitation. In accordance with the energy balance principle, we can now find the condition for sustaining regime (10.8) in terms of the work done by the forces of parametric excitation. In agreement with Carnot's theorem and Sect. 9.4, we find that $E(J^0, \varphi) = (J^0)^2 \frac{2m(1+R)}{1-R} + \int_0^T [L^{-1}(p)x_0(t) - \mu c(t) + \varphi]x_0(t)dx_0(t)dt = 0$ (10.10) where we denote

$$x_0(t) = -J^0 \chi(t) \approx x(t)$$

$$J^0 = -\Delta \left[\sum_{j=1}^{(J)} A_j (2\Omega_j)^{-1} \cot(\frac{1}{2}\Omega_j T) \right]^{-1} \quad (10.11)$$

Substitution of (10.9) and (10.11) into (10.10) and consequent integration yields the equation for the definition of the unknown phase φ

$$E_d + \mu(J^0)^2 \sum_{k=1}^K \sum_{j=1}^{(J)} \frac{\Omega_j \sin(\frac{1}{2}\Omega_j T) \sin(\frac{1}{2}\Omega_j T)}{c_k k \omega A_j A_q \sin(\varphi_k + k\omega\varphi)} \times \left[\frac{\sin[\frac{1}{2}(\Omega_j + \Omega_q)T]}{\Omega_j + \Omega_q} + \frac{(\Omega_j + \Omega_q)^2 - k^2 \omega^2}{(\Omega_j + \Omega_q)^2 - k^2 \omega^2} \right] = 0 \quad (10.12)$$

where E_d denotes the work done by the forces of damping on motion (10.11). If, for example, the energy losses in the system are described using Raleigh's dissipative function, the corresponding calculations are performed in Sect. 9.6 (see (9.27)). Eq. (10.12) may be obtained also in accordance with the procedure of Sect. 9.8 by calculation of the series.

It is seen that both terms involved in (10.12) are proportional to the value $(J^0)^2$. Hence, at the frequencies corresponding to an infinite impulse, and defined by the equation

$$\sum_{j=1}^{(J)} A_j (2\Omega_j)^{-1} \cot \frac{\omega}{2\Omega_j} = 0 \quad (10.13)$$

the linear and impact damping may not be sufficient to stabilise the system. This leads to the appearance of infinite amplitudes. The restriction of the above amplitudes may be obtained by accounting for nonlinear friction [25,155]. The influence of the external source may be accounted for in the energy balance equation after the combination of (10.12) with the expressions obtained in Sect. 9.6. Let the external and parametric excitations be sinusoidal and cophasal. Then, in (10.1) we can let $c(t) = c_1 \cos(\omega t + \varphi)$, $P(t) = P_1 \cos(\omega t + \varphi)$, where φ is the unknown phase. The existence condition of resonant regimes follows from the energy balance equation and takes the form $|\sin \varphi| \leq 1$

where we denote $\Omega_{\pm}^{jq} = \Omega_j \pm \Omega_q$.

For systems with symmetric impact nonlinearity instead of (10.11), (10.12), etc., after standard manipulations with the symmetric PGF in the form

$$\chi^*(t) = \sum_{j=1}^{(j)} \frac{A_j \sin[\Omega_j(t - 1/4T)]}{2\Omega_j \cos(1/4\Omega_j T)}, \quad 0 \leq t < T/2,$$

for symmetric regimes $x(t + 1/2T) = -x(t)$ which take place only if $c(t + 1/2T) = c(t)$, we find the formulae

$$x(t) \approx x_0(t) = -x_0^*(t), \quad x_0^*(t) = \Delta \left[\sum_{j=1}^{(j)} A_j (2\Omega_j)^{-1} \tan(1/4\Omega_j T) \right]^{-1}$$

$$E_d + \mu(J_0)^2 \sum_{k=1}^{\infty} \sum_{j=1}^{(j)} \Omega_j \cos(1/4\Omega_j T) \cos(1/4\Omega_k T) \left[\frac{\sin(1/4\Omega_{+}^{jq} T)}{\sin(1/4\Omega_{+}^{jk} T)} + \frac{(\Omega_{+}^{jq})^2 - 4k^2\omega^2}{(\Omega_{+}^{jk})^2 - 4k^2\omega^2} \right] = 0$$

etc. The regimes $x(t + T/2) = -x(t)$ may exist also under the combined excitation. This requires $P(t + T/2) = -P(t)$.

The energy balance equations for the MDOF systems, therefore, are quite analogous to the corresponding equations for the SDOF system. Similarly to the case of the SDOF system, the problem of matching the force and the parametric excitation becomes relevant in this case. Almost similarly to the simplest systems, the presence of few harmonics in the excitation disturbance may cause the appearance of few pairs of resonant regimes. These phenomena were discussed above. Equation (10.12) or the third equation of (10.15) indicate, in particular, that at resonance the energy balance is controlled not only by the frequencies at which the backbone curve possesses the asymptotes (roots of Eq. (10.13)). Similarly to the case of SDOF system, the energy balance may be controlled by the frequencies of the linear

10.3. Auxiliary Analysis

approximations must be accounted for.

parametric instability. It is known [25] that if ν is the frequency of the periodic parametric excitation, then when $\nu \approx 2\Omega_j k^{-1}$ or when $\nu \approx |\Omega_j \pm \Omega_q| k^{-1}$, $k = 1, 2, \dots$ in the linear parametric system the build up of simple or combination resonance is possible. The development process of the resonance is accompanied by significant energy "pumping" from the source into the system. This causes different types of complex regimes of motion. Solutions (10.11) of the periodic single-impact regimes do not contain information about the periodic excitation. To correct this, the higher

The present analysis follows the scheme of §7. In order to study the behaviour of system (10.7) or (10.1) we have to construct representation (7.12) for the system with an arbitrary number of degrees of freedom. Here the friction is considered in the energy balance equation only, the unknown phase is related to the parametric excitation, and the instant of the impact coincides with the initial time reference. In this section we obtain the formula for the second repetitive kernel which is involved in (7.12). Accounting for (7.14) and (2.20) we find that

$$\kappa_2(t + \phi, \phi) = \sum_{q=1}^{\infty} \sum_{j=1}^{(j)} \sum_{k=1}^{(k)} \frac{c_k A_j A_q}{4\Omega_j \Omega_q \sin(1/2\Omega_j T) \sin(1/2\Omega_q T)} \times \left\{ \int_0^t \cos[\Omega_j(t - \tau - 1/2T)] \cos[\Omega_q(\tau - 1/2T)] \times \cos(k\omega\tau + \phi_k + k\omega\phi) d\tau + \int_T^t \cos[\Omega_j(t - \tau + 1/2T)] \right.$$

$$\left. \times \cos[\Omega_q(\tau - 1/2T)] \cos(k\omega\tau + \phi_k + k\omega\phi) d\tau \right\}$$

Integration and elementary rearrangements yield

(10.16)

In the second approximation for the PPGF we obtain the representation defining formula (7.12)

(10.17)

$$(10.18) \quad \left[(\phi, \phi + t)_*^2 M + (t)_* \chi \right] f = (t)_* x$$

the first repetitive kernel

10.4.

(61.01)

(10.18), the condition $|x(t)| \leq \Delta$ does not hold, respectively).

Information about the behaviour of the resonant regimes is given by the interaction impulse J (the phase ϕ may be obtained from energy balance condition (10.12))

The obtained relationships may be applied to the calculation of engineering structures. Some of the relevant dynamic models are considered in the following sections. If it is desired to analyse more complex regimes (subperiodic, for example) the above relationships are still valid. Only substitutions for the period $T_1 = lT$ and for the frequency $\omega_1 = \omega l^{-1}$ are required.

10.5. Parametric Vibration of an Oscillator Suspended Inertially Inside a Container

In the present section we consider the problem which was analysed in [17]¹ using the stitching method in the case of a pure forced excitation. In Fig. 10.1 the dynamic model contains two interacting elements (suspension and container) and a linear oscillator which is mounted upon the suspension.

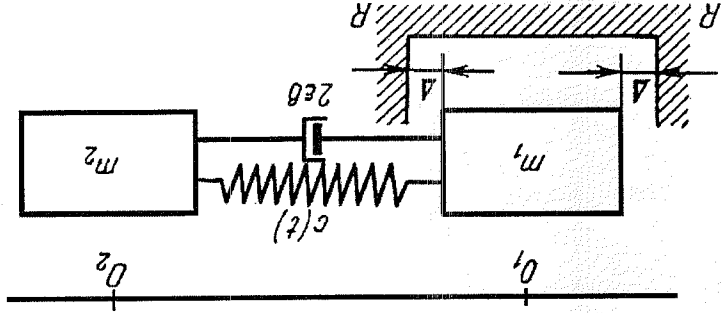


Fig. 10.1

The bodies of the suspension m_1 and of the oscillator m_2 are connected by elastic and damping elements with coefficients c and $2\epsilon b$. We assume that the elasticity possesses a periodic component which is proportional to $\mu \cos(2\omega t + \varphi)$, where the phase φ is due to the choice of the initial time reference, $\mu = O(\epsilon)$. Assume that the left-hand and right-hand side restitution ratios are similar. The motion of each element of the system will be counted relative to a stationary datum corresponding to the symmetric disposition of the suspension in the container and the nondeformed condition of the elastic element. The analysis will be restricted to the study of the case of periodic regimes which involve consequent collisions of right-hand and left-hand side surfaces of the suspension against the internal walls of the container.

¹ The analysis mentioned is contained in the Russian edition of the monograph only.

$$J = -\frac{\Delta}{\Delta} \frac{\chi(0) + \mu k \kappa^2(\varphi, \varphi)}{2} \left\{ \sum_{j=1}^{(j)} \frac{A_j}{\Omega_j T} \cot \frac{2\Omega_j}{2} + \mu \sum_{k=1}^{\infty} \sum_{j=1}^{(j)} \sum_{q=1}^{(q)} \frac{c_k A_j A_q}{4\Omega_j \Omega_q} \cos(k\omega\varphi + \varphi_k) \right. \\ \left. \times \left[\frac{\Omega_{(+)}^{(q)}}{\Omega_{(+)}^{(q)}} \frac{\cot \frac{\Omega_{(+)}^{(q)}}{2} - k^2 \omega^2}{\cot \frac{\Omega_{(+)}^{(q)}}{2} + \cot \frac{\Omega_{(+)}^{(q)}}{2}} + \frac{\Omega_{(-)}^{(q)}}{\Omega_{(-)}^{(q)}} \frac{\cot \frac{\Omega_{(-)}^{(q)}}{2} - k^2 \omega^2}{\cot \frac{\Omega_{(-)}^{(q)}}{2} - \cot \frac{\Omega_{(-)}^{(q)}}{2}} \right] \right\}^{-1}$$

(10.20) Parametric forces, therefore, may sustain the quasi-free vibration and may also generate the motions which are defined by the nature of excitation. It is evident from (10.20) that the equation for the frequencies of infinite impulses may be written in the form

$$\sum_{j=1}^{(j)} A_j (2\Omega_j)^{-1} \cot \left(\frac{1}{2} \Omega_j T \right) + \mu B (\Omega_j; \Omega_{(+)}^{(q)}, \Omega_{(-)}^{(q)}) = 0 \quad (10.21)$$

If the value of the parameter B involved in the latter equation is $O(1)$ and $\mu = O(\epsilon)$, then Eq. (10.13) describes accurately the frequencies of large impulses. If the above parameter B is not small, the frequencies of large impulses must be found from Eq. (10.21). An increase in this parameter may be due to the phenomenon of linear parametric instability. As follows from the energy balance equation, the force source may sustain the vibration (if both force and parametric sources are matched) or suppress the action of a parametric source in particular frequency spans (see Sect. 7.5).

For the study of the stability of periodic regimes using the energy balance method (Sect. 7.6) we can use Eq. (10.12) and relationship (10.20). Under sinusoidal excitation we obtain two resonant regimes. The regime with the smaller value of impulse usually is unstable. Calculation of the repetitive kernels reveals the dynamic effects of higher orders, grazing regimes at the boundary of the zone of linear parametric instability, etc. Similar conclusions may be drawn with respect to the behaviour of the symmetric system. For example, (10.20) and (10.21) in the symmetric case take the form

$$J = \Delta \left\{ \sum_{j=1}^{(j)} \frac{A_j}{\Omega_j T} \tan \frac{2\Omega_j}{4} + \mu \sum_{k=1}^{\infty} \sum_{j=1}^{(j)} \sum_{q=1}^{(q)} \frac{c_k A_j A_q}{4\Omega_j \Omega_q} \cos(\varphi_{2k} + 2k\omega\varphi) \right. \\ \left. \times \left[\frac{\Omega_{(+)}^{(q)}}{\Omega_{(+)}^{(q)}} \frac{\cot \frac{\Omega_{(+)}^{(q)}}{2} - 4k^2 \omega^2}{\cot \frac{\Omega_{(+)}^{(q)}}{2} + \tan \frac{\Omega_{(+)}^{(q)}}{4}} + \frac{\Omega_{(-)}^{(q)}}{\Omega_{(-)}^{(q)}} \frac{\cot \frac{\Omega_{(-)}^{(q)}}{2} - 4k^2 \omega^2}{\tan \frac{\Omega_{(-)}^{(q)}}{4} - \tan \frac{\Omega_{(-)}^{(q)}}{4}} \right] \right\}^{-1} \quad (10.22)$$

The equations of motion of the above system take the form

$$\begin{aligned} m_1 \ddot{x}_1 + 2\epsilon b(\dot{x}_1 - \dot{x}_2) + c[1 + \mu \cos(2\omega t + \phi)](x_1 - x_2) + \Phi_1(x_1, \dot{x}_1) = 0 \\ m_2 \ddot{x}_2 + 2\epsilon b(\dot{x}_2 - \dot{x}_1) + c[1 + \mu \cos(2\omega t + \phi)](x_2 - x_1) = 0 \end{aligned} \quad (10.23)$$

Along with the variables x_1 and x_2 , for our further analysis it is convenient to introduce the new set of variables y and z , related to the initial set x_1, x_2 as follows

$$(m_1 + m_2)x_1 = y + m_2 z, \quad (m_1 + m_2)x_2 = y - m_2 z$$

Since that the impact coincides with the initial time reference we find that

$$\Delta = \frac{m_1 + m_2}{y(0) + m_2 z(0)}$$

and the equations of motion in the new variables take the form

$$\ddot{z} + 2\epsilon b\dot{z} + \Omega^2[1 + \mu \cos(2\omega t + \phi)]z + \dot{m}_1^{-1}\Phi_1\left(\frac{m_1 + m_2}{y + m_2 z}, \frac{m_1 + m_2}{\dot{y} + m_2 \dot{z}}\right) = 0$$

(10.24)

$$\ddot{y} + \Phi_1\left(\frac{y + m_2 z}{m_1 + m_2}, \frac{\dot{y} + m_2 \dot{z}}{m_1 + m_2}\right) = 0$$

The first of these equations describes the evolution of the relative approaching of the oscillator and suspension masses, the second describes the saw-like motion of the system "oscillator-suspension" as a solid structure. In the above equations we denote

$$\Omega^2 = c/m, \quad b_0 = b/m, \quad m = m_1 m_2 (m_1 + m_2)^{-1}$$

The system in the form (10.24) will be used for simplification. It indicates that all nonconservative forces except the impact damping are acting on the motion $z(t)$.

In accordance with the accepted scheme of analysis, define the solution of the degenerative system for $b = \mu = 1 - R = 0$. For the linear part of the system we have to construct the system of PGFs. The natural frequencies of the linear system may be obtained immediately from (10.24): $\Omega_1 = 0$ (the system does not possess the full damping feature) and $\Omega_2 = \Omega$ which is the frequency of the relative vibration of the system "oscillator-suspension". The above PGFs are the solution of the system

$$m_1 \ddot{x}_1 + c(x_1 - x_2) = \delta_{T/2}(t), \quad m_2 \ddot{x}_2 + c(x_2 - x_1) = 0$$

which is equivalent to

$$\ddot{y} = \delta_{T/2}(t), \quad \ddot{z} + \Omega^2 z = m_1^{-1} \delta_{T/2}(t)$$

Hence, for $0 \leq t < T/2$ we have

$$y(t) \equiv \chi_y^*(t) = t/2 - T/8, \quad z(t) \equiv \chi_z^*(t) = \frac{1}{\sin[\Omega(t - T/4)]} \frac{2\Omega m_1 \cos(\Omega T/4)}{\sin[\Omega(t - T/4)]}$$

Thus, in agreement with the formulae of coordinate transformations we find

$$\begin{aligned} x_1(t) \equiv \chi_{11}^*(t) = \frac{1}{m_2} \frac{\sin[\Omega(t - T/4)]}{\sin[\Omega(t - T/4)]} + \left(\frac{2}{t} - \frac{8}{T}\right) \left[\frac{m_1 + m_2}{2\Omega} \frac{\cos(\Omega T/4)}{\cos(\Omega T/4)} - \left(\frac{t}{2} - \frac{8}{T}\right) \frac{1}{\sin[\Omega(t - T/4)]}\right] \\ x_2(t) \equiv \chi_{21}^*(t) = \frac{1}{m_1 + m_2} \left[\left(\frac{t}{2} - \frac{8}{T}\right) \frac{m_1 + m_2}{2\Omega} \frac{\cos(\Omega T/4)}{\cos(\Omega T/4)} - \frac{1}{\sin[\Omega(t - T/4)]}\right] \end{aligned}$$

The solution of the conservative vibro-impact problem in terms of initial and new coordinates takes the form

$$x_1(t) = -J^0 \chi_{11}^*(t)$$

$$J^0 = -\frac{\Delta}{\Delta(m_1 + m_2)} \left(\frac{8}{T} + \frac{m_2}{2\Omega m_1} \tan \frac{4}{\Omega T} \right)^{-1}$$

$$x_2(t) = -J^0 \chi_{21}^*(t), \quad y(t) = -J^0 \chi_y^*(t), \quad z(t) = -J^0 \chi_z^*(t)$$

In monograph [17], detailed analysis of the obtained solutions was carried out. It appears that it is feasible to study the single-impact regimes in the frequency band $\Delta = \left\{ \frac{1}{2} \Omega < \omega < \Omega_*, \Omega \leq \omega < \infty \right\}$, where the frequency Ω_* corresponds to the case of equality in the relationship, establishing the positivity of the impact impulse

$$\frac{8}{T} + \frac{2\Omega m_1}{m_2} \tan \frac{4}{\Omega T} \geq 0, \quad T = \frac{2\pi}{\omega}$$

Hence, $J^0 \rightarrow \infty$ as $\omega \rightarrow \Omega_*$, $J^0 \rightarrow 0$ as $\omega \rightarrow \Omega$ (grazing process) and, finally, $J^0 \rightarrow \infty$ as $\omega \rightarrow \infty$.

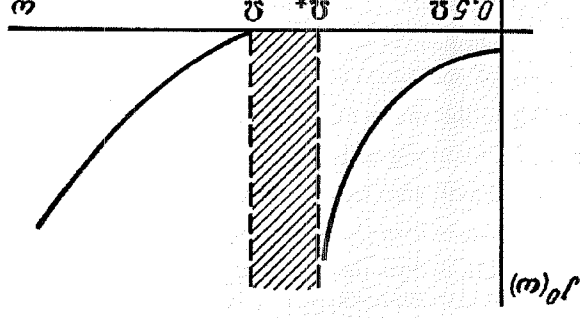


Fig. 10.2

Outside the frequency domain Δ mentioned above the geometrical condition $|x_1(t)| \leq \Delta$ does not hold. Hence, the frequency band Δ is thought of to be the domain of natural frequencies of the conservative system. The graph of the dependence

$J^0(\omega)$ is depicted in Fig. 10.2. The hatched area corresponds to the supplement to the

domain Δ . From above, the resonant frequencies outside the zone of linear parametric instability are $\omega = \Omega_*$ and $\omega = \infty$. We can also expect the development in system (10.23) of the T -periodic resonant regimes at the frequency $\omega \approx \Omega$ due to the periodic excitation of the frequency 2ω .

To estimate this, we find the solutions of the second approximation. In the coordinates y, z we find

$$y(t) = -J(t/2 - T/8), \quad z(t) = -J \left[\chi_z^*(t) - \mu \Omega^2 \kappa_z^*(t + \varphi, \varphi) \right]$$

where in accordance with (10.24) for the calculation of the repetitive kernel κ_{zz} we can use the results of the calculations carried out for the SDOF system. By making use of the values of the coefficients which are involved in the problem and formula (7.19) (particular case of (10.19)) we find

$$\kappa_{zz}^*(t + \varphi, \varphi) = \frac{\cos(\omega t + \varphi)}{4\Omega^2 m_1 \cos(\Omega T/4)} \left\{ -\frac{\omega}{1} \sin \omega t \cos \left[\Omega \left(t - \frac{4}{T} \right) \right] \right.$$

$$\left. + \frac{1}{4} \frac{2(\omega - \Omega)}{\Omega T} \sin \left[(\omega - \Omega)t + \frac{4}{\Omega T} + \frac{2(\omega + \Omega)}{1} \sin \left[(\omega + \Omega)t - \frac{4}{\Omega T} \right] \right] \right\}$$

Now the solutions in terms of the initial variables take the form

$$x_1(t) = -(m_1 + m_2)^{-1} J \left[\chi_y^*(t) + m_2 \chi_z^*(t) - \mu \Omega^2 m_2 \kappa_z^*(t + \varphi, \varphi) \right]$$

$$x_2(t) = -(m_1 + m_2)^{-1} J \left[\chi_z^*(t) - m_1 \chi_z^*(t) + \mu \Omega^2 m_1 \kappa_z^*(t + \varphi, \varphi) \right]$$

The condition $x_1(0) = \Delta$ yields

$$J = \frac{\Delta(m_1 + m_2)}{\frac{1}{8} T + \frac{m_2}{m_1} \left[1 + \frac{\mu \Omega^2 \cos \varphi}{2(\omega^2 - \Omega^2)} \right] \tan \frac{4}{\Omega T}} \quad (10.27)$$

In order to estimate the value of the unknown phase we construct the energy balance equation. It is worth noting that the process of energy dissipation and replenishment is described by the first equation (10.24) and, hence, the desired equation will be similar to Eq. (7.43) obtained for the SDOF system. With accepted accuracy, by making use of solutions in the form of (10.25) and after manipulations similar to that in §7 we find

Two values for the unknown phase follow from (10.28). The existence condition of the considered regimes takes the form

$$\rho \frac{1-R}{1+R} + \frac{2 \cos^2(\frac{1}{4} \Omega T)}{2 \cos^2(\frac{1}{4} \Omega T)} = \frac{2(\Omega^2 - \omega^2)}{\mu \Omega^2 \omega \sin \varphi} \tan \frac{4}{\Omega T}, \quad \rho = \frac{m}{m_1} \quad (10.28)$$

$$\mu \geq 2 \left(\frac{\omega}{\Omega} - \frac{\omega}{\Omega} \right) \left(\frac{1-R}{1+R} + \varepsilon b_0 \frac{2 \cos^2(\frac{1}{4} \pi \Omega \omega^{-1})}{\pi \Omega^2 \sin(\pi \Omega \omega^{-1})} \cot 2\omega \right) \quad (10.29)$$

Relationships (10.25)–(10.29) define the regimes under study. As mentioned above it is feasible to analyse it in the frequency domain Δ , and a detailed study of the conditions $|x_1(t)| \leq \Delta$, $J \geq 0$ must be carried out numerically.

Formula (10.27) indicates the possibility of the appearance of infinite impulses in the narrow vicinities of the frequencies $\omega = \Omega_*$, Ω and as $\omega \rightarrow \infty$. The process of frequency pulling to the values of infinite impulses is possible only if $b_0 = 1 - R = 0$. The conditions for sustaining such regimes at the frequencies close to Ω_* and Ω may be obtained from (10.29) after the substitutions $\omega = \Omega_*$ and $\omega = \Omega$. In the case of the substitution $\omega = \Omega$, condition (10.29) is simplified and takes the form $\mu \Omega^2 \geq 4 \varepsilon b_0$.

A similar relationship defines the condition of the development of linear parametric instability. Again, it is worth noting that in the zone of linear parametric resonance the vibro-impact regimes are characterised by significant instability. Hence, complex types of motion may appear in the system along with simple regimes. The energy condition of the instability yields that lower values of impulse correspond to unstable regimes.

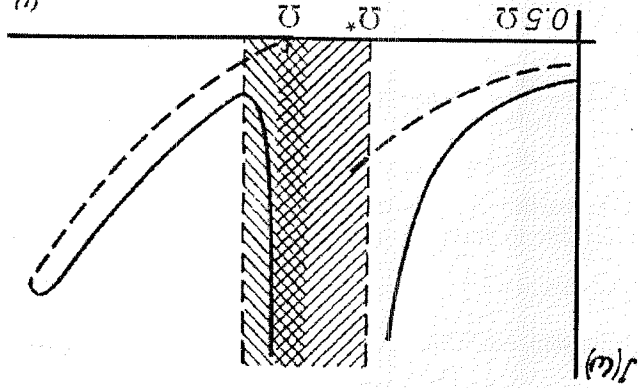


Fig. 10.3

Information about the character of the motion of suspension and the oscillator may be found in reference [17]. Figure 10.3 portrays the typical shape of the dependence $J(\omega)$. We assume now that the existence conditions of resonant regimes in the vicinities of the frequencies Ω_* and Ω are satisfied. The left hatching corresponds to the frequencies of the linear

parametric instability. The behaviour of the regimes outside the domain Λ is not reflected.

Accounting for the force sources in the right-hand sides of Eq. (10.23) may be carried out similarly to Sect. 7.9, but the corresponding calculations are not represented here. Such a problem may appear in the dynamic analysis of mechanisms with nonlinear position functions of the driven members [191]. In such mechanisms due to the imperfection (backlash) of a transmission, it is necessary to account for the possible collisions between the elements of a structure.

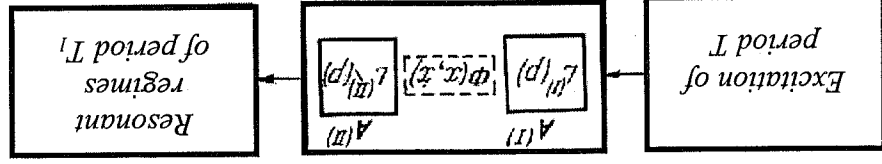


Fig. 10.4

10.6. Dynamics of Vibro-impact Mechanisms Mounted on a Vibrating Base

The methods of the present paragraph allow the analysis of the resonant regimes in the system of a general form which may be modelled as shown in Fig. 10.4. The T -periodic excitation may be either of parametric or of combined nature, the corresponding resonant regimes may be fundamental or more complex.

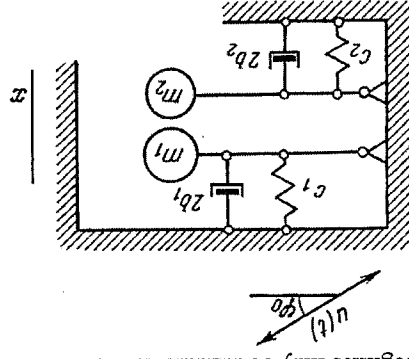


Fig. 10.5

In this section we consider briefly the dynamics of vibro-impact mechanisms mounted

on the vibrating base, as applied to two-degrees-of-freedom (TDOF) systems¹. The case of the SDOF system was analysed in §7. The estimation of the fundamental concepts for the TDOF systems allows simple analysis of higher-order system. As the first example, consider a system whose dynamic model may be reduced to a couple of interacting oscillators, see Fig. 10.5. It is assumed also that the system is subjected to sinusoidal vibration. By making use of the results of Section 7.9, the equations of motion in the case of small nonconservative forces (small vibration, small damping and quasi-elastic impact) may be reduced to the form

$$\begin{aligned} m_1 \ddot{x}_1 + \varepsilon b_1 \dot{x}_1 + c_1 [1 + \varepsilon \mu_1 \cos(vt + \varphi)] x_1 &= \varepsilon h_1 \cos(vt + \varphi) - \Phi_1(x, x) \\ m_2 \ddot{x}_2 + \varepsilon b_2 \dot{x}_2 + c_2 [1 + \varepsilon \mu_2 \cos(vt + \varphi)] x_2 &= \varepsilon h_2 \cos(vt + \varphi) + \Phi_1(x, x) \end{aligned} \quad (10.30)$$

The meaning of the notation is evident from Fig. 10.5 and Sect. 7.9, and v is the frequency of vibration, $\mu_j, h_j \sim v^2$, $j = 1, 2$ (in the relative motion the translational force of inertia is involved). The impact force $\Phi_1(x, x)$ depends on the relative coordinate $x = x_1 - x_2$, and the impact takes place at $x = \Delta$.

If the direction of vibration is perpendicular to the levers of the mechanism $\varphi_0 = \pi/2$, $\mu_1 = \mu_2 = 0$, we have the case of forced vibration which was studied in detail by Brunstein and Kobrinsky [77].

If $\varphi_0 = 0$, then $h_1 = h_2 = 0$, and we have the case of parametric vibration. If the parameters of both equations of (10.30) are equal, or if, for instance, $m_2 \rightarrow \infty$, the system may be reduced easily to the SDOF system, studied in §7.

The study of equations (10.30) in general will be carried out using the scheme widely implemented in this monograph. For example, for the regime of period $T = 2\pi\omega^{-1}$ which exists due to excitation with the frequency $v = 2\omega$ (the fundamental parametric or the second-order subperiodic resonance).

Consider the conservative approximation when $\varepsilon = 0$

$$m_1 \ddot{x}_1 + c_1 x_1 = -\Phi_0(x, x), \quad m_2 \ddot{x}_2 + c_2 x_2 = \Phi_0(x, x)$$

The operator equation for the relative coordinate takes the form

$$x = - \left[\frac{m_1^{-1}}{p^2 + \Omega_1^2} + \frac{m_2^{-1}}{p^2 + \Omega_2^2} \right] \Phi_0(x, x), \quad \Omega_j^2 = c_j m_j^{-1}, \quad j = 1, 2 \quad (10.31)$$

From the above equation for $0 \leq t < T$ we find

$$x = -j^0 \left[\frac{2\Omega_1 m_1 \sin(\frac{1}{2}\Omega_1 T)}{1 - \cos[\Omega_1(t - \frac{1}{2}T)]} + \frac{2\Omega_2 m_2 \sin(\frac{1}{2}\Omega_2 T)}{1 - \cos[\Omega_2(t - \frac{1}{2}T)]} \right] \quad (10.32)$$

¹ See also reference [109].

The condition $x(0) = \Delta$ yields

$$J^0 = -\Delta \left[\frac{1}{\Omega_1 T} \cot \frac{2\Omega_1 m_1}{2} + \frac{1}{\Omega_2 T} \cot \frac{2\Omega_2 m_2}{2} \right] \quad (10.33)$$

The equation for the definition of the frequencies of large impulses takes the form

$$\frac{1}{\pi \Omega_1} \cot \frac{\omega}{2\Omega_1 m_1} + \frac{1}{\pi \Omega_2} \cot \frac{\omega}{2\Omega_2 m_2} = 0 \quad (10.34)$$

For the coordinates x_1 and x_2 we find

$$x_1 = -J^0 \frac{\cos[\Omega_1(t-T/2)]}{\cos[\Omega_2(t-T/2)]}, \quad x_2 = J^0 \frac{2\Omega_1 m_1 \sin(\Omega_1 T/2)}{2\Omega_2 m_2 \sin(\Omega_2 T/2)}$$

It is feasible to consider the obtained solutions in the frequency range $\omega > \Omega_1$ (for definiteness $\Omega_1 < \Omega_2$). This restriction is caused, as usual, by the geometrical condition $x = x_1 - x_2 \leq \Delta$. If $\omega < \Omega_1$ the motion takes the form as portrayed in Fig. 10.6a.

In the frequency band $\Omega_1 < \omega < \Omega_2$, the motions for which the condition $x \leq \Delta$ does not hold are both possible (see Fig. 10.6b). Finally, in the high-frequency span $\omega > \Omega_2$ the motions are similar to that in Fig. 10.6c,d and our restrictions are satisfied. The relationship between the frequencies Ω_1 and Ω_2 governs the type of the process developed in the system [77].

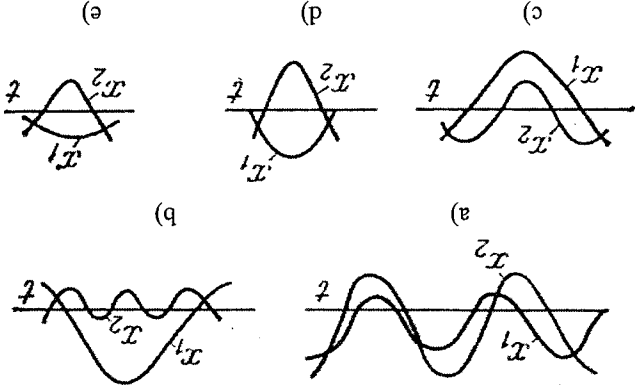


Fig. 10.6

If $\omega = \Omega_1$ or Ω_2 , we have $J^0 = 0$. At the natural frequencies of the linear part of the system the grazing regime takes place. The domains of existence of the regimes with clearance and preload are altered (see Fig. 9.2a). The frequencies of the infinite impulses, Ω_1^* and Ω_2^* , are the roots of Eq. (10.34) satisfying the condition $\omega > \Omega_1$.

These frequencies, evidently, are related to the natural frequencies as

$$\Omega_1 < \Omega_1^* < \Omega_2 < \Omega_2^* < \infty.$$

An analysis of the conservative approximation gives information about the envelopes of the studied process. The small vibration defined in the framework of the second approximations is also superimposed on the above envelopes. Bearing in mind the influence of the viscous damping on the energy balance equation, we let $b_j = 0$ initially. The desired solutions may be obtained using general formulae, represented at the beginning of the present paragraph. However, these solutions may be obtained in a more simple manner by making use of the integral equation of periodic regimes written for each coordinate of the system (compare with Sections 9.2 and 7.3). Denote by $\kappa_j(t, s; \mu)$ the PPGF of SDOF systems representing the mechanism under consideration. By making use of (7.9) and (9.4) we find

$$x_j(t) = f_j(t) - (-1)^{j+1} \int_0^t \kappa_j(t, s; \mu) \Phi_j[x(s), x(s)] ds, \quad j = 1, 2 \quad (10.35)$$

where $f_j(t)$ is the solution of the corresponding linear problems

$$m_j \ddot{f}_j + c_j [1 + \varepsilon \mu_j \cos(2\omega t + \varphi)] f_j = \varepsilon h_j \cos(2\omega t + \varphi)$$

defined similarly to that in Sect. 7.9. After representation of the solutions in the form $f_j(t) \approx a_{0j} + a_{1j} \cos(\omega t + \frac{1}{2}\varphi) + a_{2j} \cos(2\omega t + \varphi) + \dots$

we substitute this into the equation of motion and equate similar terms. By doing so, we find

$$a_{1j} = 0, \quad a_{2j} = \frac{m_j \left[1 - \frac{1}{2} \varepsilon^2 \mu_j^2 (\Omega_2^2 - 4\omega^2) \right]}{\varepsilon h_j}, \quad a_{0j} = -\frac{1}{2} \varepsilon \mu_j a_{2j}$$

Hence, within the accepted accuracy we have

$$f_j(t) \approx \varepsilon h_j \left[m_j (\Omega_2^2 - 4\omega^2) \right]^{-1} \cos(2\omega t + \varphi)$$

Subtract the second equation of (10.35) from the first and make use of the calculations of §7 (or the more general calculations of the present paragraph). Accounting for (7.15) (or (10.16)) and the values of the parameters involved yields for $0 \leq t < T$

$$x(t) = f(t) - \int_0^t \frac{1}{2} \frac{\sum_{j=1}^2 2\Omega_j m_j \sin \frac{1}{2} \Omega_j T}{\cos \Omega_j (t - \frac{1}{2} T)} \times \left[\frac{4m_j \omega \sin \frac{1}{2} \Omega_j T}{\varepsilon \mu_j} \left[\sin \omega t \cos(\omega t + \varphi) \sin \Omega_j (t - \frac{1}{2} T) + \omega \cos(\omega t + \varphi) \right] \right] \times \left[\frac{2\omega + 2\Omega_j}{\cos[\omega + \Omega_j (t - \frac{1}{2} T)]} - \frac{2\omega - 2\Omega_j}{\cos[\omega - \Omega_j (t + \frac{1}{2} \Omega_j T)]} \right] \quad (10.36)$$

where $f(t) = f_1(t) - f_2(t) = a_{01} - a_{02} + (a_{21} - a_{22}) \cos(2\omega t + \phi) + \dots$ From (10.36) and from the condition $x(0) = \Delta$ it follows that

$$j = -\Delta(\phi; \omega) \quad \times \left[\frac{1}{2\Omega_1 m_1} \cot \frac{\omega}{2\Omega_1^2 - 2\omega^2} \left(1 - \frac{\Omega_1^2 \varepsilon \mu_1 \cos \phi}{\Omega_2^2 \varepsilon \mu_2 \cos \phi} \right) + \frac{1}{2\Omega_2 m_2} \cot \frac{\omega}{2\Omega_2^2 - 2\omega^2} \right]^{-1} \quad (10.37)$$

Here $\Delta_0(\phi; \omega) = \Delta - f(0)$. In order to define the representations for the coordinates x_1 and x_2 it is necessary to make the substitution $f(t) \rightarrow f_j(t)$ in (10.36) and to keep the term with index $j = 1$ or $j = 2$ in the sum involved in this equation.

The expression (10.37) indicates that in the vicinities of the frequencies Ω_1 and Ω_2 corresponding to the centres of the domains of the fundamental linear resonance the appearance of intensive collisions is most likely. As noted above, the motions in these domains may be much more complex.

Whilst constructing the energy balance equation we have to keep in mind that all forces, with the exception of the forces of impact damping, are acting on the absolute displacements x_1 and x_2 . Hence, the above equation becomes

$$j \left[\frac{2m(1+R)}{1-R} + \sum_{j=1}^2 \frac{4m_j^2 \sin^2(\frac{1}{2}\Omega_j T)}{\varepsilon b_j(T - \Omega_j^{-1} \sin \Omega_j T)} \right] = \left[\sum_{j=1}^2 \frac{2\varepsilon h_j \omega m_j^{-1}}{4\omega^2 - \Omega_j^2} \right] + \frac{4}{1} j \varepsilon \mu_j \omega \Omega_j m_j^{-1} \cot \frac{\omega}{\pi} \sin \phi \quad (10.38)$$

From Eq. (10.38) we can obtain the unknown phase, the existence condition of the considered regimes and the instability energy condition. It is seen here that accounting for the force source does not affect significantly the character of the motion. In general, the dependence $j(\omega)$ is subjected to the influence of numerous parameters of the system. The conducted analysis allows the estimation of the frequency ranges where the appearance of intensive vibration with even infinite impulses is possible. Hence, the considered system may be characterised by the *resonant frequencies diagram*, as shown in Fig. 10.7.

The hatched bands in this diagram correspond to the frequencies where the vibration of the considered mechanism is intensive. The width of these bands is of order ε . The types of regimes (with a clearance $\Delta > 0$ or with a preload $\Delta < 0$) that are the most likely to arise here are indicated in the blank areas.

It may be shown that for $\Delta = 0$ the resonant regimes are localised mostly in the hatched areas.

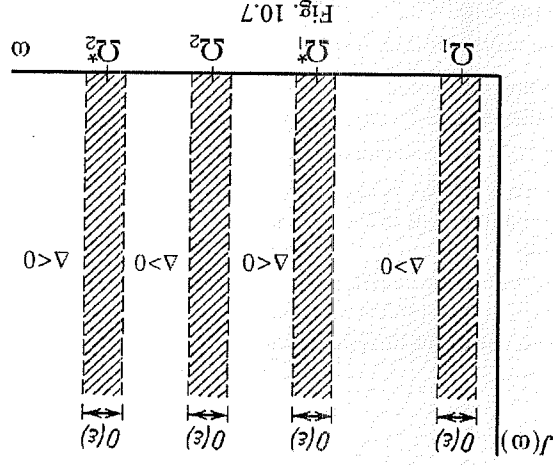


Fig. 10.7

Similarly, we can consider the problem of the construction of motion whose frequency equals that of the excitation force $\nu = \omega$. The excitation in this case may sustain intensive vibration in the vicinity of the frequencies $\omega \approx \Omega_j^*$, but the present approximation, apparently, is not capable of indicating the resonant regimes in the vicinity $\omega \approx \Omega_j$. These regimes correspond to the first lateral resonance. Hence, the diagram of resonant frequencies along with the frequency bands of ε order in the vicinities $\omega \approx \Omega_j^*$ must contain bands of order ε^2 in the vicinity of the first lateral resonance (see Fig. 10.8) also. It is known that for the development of the first lateral resonance the damping order must be lower than that of the modulation: the difference must be one order at least.

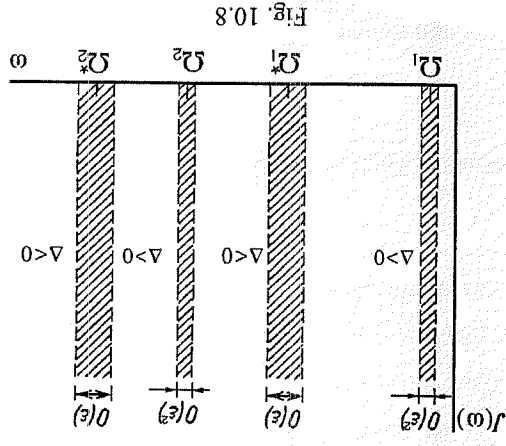


Fig. 10.8

In the system considered, as expected, the combination resonances are absent. The interaction of the two bodies is due to the impact only.

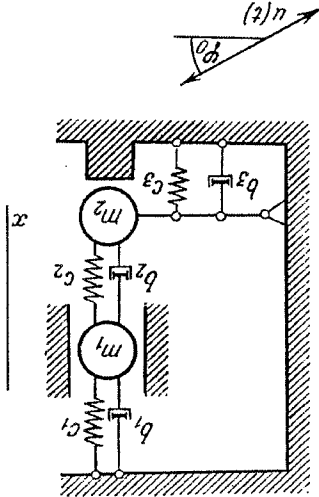


Fig. 10.9

Now consider the model in Fig. 10.9. Without loss of generality, we let $m_1 = m_2 = 1$ and reduce the equations of motion to the form

$$\ddot{x}_1 + \omega_1^2 x_1 - \alpha x_2 + 2\epsilon b_{01} \dot{x}_1 - \epsilon b_1 \dot{x}_2 = \epsilon h_1 \cos(\nu + \phi)$$

$$\ddot{x}_2 + \omega_2^2 [1 + \epsilon \mu \cos(\nu + \phi)] x_2 - \alpha x_1 + 2\epsilon b_{02} \dot{x}_2 - \epsilon b_2 \dot{x}_1 + \Phi(x_2, \dot{x}_2) = \epsilon h_2 \cos(\nu + \phi)$$

Here $\omega_1 = \sqrt{c_1 + c_2}$, $\omega_2 = \sqrt{c_2 + c_3}$ are the partial frequencies [108], $\alpha = c_2$ is the coupling coefficient, $2\epsilon b_{01} = b_1 + b_2$, $2\epsilon b_{02} = b_2 + b_3$, ϵb_1 and the coefficients μ and $h_k \sim \nu^2$ are calculated as described above (see Sect. 7.9).

In the linear conservative system the matrix of the transformation from the coordinates x_1 and x_2 to the normal coordinates takes the form [82]

$$\Theta_{jk} = \begin{bmatrix} \cos \phi_0 & -\sin \phi_0 \\ \sin \phi_0 & \cos \phi_0 \end{bmatrix}, \cot 2\phi_0 = \frac{\omega_2^2 - \omega_1^2}{2\alpha}$$

The natural frequencies of the linear system,

$$\Omega_{1,2}^2 = \frac{1}{2} \left[\omega_1^2 + \omega_2^2 \pm \sqrt{(\omega_1^2 - \omega_2^2)^2 + 4\alpha^2} \right]$$

in a combination with the elements of the above matrix, define the dynamic

compliance operator at the point x_2

$$L_{22}^{(2)}(p) = \sum_{j=1}^2 \frac{p^2 + \Omega_{2j}^2}{\Theta_{2j}^2}$$

and the transfer dynamic compliance operator from the point x_2 into the point x_1

$$L_{21}^{(2)}(p) = \sum_{j=1}^2 \frac{p^2 + \Omega_{2j}^2}{\Theta_{1j}^2 \Theta_{2j}^2}$$

From Eq. (10.39) we obtain equations in the form of (10.1) (friction is neglected)

$$x_2(t) = L_{22}^{(2)}(p) \{ \epsilon h_2 \cos(\nu + \phi) - \omega_2^2 \epsilon \mu \cos(\nu + \phi) x_2 - \Phi(x_2, \dot{x}_2) \} + L_{12}^{(2)}(p) [\epsilon h_1 \cos(\nu + \phi)]$$

The approximate definition of the function $x_2(t)$ is given by the method described in Sections 10.1–10.4. Representation (10.11) takes the form

$$x_{02}(t) = -J_0 \sum_{j=1}^2 \frac{2\Omega_j}{\Theta_{2j}^2} \cos[\Omega_j(t - \frac{1}{2}T)] \sin(\frac{1}{2}\Omega_j T)$$

and the expression for the impulse is similar to (10.33),

$$J_0 = -\Delta \left(\frac{\Theta_{21}^2}{2\Omega_1} \cot \frac{\Omega_1 T}{2} + \frac{\Theta_{22}^2}{2\Omega_2} \cot \frac{\Omega_2 T}{2} \right)^{-1}$$

Hence, the resonant frequencies outside the zones of linear instability are also defined from Eq. (10.34).

The relationships of the second approximation (10.17) are defined by the repetitive kernel (10.16) at $j, q = 1, 2$. Here it is necessary to choose the ratio of the excitation frequency ν to the frequency ω of the regime and to consider the values of the form factors Θ_{jk} . The representation for the impulse of the impact is given by (10.20). The existence condition of the regimes follows from (10.12) and (9.27) (see (10.14)). The construction of the instability energy condition and the representation of the motion $x_1(t)$ may be carried out in a standard manner.

Since the linear part of the system embodies two coupled degrees of freedom, the combination resonances will affect the diagram of the resonant frequencies (see the previous problem). By making use of the above calculation scheme, we can obtain a description of the considered regimes in the systems of higher order (see Fig. 10.4). Similar problems combine these two examples to some extent.

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in the sense of *Lebesgue-Stieltjes*. The integrating function $u(t)$ is assumed to be $u(t) \in BV(J)$, if $u(t)$ is the scalar function defined for $t \in J$ and possessing the following features:

- (a) $u(t)$ is continuous from the right and possesses a limited variation in every compact subinterval of the interval J .
- (b) The discontinuities $t_1 < t_2 < \dots (t_1 \geq t_0 \geq 0)$ of the function $u(t)$ possess the single limiting point $+\infty$.

The functions $x(t)$ are supposed to be the functions defined on J with values in $B_x(K)$, continuous from the right and possessing the same points of first-order discontinuity as $u(t)$.

The existence conditions of integral (AI 1.1) are:

- $f(t, x)$ is a function defined on G with values in $\mathbb{R}^n$;
- $f(t, x)$ possesses a bounded norm and is continuous with respect to x uniformly in time t ;
- $f(t, x)$ possesses in time t at most a denumerable set of first-order discontinuities.

Further, if the existence of integral (AI 1.1) is not reserved we particularly assume that the above conditions hold.

For the function $f(t, x)$ defined on G and integrable with respect to $u(t) \in BV(J)$ we set

$$S_x(f) = \sup_{|t_2 - t_1| \leq 1} \left| \int_{t_1}^{t_2} f(s, x(s)) du(s) \right|, \quad x \in B_x(K)$$

Lemma 1. Assume that the function $f(t, x)$ is defined on G , and is continuous in x uniformly with respect to $t \in J$. Assume that the function $x(t)$ is continuous from the right, possesses the values in $B_x(K)$ and is a function with a limited variation in every compact subinterval J . Assume that the function $f(t, x)$ possesses the same discontinuities as the function $u(t) \in BV(J)$ and the function $f(t, x(t))$ is integrable with respect to $u(t)$.

Then for each $\eta > 0$ we may find $\varepsilon > 0$ such that

$$\sup_{|t_2 - t_1| \leq 1} \left| \int_{t_1}^{t_2} f(s, x(s)) du(s) \right| < \eta, \quad (t_1, t_2) \in [0, T], \quad 0 < T < \infty$$

if $S_x(f) < \varepsilon$.

Proof. From the conditions of Lemma 1 for each $\eta > 0$ we can find $\delta > 0$ such

This monograph is devoted to an investigation of mechanical systems with strong threshold nonlinearities and developing the proper methods for their analysis.

On its completion we would like to discuss the problem of the validity of the methods used.

The level of rigour of our representation is said to be "physical" from the point of view of modern mathematics. And it is really so, since some approximate approaches used for the solution of the corresponding problems were based mainly on an intuitive appreciation of the nature of the required solutions. It was the only technique which enabled delivery of the desired result.

Therefore, some of our approaches are valid only in the "mechanical" sense. In the context of modern mathematics, however, the above approaches have to be justified. In particular, the assumption of the agreement of the obtained solutions with the initial differential equations of motion is still open. This also holds true for the problem of singularisation and for some other approaches used by the authors in this monograph.

The construction of the rigorous mathematical theories supporting the application of the approximate methods lies beyond the scope of the scientific interests of the authors. The authors objective was to create, first of all, the approaches which are correct in the "physical" sense. At the same time, the authors believe that the "problem of mathematical substantiation" is important. The solution to the problem not only supplies the necessary guarantees of accuracy for the corresponding analysis, it also allows an assessment of the applicability boundaries for different approximate methods and formalisation of the computational algorithms.

VSh Burd kindly provided the material in the present appendix. The interested reader will find the proofs of the theorems underlying the application of the averaging method in the systems described by the differential equations in generalised functions here. In particular, the problem of closeness of the exact and the averaged solutions on the infinite time interval is solved. Such systems were frequently used in §5.6. In a different context, the problems of substantiation of the averaging method as applied to such systems were considered in [152, 153]. Dealing with the material of the present appendix requires a fundamental knowledge of the *Lebesgue-Stieltjes* integral and of the properties of functions with limited variation.

(1) We will use the following notation: $|x|$ is the norm of the element $x \in \mathbb{R}^n$, J is the interval $[0, \infty]$, $B_x(K) = \{x: x \in \mathbb{R}^n, |x| \leq K\}$, $G = J \times B_x(K)$. Consider here the integrals

$$\int_{t_1}^{t_2} f(s, x(s)) du(s), \quad (t_1, t_2) \in J$$

(AI 1.1)

each subinterval with length smaller than unity, the function $x^0(t)$ takes at most k different values (k depends on δ exclusively). Let $x_j(t)$ ($j=1, \dots, k$) be the values of $x(t)$ in the interval $|t_2 - t_1| \leq 1$. By setting $\varepsilon = \eta/(2k)$, we find that

$$\left| \int_{t_1}^{t_2} f(s, x(s)) ds \right| \leq \left| \int_{t_1}^{t_2} f(s, x^0(s)) ds \right| + \left| \int_{t_1}^{t_2} f(s, x^0(s)) du(s) \right| + \left| \int_{t_1}^{t_2} f(s, x^0(s)) du(s) \right| \leq \frac{\eta}{2} + \sum_{j=1}^k \left| \int_{t_1}^{t_2} f(s, x_j) du(s) \right| \leq \frac{\eta}{2} + \frac{2k}{\eta} \cdot k = \eta$$

The last inequality is valid for arbitrary t_1, t_2 satisfying the inequality $|t_2 - t_1| \leq 1$.

□

(2) Consider the differential equation in generalised functions in $\mathbb{R}^n$

$$Dx(t) = X(t, x) + R(t, x) Du(t) \quad (\text{AI 2.1})$$

where the functions $X(t, x)$ and $R(t, x)$ are defined in G , $u(t) \in BV(J)$, and $Dx(t)$ and $Du(t)$ are the generalised derivatives of the functions $x(t)$ and $u(t)$, respectively.

We sought of the solution of Eq. (AI 2.1) in the form of the function $x(t, t_0, x_0)$ which is:

- defined on the interval I , where the left end t_0 satisfies the condition $x(t_0) = x_0$;
- continuous from the right;
- a function with limited variation in I ;
- possesses the generalised derivative in (t_0, T) , $T \in I$,
- satisfies condition (AI 2.1).

It is known¹ that $x(t)$ is the solution of Eq. (AI 2.1) (crossing the point (t_0, x_0)) in I if and only if $x(t)$ satisfies the integral equation

$$x(t) = x_0 + \int_{t_0}^t X(s, x(s)) ds + \int_{t_0}^t R(s, x(s)) du(s), \quad t \in I \quad (\text{AI 2.2})$$

where:

- for each function $x(t)$, which is continuous from the right and possesses the bounded variation on I , the function $X(t, x(t))$ is integrable;
- the function $R(t, x(t))$ is integrable on I with respect to $u(t)$;
- the second integral is considered on the interval $[t_0, t]$.

¹ See Pandit SG, Deo SG, Differential systems involving impulses, Lect. Notes in Math, 954, Springer-Verlag, 1982.

It is evident that the function $x(t)$ is the solution to Eq. (AI 2.2) and possesses discontinuities at the same points as $u(t)$. Along with Eq. (AI 2.1) we consider the undisturbed ordinary differential equation in $\mathbb{R}^n$

$$dy/dt = X(t, y) \quad (\text{AI 2.3})$$

We assume that Eq. (AI 2.3) possesses the solution $\psi(t, t_0, \xi_0)$ ($\psi(t_0, t_0, \xi_0) = \xi_0$) which is defined for all $t \geq t_0 \geq 0$ belonging to the set G along with the ρ -neighbourhood ($\rho > 0$).

Theorem 1. Assume that the function $X(t, x)$ satisfies the Lipschitz condition:

$$|X(t, x_1) - X(t, x_2)| \leq L|x_1 - x_2|, \quad x_1, x_2 \in B_x(K), \quad t \in J$$

Assume that the function $R(t, x)$ is continuous in x and uniform with respect to time $t \in J$.

Also, let the solution $\psi(t, t_0, \xi_0)$ of Eq. (AI 2.3) be *uniformly asymptotically stable*. Then for arbitrary $\varepsilon > 0$ ($0 < \varepsilon < \rho$) it is possible to find numbers $\eta_1(\varepsilon)$, $\eta_2(\varepsilon)$ such that for all solutions $x(t, t_0, x_0)$ to Eq. (AI 2.1) ($x(t_0, t_0, x_0) = x_0$), defined at $t \geq t_0$ with values in $B_x(K)$, and with initial conditions, satisfying the inequality $|x_0 - \xi_0| < \eta_1(\varepsilon)$, and for all $R(t, x)$ satisfying the inequality $S_x(R) < \eta_2(\varepsilon)$, the inequality

$$|x(t, t_0, x_0) - \psi(t, t_0, \xi_0)| < \varepsilon \quad (\text{AI 2.4})$$

holds for all times $t \geq t_0$.

Proof. Let $y(t, t_0, x_0)$ be the solution of Eq. (AI 2.3) with the same initial conditions as the solution $x(t, t_0, x_0)$ of Eq. (AI 2.1). From the conditions of the theorem we find the inequality

$$|x(t, t_0, x_0) - y(t, t_0, x_0)| \leq L \int_{t_0}^t |x(s, t_0, x_0) - y(s, t_0, x_0)| ds + F(t) \quad \text{where } F(t) = \left| \int_{t_0}^t R(s, x(s, t_0, x_0)) du(s) \right|.$$

The well-known integral inequality² yields

$$|x(t, t_0, x_0) - y(t, t_0, x_0)| \leq F(t) + L \int_{t_0}^t e^{L(t-s)} F(s) ds$$

From the latter expression for $t_0 \leq t \leq t_0 + T$ we find that

² See, for example, Barbashin EA, Introduction into the theory of stability, Moscow, Nauka, 1967 (in Russian).

$$\left| \mathbf{x}(t, t_0, \mathbf{x}_0) - \mathbf{y}(t, t_0, \mathbf{x}_0) \right| \leq (T+1) \left(1 + LTe^{LT} \right) \sup_{|t_2 - t_1| \leq T} \left| \int_{t_1}^{t_2} \mathbf{R}(s, \mathbf{x}(s, t_0, \mathbf{x}_0)) du(s) \right|$$

By virtue of the uniform asymptotic stability of the solution $\psi(t_0, t_0, \xi_0)$ of Eq. (AI 2.3) there exist numbers $\delta < \varepsilon$ and $T > 0$ such that, from the inequality

$$(AI 2.5) \quad \left| \mathbf{y}(t_0 + T, t_0, \mathbf{x}_0) - \psi(t_0 + T, t_0, \xi_0) \right| < \delta/2, \quad t \geq t_0$$

From Lemma 1, the number $\eta_2(\varepsilon)$ may be chosen to satisfy the inequality

$$(AI 2.6) \quad \left| \mathbf{x}(t, t_0, \mathbf{x}_0) - \mathbf{y}(t, t_0, \mathbf{x}_0) \right| < \delta/2, \quad t_0 \leq t \leq t_0 + T$$

Then $\left| \mathbf{x}(t, t_0, \mathbf{x}_0) - \psi(t, t_0, \xi_0) \right| < \varepsilon/2 + \delta/2 < \varepsilon$, $t_0 \leq t \leq t_0 + T$. From (AI 2.5) and

(AI 2.6) we find, successively

$$\left| \mathbf{x}(t_0 + T, t_0, \mathbf{x}_0) - \psi(t_0 + T, t_0, \mathbf{x}_0) \right| < \delta$$

The proof of the theorem that follows is almost identical to that of Lemma 6.3.

Remark 1. It was assumed in the statement of the theorem that the solution $\mathbf{x}(t, t_0, \mathbf{x}_0)$ is defined for $t \geq t_0$ and belongs to $B_x(K)$. If the conditions of the local existence theorem for the solutions of Eq. (AI 2.1) hold, and if $S_x(R)$ is small enough for $\mathbf{x} \in B_x(K)$, then the solution $\mathbf{x}(t, t_0, \mathbf{x}_0)$ with the initial condition chosen close enough by a norm to that of the solution $\psi(t, t_0, \xi_0)$ of Eq. (AI 2.3) will be defined for all times $t \geq t_0$ and will not leave the sphere $B_x(K)$.

Remark 2. If the solution $\psi(t, t_0, \xi_0)$ of Eq. (AI 2.3) is uniformly and asymptotically stable with respect to the part of the variables $\psi_1, \dots, \psi_k$ ($k < n$), then the inequality in (AI 2.4) in the statement of Theorem 1 must be substituted using the inequality

$$\left| \mathbf{x}_i(t, t_0, \mathbf{x}_0) - \psi_i(t, t_0, \xi_0) \right| < \varepsilon, \quad i = 1, \dots, k$$

(3) We will apply Theorem 1 to the solution of the problem of averaging on the infinite interval. The following scheme will be used. The studied equation in $\mathbf{R}^n$ takes the form

$$(AI 3.1) \quad D\mathbf{x}(t) = \mathbf{R}(t, \mathbf{x}, \varepsilon) Du(t, \varepsilon)$$

where ε is a small positive parameter. It is shown that the following limiting equality holds,

$$(AI 3.2) \quad \lim_{\varepsilon \rightarrow 0} \sup_{|t_2 - t_1| \leq T} \left| \int_{t_1}^{t_2} \mathbf{R}(s, \mathbf{x}, \varepsilon) du(s, \varepsilon) - \int_{t_1}^{t_2} \mathbf{X}(s, \mathbf{x}) ds \right| = 0, \quad \mathbf{x} \in B_x(K)$$

that the right-hand side of Eq. (AI 3.1) is converging in the integral sense as $\varepsilon \rightarrow 0$ to $\mathbf{X}(t, \mathbf{x})$ if the limiting equality (AI 3.2) holds.

Consider first the differential equation in $\mathbf{R}^n$ (containing fast and slow times) in the standard form

$$(AI 3.3) \quad D\mathbf{x}(t) = \varepsilon \mathbf{X}(t, \tau, \mathbf{x}, \varepsilon) Du(t), \quad \tau = \varepsilon t$$

where ε is a small positive parameter, $\varepsilon \in [0, \varepsilon_0]$, the function $\mathbf{X}(t, \tau, \mathbf{x}, \varepsilon)$ with the values in $\mathbf{R}^n$ is defined for $t \in J$, $\mathbf{x} \in B_x(K)$, $u(t) \in BV(J)$ and, also $u(t)$ is bounded at J .

Theorem 2. Assume that:

(a) The function $\mathbf{X}(t, \tau, \mathbf{x}, \varepsilon)$ is uniformly continuous in each of the variables $\tau, \mathbf{x}, \varepsilon$ with respect to the other variables.

(b) $\left| \mathbf{X}(t, \tau, \mathbf{x}, \varepsilon) \right| \leq M < \infty$, $(t, \mathbf{x}) \in G$, $\varepsilon \in [0, \varepsilon_0]$.

(c) The limit

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_{t_0+T}^{t_0+T+T} \mathbf{X}(s, \tau, \mathbf{x}, 0) du(s) = \mathbf{X}(\tau, \mathbf{x}), \quad (\tau, \mathbf{x}) \in G$$

exists uniformly for $t \in J$.

(d) The function $\mathbf{X}(\tau, \mathbf{x})$ satisfies the Lipschitz condition

$$\left| \mathbf{X}(\tau, \mathbf{x}_1) - \mathbf{X}(\tau, \mathbf{x}_2) \right| \leq L \left| \mathbf{x}_1 - \mathbf{x}_2 \right|, \quad \mathbf{x}_1, \mathbf{x}_2 \in B_x(K), \quad t \in J$$

and is continuous in τ uniformly with respect to $\mathbf{x} \in B_x(K)$.

(e) The equation in $\mathbf{R}^n$

$$(AI 3.4) \quad dx/d\tau = \mathbf{X}(\tau, \mathbf{x})$$

possesses the uniformly asymptotically stable solution $\mathbf{x} = \psi(\tau, t_0, \xi_0)$

(uniformly asymptotically stable with respect to variables $\psi_1, \dots, \psi_k$, $k < n$) which is located (together with the ρ -neighbourhood, $\rho > 0$) in the set G . Then for arbitrary α , $0 < \alpha < \rho$, there exist numbers $\varepsilon_1(\alpha)$, $0 < \varepsilon < \varepsilon_0$ and $\beta(\alpha)$ such that for all $0 < \varepsilon < \varepsilon_1$ the solution $\phi(t, t_0, \mathbf{x}_0)$ of Eq. (AI 3.3) defined for $t \geq t_0$ for which $\left| \mathbf{x}_0 - \xi_0 \right| < \beta(\alpha)$ will satisfy the inequality

$$\left| \phi(t, t_0, \mathbf{x}_0) - \psi(t, t_0, \xi_0) \right| < \alpha, \quad t = 1, \dots, k, \quad t \geq t_0$$

Proof. Assuming in equation (AI 3.3) the time change $\tau = \varepsilon t$ we find

$$D\mathbf{x}(\tau) = \varepsilon \mathbf{X}\left(\frac{\tau}{\varepsilon}, \tau, \mathbf{x}, \varepsilon\right) D\mathbf{u}\left(\frac{\tau}{\varepsilon}\right)$$

(AI 3.5)

statement of the theorem. It is evident, that by virtue of the continuity of $\mathbf{X}(t/\varepsilon, \mathbf{x}, \varepsilon)$ in the fourth variable, which is uniform with respect to the other variables, it is sufficient to estimate that, for arbitrary $\delta > 0$, for small ε

$$\Pi(\varepsilon) = \sup_{|t_2 - t_1| \leq 1} \left| \int_{t_1}^{t_2} \varepsilon \mathbf{X}(t/\varepsilon, \mathbf{x}, \varepsilon) dt - \int_{t_1}^{t_2} \varepsilon \mathbf{X}(t/\varepsilon, \mathbf{x}, 0) dt \right| < \delta$$

We choose a number $\eta > 0$ such that when $|t_1 - t_2| < \eta$ the inequalities

$$|\mathbf{X}(t/\varepsilon, \mathbf{x}, \varepsilon) - \mathbf{X}(t/\varepsilon, \mathbf{x}, 0)| < \delta/4, \quad |\mathbf{X}(t_1, \mathbf{x}) - \mathbf{X}(t_2, \mathbf{x})| < \delta/4$$

hold.

Let $g(t)$ be such a piecewise constant function defined in the interval $[t_1, t_2]$

$(|t_2 - t_1| \leq 1)$ such that $|t - g(t)| < \eta$, $t \in [t_1, t_2]$. Then

$$\Pi(\varepsilon) \leq \left| \int_{t_1}^{t_2} \varepsilon \left[\mathbf{X}\left(\frac{t}{\varepsilon}, \mathbf{x}, 0\right) - \mathbf{X}\left(\frac{t}{\varepsilon}, \mathbf{x}, \varepsilon\right) \right] dt \right| + \left| \int_{t_1}^{t_2} \varepsilon \mathbf{X}\left(\frac{t}{\varepsilon}, \mathbf{x}, 0\right) dt - \int_{t_1}^{t_2} \varepsilon \mathbf{X}(g(t), \mathbf{x}, 0) dt \right|$$

$$+ \left| \int_{t_1}^{t_2} \varepsilon \mathbf{X}(g(t), \mathbf{x}) dt - \int_{t_1}^{t_2} \varepsilon \mathbf{X}(g(t), \mathbf{x}, 0) dt \right| \leq \frac{\eta}{\delta} + \sum_{k=1}^n \left| \int_{\sigma_k}^{\sigma_{k+1}} \varepsilon \mathbf{X}(t/\varepsilon, \mathbf{x}, 0) dt \right|$$

$$- \int_{\sigma_k}^{\sigma_{k+1}} \varepsilon \mathbf{X}(t/\varepsilon, \mathbf{x}, \varepsilon) dt$$

where $t_1 = \sigma_0 < \sigma_1 < \dots < \sigma_n = t_2$. Now have to demonstrate that the inequality

$$\left| \int_{\sigma_k}^{\sigma_{k+1}} \varepsilon \mathbf{X}(t/\varepsilon, \mathbf{x}, 0) dt - \int_{\sigma_k}^{\sigma_{k+1}} \varepsilon \mathbf{X}(t/\varepsilon, \mathbf{x}, \varepsilon) dt \right| < \delta/(2n)$$

holds for small enough ε .

The latter inequality follows from the limiting equality

$$\lim_{\varepsilon \rightarrow 0} \int_{\sigma_k}^{\sigma_{k+1}} \varepsilon \mathbf{X}(t/\varepsilon, \mathbf{x}, 0) dt - \int_{\sigma_k}^{\sigma_{k+1}} \varepsilon \mathbf{X}(t/\varepsilon, \mathbf{x}, \varepsilon) dt = 0 \quad (\text{AI 3.6})$$

It may be seen easily that the limiting equality (AI 3.6) follows from the third condition of the theorem. In conclusion we note that in the case when the solution $\psi(t/\varepsilon, t_0, \xi_0)$ to the Eq. (AI 3.4) is uniformly asymptotically stable with respect to a part of the variables, it is necessary to use Remark 2 to Theorem 1.

(4) The above approach allows the analysis of the closeness of the solutions of exact and averaged equations on the infinite time interval in systems with fast and slow variables. Consider, for example, the following system of differential equations with the fast phase

$$D\mathbf{x}(t) = \varepsilon \mathbf{X}(\mathbf{x}, \mathbf{y}, \varepsilon) D\mathbf{u}(\mathbf{y}), \quad \frac{d\mathbf{y}}{dt} = \omega(\mathbf{x}) + \varepsilon \mathbf{Y}(\mathbf{x}, \mathbf{y}, \varepsilon) \quad (\text{AI 4.1})$$

where $\mathbf{x}$ is a n -dimensional vector, $\mathbf{y}$ is a scalar function, and ε is a small positive parameter in the interval $[0, \varepsilon_0]$, $D\mathbf{x}(t)$, $D\mathbf{u}(\mathbf{y})$ are generalised

derivatives of the functions $\mathbf{x}(t)$ and $\mathbf{u}(\mathbf{y})$. Suppose that the function $\mathbf{X}(\mathbf{x}, \mathbf{y}, \varepsilon)$ with the values in $\mathbf{R}^n$ and the scalar function $\mathbf{Y}(\mathbf{x}, \mathbf{y}, \varepsilon)$ are periodic in the fast variable $\mathbf{y}$ with period 2π , and $\mathbf{u}(\mathbf{y})$ is a scalar 2π -periodic function with a bounded variation.

Theorem 3. Assume:

(a) The functions $\mathbf{X}(\mathbf{x}, \mathbf{y}, \varepsilon)$ and $\mathbf{Y}(\mathbf{x}, \mathbf{y}, \varepsilon)$ are defined for $\mathbf{x} \in B_x(K)$, $\mathbf{y} \in [-\infty, \infty]$, $\varepsilon \in [0, \varepsilon_0]$ and are continuous in the variables $\mathbf{x}, \varepsilon$ uniformly with respect to the other variables.

(b) The function $\omega(\mathbf{x})$ for $\mathbf{x} \in B_x(K)$ satisfies the inequality $\omega(\mathbf{x}) \geq c > 0$, where c is a constant, and the Lipschitz condition,

(c) There exists a constant M such that

$$|\mathbf{X}(\mathbf{x}, \mathbf{y}, \varepsilon)| \leq M, \quad |\mathbf{Y}(\mathbf{x}, \mathbf{y}, \varepsilon)| \leq M$$

$$\mathbf{x} \in B_x(K), \quad \mathbf{y} \in [-\infty, \infty], \quad \varepsilon \in [0, \varepsilon_0]$$

(d) The function

$$\mathbf{X}(\mathbf{x}) = \frac{1}{2\pi} \int_0^{2\pi} \mathbf{X}(\mathbf{x}, \mathbf{y}, 0) d\mathbf{u}(\mathbf{y})$$

satisfies the Lipschitz condition:

$$|\mathbf{X}(\mathbf{x}_1) - \mathbf{X}(\mathbf{x}_2)| \leq L_1 |\mathbf{x}_1 - \mathbf{x}_2|, \quad \mathbf{x}_1, \mathbf{x}_2 \in B_x(K)$$

(e) The equation

$$d\mathbf{x}/d\tau = \mathbf{X}(\mathbf{x})$$

possesses the uniformly asymptotically stable solution $\psi(t/\varepsilon, t_0, \xi_0)$ which belongs, together with the ρ -neighbourhood ($\rho > 0$), to the set $B_x(K)$.

Then, for arbitrary α , $0 < \alpha < \rho$, there exist numbers $\varepsilon_1(\alpha)$, $0 < \varepsilon_1 < \varepsilon_0$ and $\beta(\alpha)$ such that for all $0 < \varepsilon < \varepsilon_1$ the slow variables of the solution to system (AI 4.1) for which $|\mathbf{x}_0 - \xi_0| < \beta(\alpha)$ satisfy the inequality

$$|\mathbf{x}(t/\varepsilon, t_0, \mathbf{x}_0) - \psi(t/\varepsilon, t_0, \xi_0)| < \alpha, \quad t \geq t_0$$

Proof. In system (AI 4.1) after the changes $\tau = \varepsilon t$, $\alpha = \varepsilon \mathbf{y}$ we obtain the system

$$D\mathbf{x}(\tau) = \varepsilon \mathbf{X}\left(\mathbf{x}, \frac{\varepsilon}{\alpha}, \varepsilon\right) D\mathbf{u}\left(\frac{\varepsilon}{\alpha}\right), \quad \frac{d\tau}{d\alpha} = \omega(\mathbf{x}) + \varepsilon \mathbf{Y}\left(\mathbf{x}, \frac{\varepsilon}{\alpha}, \varepsilon\right) \quad (\text{AI 4.3})$$

From the second and third conditions of the theorem it follows that for small ε the function $\alpha(\tau)$ is monotonic and, therefore, the variable α may be treated as an independent variable instead of τ for small ε . Now system (AI 4.1) takes the form

$$D\mathbf{x}(\alpha) = \varepsilon \frac{\mathbf{X}(\mathbf{x}, \alpha/\varepsilon, \varepsilon)}{\omega(\mathbf{x})} D\mathbf{u}\left(\frac{\varepsilon}{\alpha}\right) + \varepsilon^2 \mathbf{X}_1\left(\mathbf{x}, \frac{\varepsilon}{\alpha}, \varepsilon\right) D\mathbf{u}\left(\frac{\varepsilon}{\alpha}\right)$$

$$(A1\ 4.4) \quad \frac{d\tau}{d\alpha} = \frac{1}{\omega(x)} + \varepsilon Y\left(x, \frac{\varepsilon}{\omega(x)}, \varepsilon\right)$$

where the functions $X(x, \alpha/\varepsilon, \varepsilon)$ and $Y(x, \alpha/\varepsilon, \varepsilon)$ possess the same features as the functions $X(x, \alpha/\varepsilon, \varepsilon)$ and $Y(x, \alpha/\varepsilon, \varepsilon)$, respectively. It may be seen easily that the right-hand sides of system (A1 4.4) are converging in the integral sense to the right-hand sides of the system

$$(A1\ 4.5) \quad \frac{dx}{d\alpha} = \frac{\omega(x)}{1} X(x), \quad \frac{d\alpha}{d\tau} = \frac{\omega(x)}{1}$$

which, in time τ , takes the form

$$(A1\ 4.6) \quad \frac{dx}{d\tau} = X(x), \quad \frac{d\tau}{d\alpha} = \omega(x)$$

The solution of system (A1 4.6) corresponding to the solution $x = \psi(\tau, t_0, \xi_0)$ of Eq. (A1 4.2) is uniformly asymptotically stable in the variable x . The corresponding solution of system (A1 4.5) possesses the same feature. After the application of Theorem 2 to system (A1 4.4) we obtain affirmation of the theorem. Theorem 3 gives a substantiation for the averaging method used in the monograph.

□

On the Analysis of Resonant Vibration of Vibro-impact Systems Using the Averaging Technique

Here the averaging technique is applied to the calculation of resonance processes in vibro-impact systems. As distinct from the method given in this monograph and in [14] which is based on results of [119], this technique is based on the use of the results of [190]. The small periodic perturbations of conservative SDOF vibro-impact systems are considered. Existence theorems and the stability of periodic solutions are derived.

The linear oscillator with the equation of motion

$$\ddot{x} + \Omega^2 x = 0 \quad (\Omega = \text{const})$$

is considered. The solid stationary limiter is installed at the point $x = \Delta$. It is suggested that in the system an elastic impact occurs when the coordinate x takes the value Δ . If $x(t_a) = \Delta$, then

$$\dot{x}(t_a - 0) = -\dot{x}(t_a + 0)$$

For clarity, the case $\Delta > 0$ will be considered. We note some facts which will be used later on. In the presence of impacts the motion of the system is periodic. The frequency of free vibration ω satisfies the inequality $\Omega < \omega < 2\Omega$.

Let $t_a = 0$. The motion takes the form

$$x(t) = -J \left[\frac{\omega}{2\pi\Omega^2} + \frac{\pi}{\omega} \sum_{k=1}^{\infty} \frac{\Omega^2}{\cos k\omega t} - k^2 \omega^2 \right] = -J\kappa(\omega t) \quad (\text{AII 1})$$

where $J = 2x(-0) = -2\Omega\Delta \tan \frac{\omega}{2\Omega}$ is the impulse of the impact. The solution of the vibro-impact system is the periodic function (AII 1). The substitution of (AII 1) in the equation of motion converts it into a true identity (in the sense of the theory of generalised functions),

$$\ddot{x} + \Omega^2 x + \Phi(x, \dot{x}) = 0 \quad (\text{AII 2})$$

Here, $\Phi[x(t), \dot{x}(t)] = -J\delta'(t)$, where $\delta'(t)$ is a periodic delta-function with period $T = \frac{2\pi}{\omega}$. Equation (AII 2) is the equation of motion of a vibro-impact system in a temporal representation.

Now we consider the perturbed vibro-impact system

$$\ddot{x} + \Omega^2 x + \Phi(x, \dot{x}) = \varepsilon g(x, \dot{x}, t) \quad (\text{AII 3})$$

here ε is small positive parameter, the function $g(x, \dot{x}, t)$ is sufficiently smooth with respect to the variables $x, \dot{x}$ and is periodic in time with period $\frac{2\pi}{\nu}$. Convert the variables $x, \dot{x}$ into the new impulse-phase variables J, ψ given by the formulae

$$x = -J\kappa(\psi), \quad \dot{x} = -J\omega_0(J)\kappa'(\psi) \quad (\text{AII 4})$$

where

$$\psi = \omega t, \quad \kappa(\psi) = \omega_0^{-1} \left[\frac{1}{2\pi\Omega^2} + \frac{\pi}{\omega} \sum_{k=1}^{\infty} \frac{\Omega^2}{\cos k\psi} - k^2 \omega^2 \right], \quad \Omega_0 = \Omega[\omega_0(J)]^{-1}$$

$$\omega_0(J) = -\frac{\pi\Omega}{J} - \arctan \frac{2\Omega\Delta}{J} \quad (\text{AII 5})$$

Transformation (AII 4) is not smooth. The function κ' possesses finite discontinuities with respect to ψ at the points $\psi = 2\pi l$ (l is an integer). The variables J, ψ describe the system uniquely. The transformation (AII 4) converts system (AII 3) into the system

$$\begin{aligned} \frac{dJ}{dt} &= -4\varepsilon\omega_0(J)g[-J\kappa(\psi), -J\omega_0(J)\kappa'(\psi), t] = \varepsilon X(J, \psi, t) \\ \frac{d\psi}{dt} &= \omega_0(J) - 4\varepsilon\omega_0(J)J^{-1}g[-J\kappa(\psi), -J\omega_0(J)\kappa'(\psi), t] = -J\kappa(\psi) \end{aligned} \quad (\text{AII 6})$$

The function $X(J, \psi, t)$ has finite discontinuities with respect to ψ at the points $\psi = 2\pi l$ (l is an integer). System (AII 6) is at resonance if

$$\omega_0(J) = \frac{p}{q} \nu \quad (\text{AII 7})$$

where q, p are mutually prime numbers. The solution of Eq. (AII 7) is denoted by J^{pq} and is called a resonant point. After the transformation of variables

$$\psi = \phi + \frac{p}{q} \nu t$$

system (AII 6) takes the form

$$\begin{aligned} \frac{dJ}{d\phi} &= \varepsilon X(J, \phi + \frac{p}{q} \nu t, t) \\ \frac{d\phi}{d\phi} &= \omega_0(J) - \frac{p}{q} \nu + \varepsilon Y(J, \phi + \frac{p}{q} \nu t, t) \end{aligned} \quad (\text{AII 8})$$

We consider the behaviour of the solutions of system (AII 8) in the $\mu = \sqrt{\varepsilon}$ -neighbourhood of the resonant solution J^{pq} . Substitute

$$J = J^{pq} + \mu z$$

and represent the right-hand side of the transformed system in terms of the power series expansion with respect to μ

$$\begin{aligned} \frac{dJ}{d\phi} &= \mu X(J)X(J^{pq}, \phi + \frac{p}{q} \nu t, t) + \frac{d}{q} \nu t, t) z + O(\mu^3) \\ \frac{d\phi}{d\phi} &= \mu \omega_0(J) \omega_0(J^{pq}, z + \frac{1}{2} \mu^2 \omega_0(J^{pq}) z^2 + \mu^2 X(J^{pq}, \phi + \frac{p}{q} \nu t, t) + O(\mu^3) \end{aligned} \quad (\text{AII 9})$$

System (AII 9) contains the single fast variable t . The right-hand sides of system (AII 9) are periodic in t with period $\frac{2\pi p}{\nu}$. We will make a standard transformation of the averaging method in system (AII 9), which allows the exclusion of the fast variable from the right-hand sides of system (AII 9) up to the terms in μ^2 exactly. The above transform takes the form

$$z = \xi + \mu u_1(\eta, t) + \mu^2 u_2(\eta, t) \xi, \quad \phi = \eta + \mu^2 v_2(\eta, t) \quad (\text{AII } 10)$$

where $u_1(\eta, t)$, $u_2(\eta, t)$, $v_2(\eta, t)$ are defined as periodic functions in time t with zero mean value from equations

$$\frac{\partial u_1}{\partial \eta} = X(J^{pq}, \eta + \frac{d}{b} \nu t, t) - X_0(\eta)$$

$$\frac{\partial u_2}{\partial \eta} = X_f(J^{pq}, \eta + \frac{d}{b} \nu t, t) - u_1(\eta, t) - u_1(\eta, t) \omega_{0J}(J^{pq}) - X_1(\eta)$$

$$\frac{\partial v_2}{\partial \eta} = X(J^{pq}, \eta + \frac{d}{b} \nu t, t) - u_1(\eta, t) - u_1(\eta, t) \omega_{0J}(J^{pq}) - X_0(\eta)$$

The functions $X_0(\eta)$, $X_1(\eta)$, $X_0(\eta)$ are defined by the formulae

$$X_0(\eta) = \frac{\nu}{2\pi p} \int_0^{2\pi p} X(J^{pq}, \eta + \frac{d}{b} \nu t, t) dt$$

$$X_1(\eta) = \frac{\nu}{2\pi p} \int_0^{2\pi p} X_f(J^{pq}, \eta + \frac{d}{b} \nu t, t) dt$$

$$X_0(\eta) = \frac{\nu}{2\pi p} \int_0^{2\pi p} X(J^{pq}, \eta + \frac{d}{b} \nu t, t) dt$$

Easy computation shows that the functions $X_0(\eta)$, $X_1(\eta)$ and $X_0(\eta)$ are periodic in η with period $\frac{2\pi}{\nu}$ and the functions $X_0(\eta)$, $X_1(\eta)$ have finite discontinuities with respect to η at the points $\eta = \frac{2\pi l}{\nu}$ (l is an integer). Transformation (AII

10) leads to the system

$$\frac{d\xi}{dt} = \mu X_0(\eta) + \mu^2 X_1(\eta) \xi + O(\mu^3)$$

$$\frac{d\eta}{dt} = \mu \omega_{0J}(J^{pq}) \xi + \mu^2 X_2 X_0(\eta) + \frac{1}{2} \mu^2 \omega_{0J}(J^{pq}) \xi^2 + O(\mu^3)$$

We assume the existence of a number η_0 ($0 < \eta_0 < \frac{2\pi}{\nu}$) such that

(AII 11)

$$\frac{d\xi}{dt} = \mu X_0(\eta), \quad \frac{d\eta}{dt} = \mu \omega_{0J}(J^{pq}) \xi^2$$

possesses the solution $\xi = 0$, $\eta = \eta_0$ (AII 14) Next we linearise the right-hand side of system (AII 13) on solution (AII 14) and obtain the matrix

$$A_0(\mu) = \begin{pmatrix} \mu X_{0\eta}(\eta_0) & 0 \\ \mu \omega_{0J}(J^{pq}) & 0 \end{pmatrix}$$

It follows from formula (AII 5) that $\omega_{0J}(J^{pq}) > 0$. Therefore, if

$$X_{0\eta}(\eta_0) > 0$$

then the matrix $A_0(\mu)$ has real eigenvalues of different signs. The classical averaging theorem of Bogoliubov, as applied on the infinite interval, yields the existence of the periodic solution with period $\frac{2\pi p}{\nu}$ for system (AII 9) for sufficiently small μ and yields the instability of this solution. We apply the obtained result to system (AII 3).

Theorem 1. Assume that there exists the number η_0 satisfying equality (AII 12) and inequality (AII 15). Then in the $\sqrt{\varepsilon}$ -neighbourhood of the resonant point J^{pq} if ε is sufficiently small there exists a unique periodic solution of system (AII 3) with period $\frac{2\pi p}{\nu}$ which is unstable.

Now assume that instead of inequality (AII 15) the following inequality holds

$$X_{0\eta}(\eta_0) < 0$$

(AII 16) In this case the eigenvalues of the matrix $A_0(\mu)$ are purely imaginary. For investigation of the problem of the existence and stability of the periodic solutions in the μ -neighbourhood of the resonant point J^{pq} it is necessary to consider the averaged equations of the second approximation. These equations take the form

$$\frac{d\xi}{dt} = \mu X_0(\eta) + \mu^2 X_1(\eta) \xi$$

$$\frac{d\eta}{dt} = \mu \omega_{0J}(J^{pq}) \xi + \mu^2 X_2 X_0(\eta) + \frac{1}{2} \mu^2 \omega_{0J}(J^{pq}) \xi^2$$

(AII 17) If there exists a number η_0 such that $X_0(\eta_0) = 0$ and $X_{0\eta}(\eta_0) < 0$, by the implicit function theorem, for small μ , there exists the unique $w(\mu) = [\xi(\mu), \eta_0]$ such that $\xi(0) = 0$ and $w(\mu)$ is the equilibrium point of system (AII 17). The matrix of the linear approximation of system (AII 17) takes the form

$$A_1(\mu) = \begin{pmatrix} \mu^2 X_1(\eta_0) & \mu X_{0n}(\eta_0) \\ \mu \omega_{0J}(J^{pq}) & \mu^2 X_{0n}(\eta_0) \end{pmatrix}$$

The eigenvalues of the matrix $A_1(\mu)$ have negative real parts if

$$X_1(\eta_0) + X_{0n}(\eta_0) < 0 \quad (\text{AII } 18)$$

and positive real parts if

$$X_1(\eta_0) + X_{0n}(\eta_0) > 0 \quad (\text{AII } 19)$$

It follows from Theorem 3(a)² that there exists a unique periodic solution with period $\frac{2\pi p}{\nu}$ of system (AII 9) for sufficiently small μ . This solution is asymptotically stable if inequality (AII 18) holds, and is unstable if inequality (AII 19) does not hold.

We apply the obtained statement to system (AII 3).

Theorem 2. Let the number η_0 satisfy equality (AII 12) and inequality (AII 16).

Let the inequality

$$X_1(\eta_0) + X_{0n}(\eta_0) \neq 0$$

hold. Then, if ε is sufficiently small, system (AII 3) has a unique periodic solution with period $\frac{2\pi p}{\nu}$ in the ε -neighbourhood of the resonant point J^{pq} .

This solution is asymptotically stable if

$$X_1(\eta_0) + X_{0n}(\eta_0) < 0$$

and unstable if

$$X_1(\eta_0) + X_{0n}(\eta_0) > 0$$

For example, consider the equation

$$\ddot{x} + \Omega^2 x + \Phi(x, \dot{x}) = \varepsilon [-\gamma \dot{x} + a \sin(\nu t + \delta)] \quad (\text{AII } 20)$$

where $\gamma > 0$ and Ω, a, ν, δ are real constants. Using the transformation (AII 4), we obtain the system

$$\frac{dJ}{dt} = -4\varepsilon\omega_0(J) [a \sin(\nu + \delta) + \gamma J \omega_0(J) \kappa^\psi(\psi)] \kappa^\psi(\psi)$$

$$\frac{d\psi}{dt} = \omega_0(J) - 4\varepsilon\omega_0(J) J^{-1} [a \sin(\nu + \delta) + \gamma J \omega_0(J) \kappa^\psi(\psi)] [-J \kappa(\psi)] \quad (\text{AII } 21)$$

Assume that the solution of Eq. (AII 7) exists. The substitution $\psi = \phi + \frac{d}{b} \nu$ yields system (AII 21) in the following form

yields system (AII 21) in the following form

$$\omega(J_{n1}) = \frac{n}{\nu}$$

then, if ε is sufficiently small, Eq. (AII 20) possesses n solutions which are unstable resonance periodic solutions with period $\frac{2\pi n}{\nu}$ in the $\sqrt{\varepsilon}$ -neighbourhood of the resonant point J_{n1} , and n solutions which are asymptotically stable

solution of the equation

Theorems 1 and 2 we obtain the following result. If the resonant point J_{n1} is a negative signs. Easy computation demonstrates that $X_1(\eta_{0l}) + X_{0n}(\eta_{0l}) < 0$. From and, therefore, $X_{0n}(\eta_{0l})$ has n values with positive signs and n values with

$$X_{0n}(\eta_{0l}) = \pm \frac{2a\nu^2}{\sqrt{1 - A_n^2}} \sqrt{\frac{\pi(5\Omega^2 - \nu^2)}{2}}$$

The evaluation of a derivative of the function $X_0(\eta)$ at the point η_{0l} yields

$$\eta_{0l} = \frac{n}{\delta} \pm \frac{n}{a \cos A_n} + \frac{n}{2\pi l}, \quad l = 0, \dots, n-1$$

number of values of n . If Eq. (AII 23) possesses a solution, then

Since $A_n \rightarrow \infty$ as $n \rightarrow \infty$, then Eq. (AII 23) possesses a solution for a finite

$$\cos(n\eta - \delta) = \frac{\gamma J^{pq} x n}{4\Omega^2 \nu^2} (\Omega^2 - \nu^2) \left(1 + \frac{J^{pq}}{J^2} \right) = A_n \quad (\text{AII } 23)$$

Then the number η_0 is defined by the equation

$$-\frac{\gamma J^{pq}}{4\Omega^2 \nu^2} \left(1 + \frac{J^{pq}}{J^2} \right) = \frac{2}{\gamma J^{pq}}$$

equation of system (AII 22) equals

The averaged value of the second addend in the right-hand side of the first

$$X_0(\eta) = \frac{\pi n (\Omega^2 - \nu^2)}{2a\nu^2} \cos(n\eta - \delta)$$

this average value equals

For an analysis of the resonant regimes we apply Theorems 1 and 2. The averaged value of the first addend in the right-hand side of the first equation of system (AII 22) is different from zero if, and only if, $q = 1, p = n (n = 0, 1, \dots)$. If $q = 1, p = n$

$$\times \left[-J \kappa \left(\phi + \frac{d}{b} \nu \right) \right] \int$$

$$\frac{d\phi}{dt} = \omega_0(J) - \frac{d}{b} \nu - 4\varepsilon\omega_0(J) J^{-1} \left[a \sin(\nu + \delta) + \gamma J \omega_0(J) \kappa^\psi(\phi) + \frac{d}{b} \nu \right] \quad (\text{AII } 22)$$

resonant periodic solutions with period $\frac{2\pi}{\nu}$ in the $\sqrt{\varepsilon}$ -neighbourhood of the resonant point J_{n1} .

Structure-borne Vibro-impact Resonances and Periodic Green Functions

This Appendix reviews the resonant behaviour of vibro-impact systems and the methodology for analysis of vibro-impact resonant phenomena. The primary emphasis is with the description of the new nonlinear effects and their relation to generative mechanical structures.

INTRODUCTION

Systematic impact interactions of elements in mechanical structures produce a synergistic effect on their vibration, noise radiation and load capacity. This takes place mainly because of accumulation of perturbations that can build up from impact to impact. The above impulsive loading may arise under variable forces as a result of intermittent unilateral contacts of the structure elements in the systems with backlashes in joints and kinematic pairs, during opening and closing of cracks etc. In some applications essential suppression of vibration by impacts may be achieved using specially designed stops, bumpers and impact dampers. Embedding the unilateral constraints into mechanical structures transforms the latter into the strongly nonlinear systems. Multiple impact interactions of structure elements can excite complex nonlinear resonant responses of systems. The specific features of the resonant waveforms are defined mainly by the character of the above unilateral constraints.

Introduction of *periodic Green functions* (PGF) [11,17] provides for a regular and effective tool to an analysis of these phenomena. PGF are steady-state responses of linear components of the structures, which are involved into interaction, on periodic impulse excitation. Application of PGF allows for obtaining the exact solution for the problem of periodic impact interaction in the engineering structures, revealing the fundamental features of their specific resonant behaviour and developing the efficient approximations for more complex system containing additional nonlinearities. This method generalises the previous attempts of the analysis of vibro-impact interactions using the series of Dirac δ -functions [58, 198, 199] and introduced an effective application of integral equations, along with asymptotic, frequency, and structural concepts to an analysis of vibro-impact phenomena. The *stitching method*, which is used commonly for the analysis of vibro-impact processes, necessitates the knowledge of the *general solution* of appropriate differential equations describing the motion between impacts [77]. Contrary to this, the PGF method produces close analytical solutions without referring to the differential equations of motion and can be applied even when the full mathematical model of the system is unknown. An exercise of the above method requires only the knowledge of complex frequency response characteristics (receptances) of contacting linear subsystems, which can be obtained as a *particular solution* of the above differential equations of motion, if they are available, or even measured in the regular experiment.

STATEMENT OF THE PROBLEM

Consider two linear stationary structures 1 and 2 (Fig. AIII 1) which are loaded by constant forces G_1 . The structures contact through the rigid bodies M_1 and M_2 . The bodies are involved in the one-dimensional T -periodic motion with collisions under the external T_q -periodic excitation $P_n(\omega t)$ ($T_q = T/q = 2\pi/\omega$, ω is the angular frequency, q is multiplicity of the sub-periodic regime) which is applied to the arbitrary point of the subsystems.

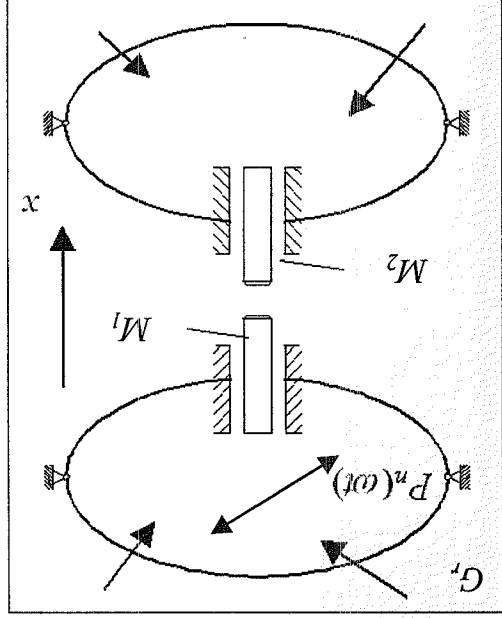


Fig. AIII 1

Let $L_{10}(i\omega)$ and $L_{20}(i\omega)$ be the complex dynamic compliances (receptances) of the bodies at contact points, Δ is the initial gap (negative Δ corresponds to interference) which results due to the action of static forces. The function $\Phi[x(t), \dot{x}(t)]$ defines the force characteristic of impact interaction, where $x(t)$ is the relative motion of the above bodies. For periodic single-impact interaction the period of which is $T = 2\pi q / \omega$ we can present the above force in the form of Fourier series

$$F(t) = \Phi[x(t), \dot{x}(t)] = \sum_{m=-\infty}^{\infty} F_m \exp\left(\frac{im\omega}{q} t\right) \quad (\text{AIII } 1)$$

whose coefficients are given by

$$F_m = \frac{1}{i\omega} \int_0^T F(t) \exp\left(-\frac{b}{i\omega} t\right) dt \quad (\text{AIII } 2)$$

The operator equations for absolute displacements of the contact elements take the form

$$x_1(t) = \sum_{l=1}^r L_{l1}(0) G_{l1}(s) R_{l1}(t) - L_{l0}(s) F(t) + \sum_{l=1}^n L_{l2}(s) R_{l2}(t) G_{l2}(s) F(t), \quad s = i\omega$$

Here the terms under the summation sign define the static deformation of the elements. Subtracting the second equation from the first and initiating time from the instance of the impact, yields

$$x(t) = g(t + \tau) - L(s) F(t) \quad (\text{AIII } 3)$$

where

$$g(t) = \sum_{l=1}^r L_{l1}(0) G_{l1}(s) + \sum_{l=1}^n L_{l2}(s) R_{l2}(t) G_{l2}(s)$$

and τ is the time lag of the excitation relative to the impact.

Taking (AIII 1) into account we find

$$x(t) = g(t + \tau) - \sum_{m=-\infty}^{\infty} L\left(\frac{1}{i\omega}\right) F^m \exp\left(\frac{1}{i\omega} t\right) \quad (\text{AIII } 4)$$

Putting (AIII 2) into (AIII 4) and changing the sequence of summation and integration, yields

$$x(t) = g(t + \tau) - \int_{-T}^0 \chi(t - \tau) \Phi[x(\tau), x(\tau)] d\tau \quad (\text{AIII } 5)$$

where

$$\chi(t) = \frac{1}{L} \sum_{m=-\infty}^{\infty} L\left(\frac{1}{i\omega}\right) \exp\left(\frac{b}{i\omega} t\right) \quad (\text{AIII } 6)$$

is a *periodic Green function*. This is a steady-state response of linear system with operator $L(s)$ on the T -periodic excitation

$$\delta^T(t) = \sum_{V=-\infty}^{\infty} \delta(t - VT)$$

where $\delta(t)$ is Dirac function.

For the stationary linear system with fractional-rational operator

$$L(s) = \frac{h_0 s^{2n-2} + h_1 s^{2n-3} + \dots + h_{2n-2}}{d_0 s^{2n} + d_1 s^{2n-1} + \dots + d_{2n}} = \frac{D(s)}{R(s)} \quad (\text{AIII } 7)$$

it is possible to sum series (AIII 6) and obtain a finite function

$$\chi(t) = \sum_{k=1}^{2n} R(s_k) \exp(s_k t) \quad t \in [0, T] \quad (\text{AIII } 8)$$

ANALYTICAL SOLUTION

Examining the integral in the right-hand side of Eq. (AIII 5), we find, using the mean value theorem and assuming continuity of periodic Green functions,

$$\int_0^T \chi(t - \tau) \Phi[x(\tau), x(\tau)] d\tau = \int_0^T \chi(t - \tau) \Phi[x(\tau), x(\tau)] d\tau = J \chi(t - \tau) \Phi[x(t), x(t)]$$

for $0 < \theta(t) < 1$. Here J is impulse of contact force and ε is a small parameter.

Noting that $\varepsilon \theta(t) \ll T$, we have for continuous periodic Green functions $\chi[t - \varepsilon \theta(t)] \approx \chi(t)$. For a momentary impact defined by Newton's hypothesis, this relation becomes exact and we have instead of (AIII 5) the finite expression

$$x(t) = g(t + \tau) - J \chi(t) \quad (\text{AIII } 9)$$

Suppose, for example, that $g(t + \tau) = a \cos(\omega t + \varphi)$, we find the unknown values of phase φ and impulse J from the impact condition:

$$x(0) = \Delta, \quad (1 + R) M \dot{x}_-(0) = J \quad (\text{AIII } 10)$$

where Δ is the static gap (interference), R is the restitution coefficient, $M = M_1 M_2 (M_1 + M_2)^{-1}$, $\dot{x}_-(0)$ is a relative velocity of colliding bodies just before the impact.

Applying conditions (AIII 10) to (AIII 8), we find

$$a \cos \varphi = \Delta + J \chi(0), \quad a \sin \varphi = -J \left[\dot{\chi}_-(0) + \frac{\omega}{1 + R} M \right]$$

The excluding φ from these equations, yields finally

$$J = \frac{-\Delta \chi(0) \pm \sqrt{\Delta^2 \chi^2(0) - \frac{\omega^2}{1 + R} M \left[\dot{\chi}_-(0) + \frac{\omega}{1 + R} M \right]^2}}{\frac{\omega^2}{1 + R} M}, \quad J \geq 0 \quad (\text{AIII } 11)$$

Then the unknown phase φ can be defined from the expressions

$$\sin \varphi = -\frac{J}{\omega} \left[\dot{\chi}_-(0) + \frac{\omega}{1 + R} M \right], \quad \cos \varphi = \frac{a}{\Delta + J \chi(0)} \quad (\text{AIII } 12)$$

Neglecting the damping in the system, we have from (AIII 11)

$$J = \frac{\chi(0)}{\Delta \pm a}$$

It is shown in [17,150], that in the case of symmetric double-side impact interaction, a representation similar to (AIII 3) of the periodic solution exists within a half period of motion. A corresponding formula for DCF in the case of

$$\chi(t) = \sum_{k=1}^n \frac{R(s_k)}{D'(s_k)} \frac{1 + \exp(s_k T/2)}{\exp(s_k t)}, \quad t \in [0, T/2] \quad (\text{AIII } 13)$$

STABILITY ANALYSIS

The disturbed motion due to small perturbation can be described as

$$x(t) = a \cos[\omega t + \varphi(t)] - \tilde{J}(t) \chi(t) \quad (\text{AIII } 14)$$

where $\varphi(t)$ and $\tilde{J}(t)$ are slowly varying functions of time.

Using the first expression (AIII 12) as the reduced condition for the balance of work done by the dissipative and exciting forces, it is possible to formulate the stability energy condition for periodic motion in the form [17]

$$\frac{d}{dt} \left\{ \frac{\tilde{J}}{\omega} \left[\chi^-(0) + \frac{(1+R)M}{1} \right] + \sin \varphi \right\}_{\tilde{J}=\tilde{J}} > 0 \quad (\text{AIII } 15)$$

Introducing two arbitrary slow functions in place of one, we can relate them by an expression similar to the second equation in (AIII 12). Differentiating (AIII 15) and using the above relation, we obtain, taking account of (AIII 12)

$$J > \frac{-\Delta \chi(0)}{\chi^-(0) + \frac{\omega}{2} \left[\chi^-(0) + \frac{1}{(1+R)M} \right]}$$

Comparing (AIII 11) with the equation obtained, we can establish that the stability condition is satisfied only with the solution, which has the positive sign before the radical.

IMPACT OSCILLATOR

We will demonstrate an application of the described method to the analysis of periodic response of the impact oscillator (Fig. AIII 2).

A steady-state response of the oscillator can be presented as

$$x(t) = a \cos(\omega t + \varphi) - J \chi(t), \quad t \in [0, \pi/\omega] \quad (\text{AIII } 16)$$

where a is the amplitude of periodic steady-state response of linear oscillator to harmonic excitation:

$$a = \frac{k |1 - (\omega/\Omega)^2|}{P}$$

$\chi(t)$ is the periodic Green function of a linear oscillator, J and φ are unknown parameters, which can be found from the impact conditions (AIII 10).

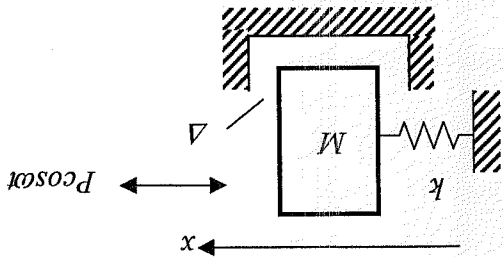


Fig. AIII 2

The periodic Green function for a linear oscillator can be found in the close form (AIII 13) by using an expression for the transfer function of linear oscillator

$$L(s) = \frac{1}{s^2 + \Omega^2}$$

This has two imaginary conjugated roots $s_{1,2} = \pm i\Omega$ and in accordance with (AIII 13) the PGF in this case takes the form

$$\chi(t) = \frac{1}{e^{i\Omega t}} \frac{1}{1} \frac{1 + e^{i\Omega T}}{2i\Omega} - \frac{1}{e^{-i\Omega T}} \frac{1 + e^{-i\Omega T}}{2i\Omega} = \frac{1}{2i\Omega} \frac{1 + \cos \frac{\Omega T}{2}}{1 + \cos \frac{\Omega T}{2}} = \frac{1}{2i\Omega} \frac{1 + \cos \frac{\Omega T}{2}}{1 + \cos \frac{\Omega T}{2}} \quad (\text{AIII } 17)$$

Using this function, the periodic solution (AIII 16) is found in the form

$$x(t) = a \cos(\omega t + \varphi) - J \frac{\sin \Omega \left(t - \frac{T}{4}\right)}{\sin \Omega T} \cos \frac{\Omega T}{4}, \quad t \in \left[0, \frac{\omega}{\pi}\right] \quad (\text{AIII } 18)$$

Applying to this expression the first impact condition from (AIII 10), $x(0) = \Delta$, we can define J as

$$J = \frac{\Delta - a \cos \varphi}{\Delta - a \cos \varphi} = \frac{1}{\tan \frac{\Omega T}{4}} \frac{4}{2\Omega} \tan \frac{2\Omega}{2\omega} \quad (\text{AIII } 19)$$

Expression (AIII 18) takes the form

$$x(t) = a \cos(\omega t + \varphi) - \frac{\Delta - a \cos \varphi}{\sin \frac{\pi \Omega}{2\omega}} \sin \left(\Omega t - \frac{\pi \Omega}{2\omega} \right), \quad t \in \left[0, \frac{\omega}{\pi}\right] \quad (\text{AIII } 20)$$

The second condition from (AIII 10) allows defining the unknown phase φ , which is not important here due to the following. As it is shown in [17], the stable periodic motions with successive impacts against each limiter exist when $\omega > \Omega$. In this frequency band the impact impulse J is increasing monotonously with an increase of ω . Contrary to this, the value of a in this frequency band is monotonously decreasing. This means that the vibro-impact resonance, which is characterised by a strong impact impulse, can be defined approximately as a function of the frequency ω in the form

$$x(t) \approx -\frac{\Delta}{\pi\Omega} \sin\left(\Omega t - \frac{\pi\Omega}{2\omega}\right), \quad t \in \left[0, \frac{\pi}{\omega}\right]$$

which contains the structure parameters of the system only. This justifies the treatment of vibro-impact resonance as *structure-borne*.

We will show that the exact solution (AIII 20) coincides with a solution, which may be obtained using the stitching method. Contrary to the PGF method, which relies on the *particular* solution (AIII 16), the stitching method requires the *general* solution for the equation of a linear oscillator

$$x(t) = C_1 \cos \Omega t + C_2 \sin \Omega t + a \cos(\omega t + \varphi) \quad (\text{AIII 21})$$

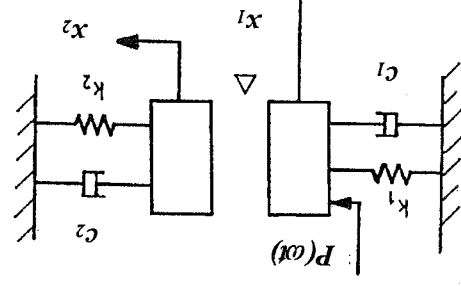
where C_1, C_2 are arbitrary constants. Applying the conditions of periodicity and impact for one period of motion between the impacts: $t = 0, x = \Delta, \quad t = \pi/\Omega, x = -\Delta$ to solution (AIII 21), we find the arbitrary constants

$$C_1 = \Delta - a \cos \varphi, \quad C_2 = -C_1 \cot(\pi\Omega/2\omega)$$

Substitution of these constants into (AIII 21) produces the solution, which coincides with (AIII 20).

TWO COLLIDING BODIES

In the case of two colliding bodies under periodic excitation (see Fig. AIII 3) two resonant modes were revealed as a result of analysis and confirmed by physical experiments (Fig. AIII 4) [221]. The motions of the bodies are shown in Fig. AIII 4 with the thick line as $x_1(t)$ and thin line as $x_2(t)$. Fig. AIII 5 shows the corresponding impulse-frequency response obtained at the sweep-up test.



Impulse J [Ns]

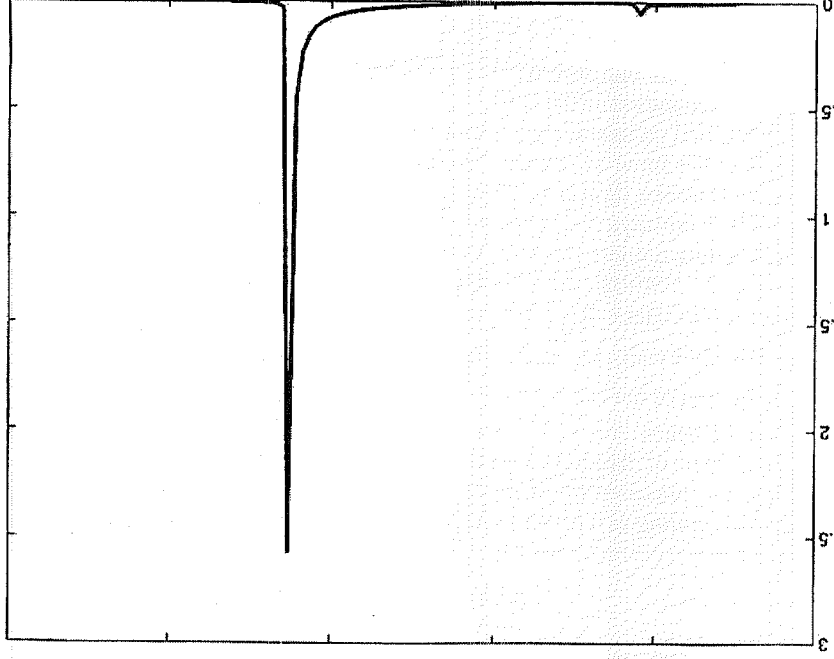
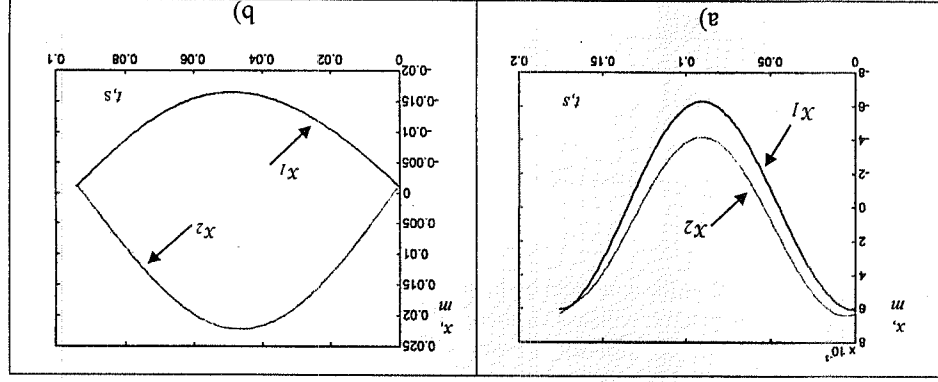


Fig. AIII 4
In the second mode, named as *clapping resonance* (Fig. AIII 4b), the bodies are involved in anti-phase motion with one strong impact per period.



$$\left(\frac{\Omega_1}{\omega}\right) \approx 1$$

In the first mode, termed as *grazing resonance* (Fig. AIII 4a), both bodies are involved in the in-phase motion with one weak contact during the period. The frequency of this resonance (Fig. AIII 5) is independent of excitation amplitude and disposed in the vicinity of the natural frequency of the first subsystem

The clapping resonance depends strongly on the amplitude of excitation and demonstrates typical non-linear behaviour with frequency pulling and jump phenomena which takes place at frequency exceeding essentially the natural frequency of the first linear subsystem (Fig. AIII 5). For the inverse sweep test the system shows the grazing resonance only.

SYSTEMS WITH MULTIPLE IMPACT PAIRS

The above procedure can be generalised for studying resonant regimes in systems with *multiple one-dimensional impact pairs* [17]. The integral equation in this case is

$$\mathbf{x}(t) = \mathbf{Q}(t + \tau) - \int_0^t \mathbf{X}(t - \tau) \mathbf{P}[\mathbf{x}(s), \mathbf{x}(s)] d\tau \quad (\text{AIII } 22)$$

where $\mathbf{x}(t)$ is a vector of the relative disposition of the elements in pairs, $\mathbf{P}(\mathbf{x}, \mathbf{x})$ is a vector of impact force characteristics, $\mathbf{\Theta}(t)$ is a vector of relative vibration of elements of the impacting pairs when impact interaction is neglected, $\mathbf{X}(t)$ is the matrix of periodic Green functions for the linear part of the system

$$\mathbf{X}(t) = \frac{1}{T} \sum_{m=-\infty}^{\infty} \mathbf{A}(im\omega) \exp(im\omega t)$$

where $\mathbf{A}(i\omega)$ is the compliance matrix.

In systems of the type examined it is possible to observe impacting modes of vibration, characterised by *simultaneous* impacts in almost all pairs. The number of impact pairs sharing vibration in various modes can vary.

For the regimes described, called *cooperative modes*, we find from (AIII 22)

$$\mathbf{x}(t) \approx -\mathbf{X}(t) \mathbf{J} \quad (\text{AIII } 23)$$

where $\mathbf{J}$ is the vector of impact impulses. From the impact conditions $\mathbf{x}(0) = \Delta$, where Δ is a vector of gap (interference) arrangements, we find

$$\mathbf{x}(t) \approx \mathbf{X}(t) \mathbf{X}^{-1}(0) \Delta \quad (\text{AIII } 24)$$

The existence conditions for regimes of type (AIII 16) and an unknown phase of excitation are found from the energy balance of the dissipative and excitation forces

$$\int_0^T \mathbf{V}^{-1}(s) \{\dot{\mathbf{x}}(t)(s)\} \dot{\mathbf{x}}(t)(s) dt + \sum_{q=1}^n \frac{2(1 + R_q) M_q}{(1 - R_q) J_q^2} \quad (\text{AIII } 25)$$

Here q is the index of the impact pair under consideration, R_q is the coefficient of restitution, J_q is the impact impulse and M_q is the reduced mass for the q -indexed pair.

EXPERIMENTAL VERIFICATION OF THE COOPERATIVE MODE

Fig. AIII 6 shows the experimental rig for the verification of the existence of a mode with simultaneous impacts in all impact pairs [216]. A rubber cord with three fixed plastic beads was stretched between the pin of the force transducer on

the left end and the pushrod of an electrodynamic shaker producing transverse sinusoidal excitation of the cord.

The beads had unilateral rigid limitation of their displacement with the possibility to register the force of contact. The pictures of cooperative mode were taken by using stroboscope flashes synchronised with the voltage applied to the exciter.

Fig. AIII 7 portrays the cooperative mode described when the set of beads hits corresponding limiters simultaneously.

Fig. AIII 8 confirms the existence of the cooperative mode by registration of contact forces. Here $\Phi_{1,2,3}$ are the signals from the load-cells, which are mounted upon the rigid wall, Φ_4 is the signal of force transducer which is proportional to the deflection of the last bead, Φ_5 is the signal of pushrod vibration.

The set of beads involved in the co-operative mode of vibration behaves like a single degree of freedom impact oscillator [17], which demonstrates frequency pulling, i.e. preservation of mode due to variation of the driving frequency and *amplitude pulling* as a preservation of mode due to the clearance variation.

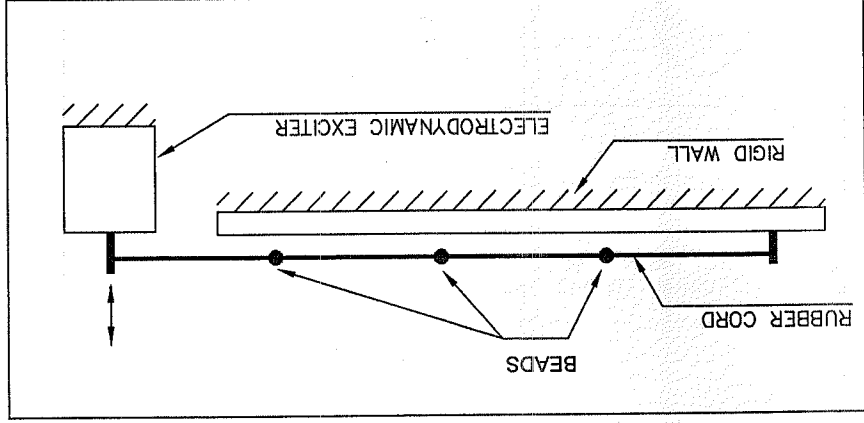


Fig. AIII 6

Both types of pulling terminate their existence with a jump from cooperative vibro-impact mode to a linear mode of vibration in which beads do not contact the limiters. There is opportunity of transition from cooperative vibro-impact mode to linear mode of vibration without impacts and vice versa by means of *hard excitation* (additional triggering impulse).

The frequency band of the cooperative modes is located in the right-hand vicinity of the natural frequencies of the linear system.

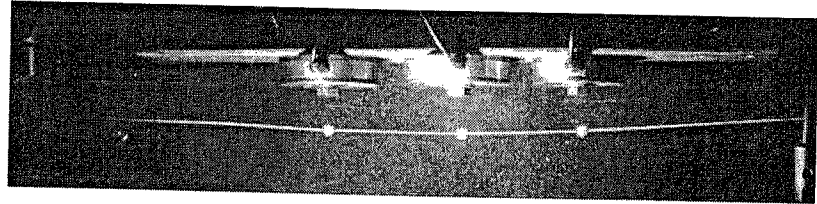
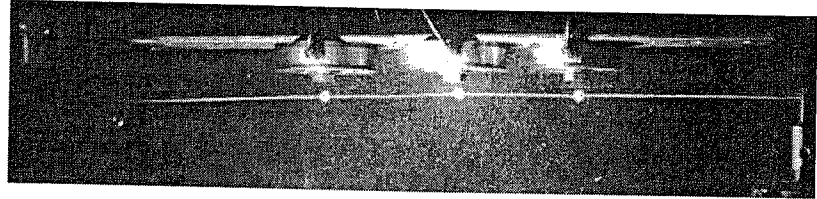
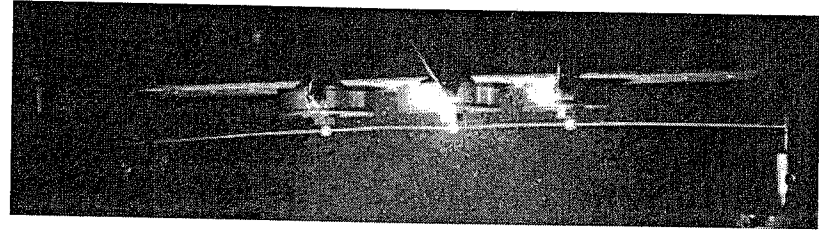
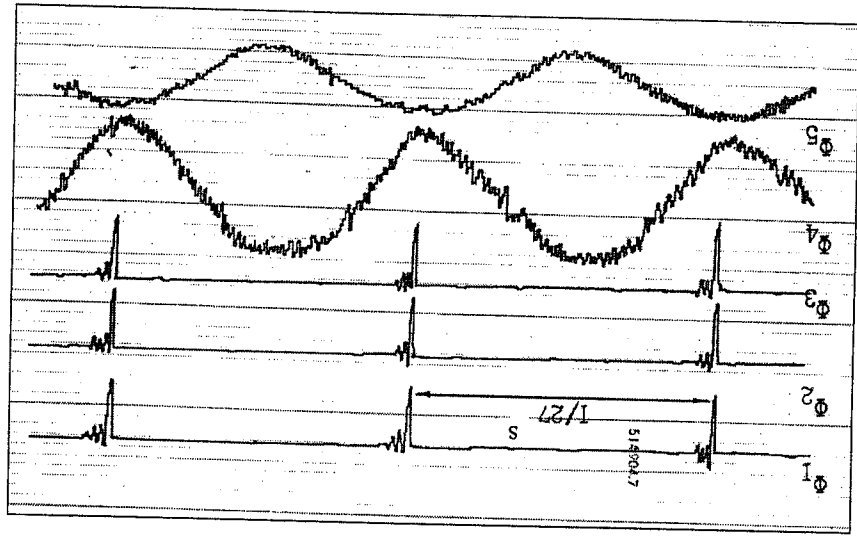


Fig. AIII 7



LOCALISED NONLINEAR STANDING WAVES

During the above experiments there were discovered *localised vibro-impact* modes of vibration (Fig. AIII 9). In this mode only one of the beads is involved in impacts against a limiter and others keep very small amplitude of vibration [216].

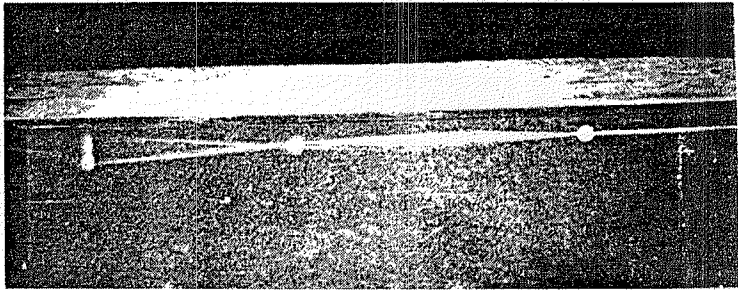
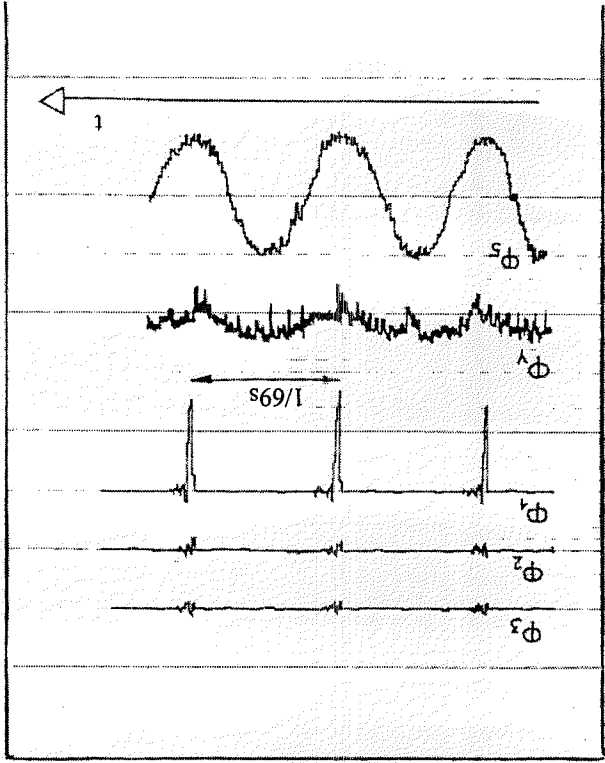


Fig. AIII 9

Figure Fig. AIII 10 shows the corresponding readouts of the sensors.



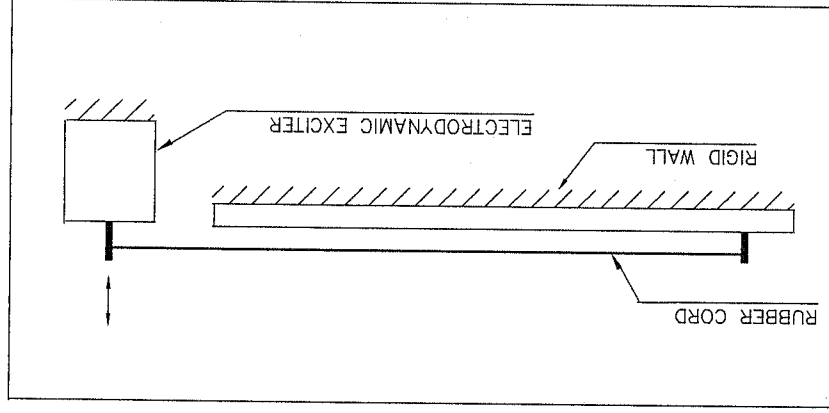
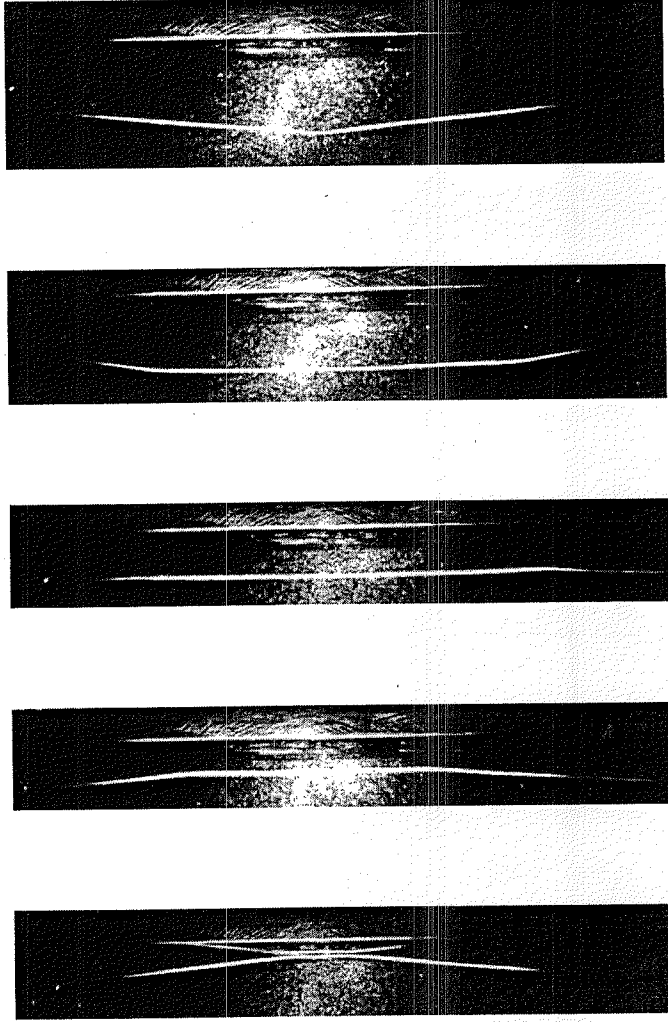


Fig. AIII 11

In such systems a significant variable length of continuous vibrating element may be involved into vibro-impact interaction with the neighbouring elements or a rigid wall.
The object of experimental study was the axially stretched rubber cord (Fig. AIII 11) located in parallel with a rigid wall that limited the transverse deflections of the cord.
As the frequency of excitation exceeded the frequency of the first resonance of linear system (the cord without motion limiter), the nonlinear mode of cord vibration took a trapezoidal form with simultaneous impact of the entire plane part against the limiter (Fig. AIII 12).
The dynamic behaviour of the cord in such a *slapping mode* is similar to that of an impact oscillator [17]. It demonstrates frequency and amplitude pulling as well as transition from slapping mode to linear mode (vibration without contact with limiter) and vice versa under action of additional impulse

Fig. AIII 12



ANALYSIS OF SLAPPING MODES

Initial theoretical study of the similar modes in strings which were involved in free vibration with special initial conditions was carried out by Amerio and Prouse,

¹ See Amerio L, Prouse G, Study of the vibration of a string vibrating against an obstacle. Rend. di Mat. Ser, 1975 6(8), No. 2.

Cabannes and Haraus¹, and Cabannes². In [241] the solution of the problem of forced vibration of a beam is obtained using the periodic Green functions. The subsequent analysis follows reference [241].

The straight uniform beam of length l is stretched parallel to the rigid limiter by axial force N (Fig. AIII 13). The following definitions will be used for description of the system: ρ is the density of beam material, E , G are elastic and shear moduli, S is cross sectional area of the beam, I is area moment of inertia, ε is a small parameter.

The beam is exposed to the action of distributed external force of density $\varepsilon P(x, t)$. It vibrates in the plane and can collide with a straight obstacle displaced parallel to the beam with a clearance Δ relative to the position of static equilibrium of the beam.

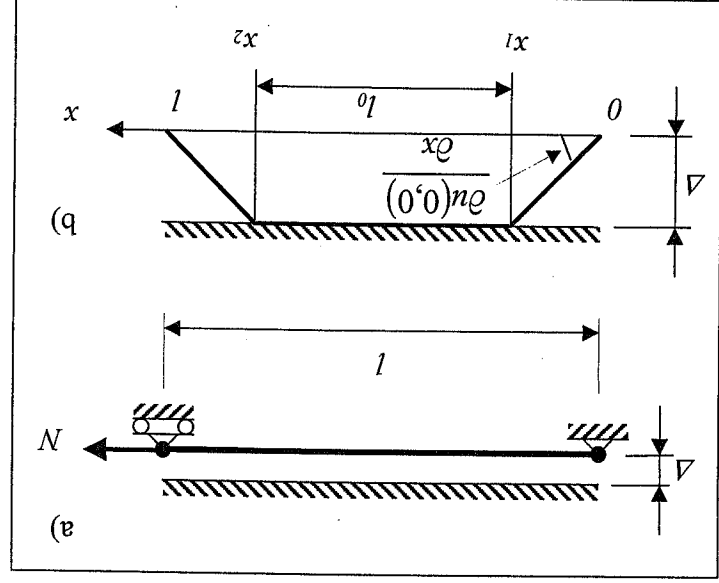


Fig. AIII 13

The Timoshenko beam equations are used

$$\rho S \frac{\partial^2 u}{\partial t^2} - (\alpha SG + N) \frac{\partial^2 u}{\partial x^2} + \alpha SG \frac{\partial^2 y}{\partial x^2} + \varepsilon P(x, t) = 0$$

$$K \frac{\partial^2 y}{\partial x^2} - \alpha SG \left(y - \frac{\partial u}{\partial x} \right) - \frac{I \rho}{S} \frac{\partial^2 y}{\partial t^2} = 0$$

with boundary conditions

(AIII 26)

¹ Cabannes H., Haraus A. Mouvements presque-periodiques vibrantees en presence

$$u(0, t) = u(l, t) = \frac{\partial u(0, t)}{\partial x} = \frac{\partial u(l, t)}{\partial x} = 0 \quad (\text{AIII 27})$$

Here $u(x, t)$ is the shape of the beam transverse deflections at instant t , $y(x, t)$ is the shape of the beam cross section rotation at instant t relative to the neutral line, $K = EI$ is bending stiffness, α is a form factor and $\Phi(u)$ is a density of impact

forces containing δ -functions. Eq. (AIII 26) allows solutions with a non-smooth shape of deflections and discontinuous distribution of velocities along the beam. Such modes can be described using *generalised functions*. An application of such functions causes an appearance of δ -functions in the left-hand side of the first Equation of (AIII 26) that compensates the impact force containing δ -functions as well.

For the T -periodic regimes with single impact over one period of excitation (slapping modes), the impact force density may be presented in the time domain as

$$\Phi[u(x, t)] = j(x) \delta_T[t - \phi(x)] \quad (\text{AIII 28})$$

where $j(x) \geq 0$ and $\phi(x)$ are the distributions of density of impact impulse and the phase of impact, respectively.

The beam vibration in the first slapping mode (as described in experiment) can be analysed using impact conditions (see Fig. AIII 13)

$$u_-(x, 0) = u_+(x, 0) = u(x, 0)$$

$$\left\{ \begin{array}{l} \frac{x_1}{\Delta} x, \quad x \in [0, x_1] \\ \Delta, \quad x \in [x_1, l - x_1] \\ \frac{x_1}{\Delta} x + \Delta + \frac{x_1}{\Delta} (l - x), \quad x \in (l - x_1, l] \end{array} \right. = \left\{ \begin{array}{l} \frac{\partial u_-(x, 0)}{\partial t} \\ \frac{\partial u_+(x, 0)}{\partial t} \\ \frac{\partial u_-(x, 0)}{\partial t} \end{array} \right. \quad (\text{AIII 29})$$

$$\frac{\partial u_-(x, 0)}{\partial t} = \left\{ \begin{array}{l} 0, \quad x \in [x_1, l - x_1] \\ \frac{\partial u_+(x, 0)}{\partial t} - R \frac{\partial u_-(x, 0)}{\partial t}, \quad x \in [0, x_1] \end{array} \right. \quad (\text{AIII 30})$$

$$j(x) = (1 + R) \rho S V_0 = j = \text{const}, \quad x \in [x_1, l - x_1]$$

$$\phi(x) = \text{const} = 0, \quad x \notin [x_1, l - x_1] \quad (\text{AIII 31})$$

Analysis of the slapping modes can be carried out using the energy balance principle as applied to the generative solution of conservative problem. For T -periodic free vibration ($\varepsilon = 0$, $R = 1$) with single elastic impact over the excitation period

$$u(x, t) = - \int_0^t \int_{x_1}^{l-x_1} \chi(x, z, t - s) \Phi[u(z, s)] ds dz, \quad t \in (0, T], \quad x \in [0, l] \quad (\text{AIII 32})$$

where $\chi(x, z, t)$ is a T -periodic Green function representing a steady state

The periodic Green function can be expressed by using the transfer dynamic compliance of the linear system $L(x, z, j\omega)$ between sections x and z as

$$\chi(x, z, t) = \frac{T}{2} \sum_{k=1}^T L(x, z, kj\omega) \cos(k\omega t) \quad (\text{AIII 33})$$

In a number of cases it is possible to neglect the rotary inertia of the cross sections and ignore, therefore, the last term in the second equation of (AIII 26). Under such an assumption, the expression for the dynamic compliance can be presented in the form of the infinite series

$$L(x, z, j\omega) = -\frac{2}{\omega^2} \sum_{m=1}^{\infty} \frac{\rho S l}{\sin \frac{\pi m x}{\pi m z}} \frac{\sin \frac{\pi m z}{\pi m z}}{\omega^2 - \Omega_m^2} \quad (\text{AIII 34})$$

with the natural frequencies Ω_m defined as

$$\Omega_m = \sqrt{\left[\left(\frac{\pi m}{l} \right)^2 \alpha K S G \left(\frac{l}{\pi m} \right)^2 + N K \left(\frac{l}{\pi m} \right)^2 + \alpha N S G \right] \left[K \left(\frac{l}{\pi m} \right)^2 + \alpha N S G \right]}$$

Substituting (AIII 34) into the (AIII 33) and using the finite expression for the Fourier series on the interval of periodicity, we find

$$\chi(x, z, t) = \frac{1}{2} \sum_{m=1}^{\infty} \frac{\rho S l}{\sin \frac{\pi m x}{\pi m z} \sin \frac{l}{\pi m z} \cos \Omega_m \left(t - \frac{T}{2} \right)} \frac{\Omega_m \sin \frac{\pi m T}{2}}{\Omega_m^2} \quad (\text{AIII 35})$$

$$x \in [0, l], t \in [0, T]$$

After substituting (AIII 28), (AIII 31) and (AIII 35) into (AIII 32) and integrating we have

$$u(x, t) = \frac{2j}{\pi} \sum_{m=1}^{\infty} \frac{\rho S \Omega_{2m-1}^2 (2m-1)}{(-1)^{m+1} \sin \pi(2m-1)l_0 \sin \pi(2m-1)x \cos \Omega_{2m-1}(t-T/2)} \frac{l}{\sin \Omega_{2m-1}T/2}$$

$$x \in [0, l], t \in [0, T] \quad (\text{AIII 36})$$

In expression (AIII 36) the unknown values j and $l_0 = 2l - x_1$ are the functions of vibration period T . For the first nonlinear impact mode (as shown in Fig. AIII 13b), from the impact condition of the middle point of the beam with coordinate $x = l/2$ which must be involved in the impact, we have

$$u(l/2, 0) = \frac{2j}{\pi} \sum_{m=1}^{\infty} \frac{\rho S \Omega_{2m-1}^2 (2m-1)}{\sin \pi(2m-1)l_0 \cot \frac{\Omega_{2m-1}T}{2}} = \Delta$$

or

$$j = \frac{\pi \rho S \Delta}{2 \sum_{m=1}^{\infty} \frac{\Omega_{2m-1}^2 (2m-1)}{\sin \pi(2m-1)l_0 \cot \frac{\Omega_{2m-1}T}{2}}} \quad (\text{AIII 37})$$

From Fig. AIII 13b it also follows that

$$l_0 = l - 2x_1 = l - \frac{\tan \partial u(0, 0) / \partial x}{2\Delta}$$

For small vibration $(\tan \partial u(0, 0) / \partial x \approx \partial u(0, 0) / \partial x)$ we find finally the equation for

$$l_0 = l \left[1 - \frac{j \sum_{m=1}^{\infty} \frac{(-1)^{m+1} \sin \frac{\pi(2m-1)l_0}{2l} \cot \frac{\Omega_{2m-1}T}{2}}{l \Delta \rho S} \right] \quad (\text{AIII 38})$$

By substituting (AIII 37) into (AIII 38), we have the following transcendental equation for the non-dimensional variable $\lambda = l_0 / l$,

$$\lambda = 1 - \frac{\pi}{2} \frac{\sum_{m=1}^{\infty} \frac{\Omega_{2m-1}^2 (2m-1)}{\sin \pi(2m-1)\lambda} \cot \frac{\pi \Omega_{2m-1} \zeta}{2}}{\sum_{m=1}^{\infty} \frac{(-1)^{m+1} \sin \pi(2m-1)\lambda}{\pi \Omega_{2m-1}^2} \cot \frac{\pi \Omega_{2m-1} \zeta}{2}} \quad (\text{AIII 39})$$

Here $\zeta = 2\pi / T\Omega_1$ is non-dimensional frequency of vibration.

In order to estimate the existence condition for the slapping mode (AIII 36) a force of excitation distributed in accordance with the first linear mode $P(x, t) = q \sin(\pi x / l) \cos(\omega t + \phi)$ was considered, where q is intensity, ϕ is the phase of excitation relative to impact. The generative solution (AIII 36) with values of $\lambda(\zeta)$ and $j(\zeta)$ being evaluated numerically according to (AIII 37), (AIII 39), was substituted into the first equation from (AIII 26). Then both parts of the equality obtained were multiplied by the function $\partial u(x, t) / \partial t$ produced analytically from (AIII 36). The equality prepared in such manner was integrated along the length of the beam and over the period of vibration $T = 2\pi / \omega$. As a result the energy balance equation between excitation and dissipation was estimated,

$$\int_T^0 \int_T^0 \Phi(u) \frac{\partial u(x, t)}{\partial t} dx dt = \varepsilon \int_T^0 \int_T^0 P(x, t) \frac{\partial u(x, t)}{\partial t} dx dt \quad (\text{AIII 40})$$

Since a considerable part of the beam is involved in simultaneous collision, the energy losses can be calculated as the impact of a rigid body with coefficient of restitution R such as $1 - R \sim \varepsilon$,

$$\int_T^0 \int_T^0 \Phi(u) \frac{\partial u(x, t)}{\partial t} dx dt = (j l_0)^2 \frac{2l_0(1+R)\rho S}{1-R} \approx \int_T^0 \int_T^0 \frac{4\rho S}{\varepsilon}$$

After substituting the excitation in the right side of Eq. (AIII 40) and integrating, the energy balance equation yields $\sin \varphi = \frac{\pi^2 \lambda (\Omega_1^2 - \omega^2)}{8g\omega \sin(\pi\lambda/2)}$. The existence condition of the slapping mode is $|\sin \varphi| \leq 1$.

INTERMITTANT INTERACTION OF CONTINUOUS SYSTEM WITH POINT LIMITERS

The analysis of steady state waves in continuous systems interacting with point limiters can be fulfilled with an application of PGF method in the following manner [226]. Take, for example, a linear one-dimensional continuous system. This can be a stretched string, beam or any other similar object with a length l and transfer dynamic compliance operator $L(x, y, p)$, where $x, y \in [0, l]$, $p = \partial/\partial t$. The specific properties of operator L are defined by the structure of the system and boundary conditions.

Let the motion of the system, $u(x, t)$, be restricted by a point limiter which is fixed at the point with the coordinates $x = a$, $y = \Delta$. This yields $u(a, t) \geq \Delta$. Assuming that the steady state T -periodic vibro-impact motion takes place under the T -periodic distributed force, we can write the equation for an unknown displacement by analogy with (AIII 5), (AIII 32) as

$$u(x, t) = u_0(x, t) - \int_0^t F(s) \chi(x, a, t - s) ds \quad (\text{AIII 41})$$

where $\chi(x, y, t)$ is the T -periodic Green function of the linear system with operator $L(x, y, p)$, $u_0(x, t)$ is the steady state solution of a linear problem of forced vibration. Unknown time instants τ and $\theta \in [0, T]$ as well as a response force $F(t)$ are defined by the additional conditions. An analytical solution of the problem is obtained in [226]. Below, we describe the experiments on discovering the proper trapezoidal modes [210].

A test rig with a stretched rubber cord sketched in Fig. AIII 14 is analogous in arrangement to that described in Fig. AIII 5. The point limiters (one or two) are performed as the thin rods with axes perpendicular to the plane of the cord vibration. There is a possibility to change a position of the limiters in the plane. The modes of the cord vibration were registered using stroboscope flashes synchronised with the voltage applied to the exciter. The results below correspond to the rubber cord with the first natural frequency equal 30 Hz. The existence of the different nonlinear steady waves in different frequency bands were registered in experiments. The wave profiles are composed from linear segments dependent on quantity and disposition of the limiters. The typical configurations are given below.

Fig. AIII 15 shows a steady state wave of frequency 34 Hz at different time instants when a single point limiter is disposed at $x = \frac{l}{2}$ and $y = \Delta = 30 \text{ mm}$. At the extreme position (Fig. AIII 15a) a cord has a form of isosceles triangle. Moving within the gap, the wave transforms itself into isosceles trapezoid (Fig. AIII 15b). The sides of the trapezoid are staying stationary and vary their length properly as a short base moves with a constant velocity by increasing in length during the motion towards the limiter and by decreasing in length due to the opposite motion.

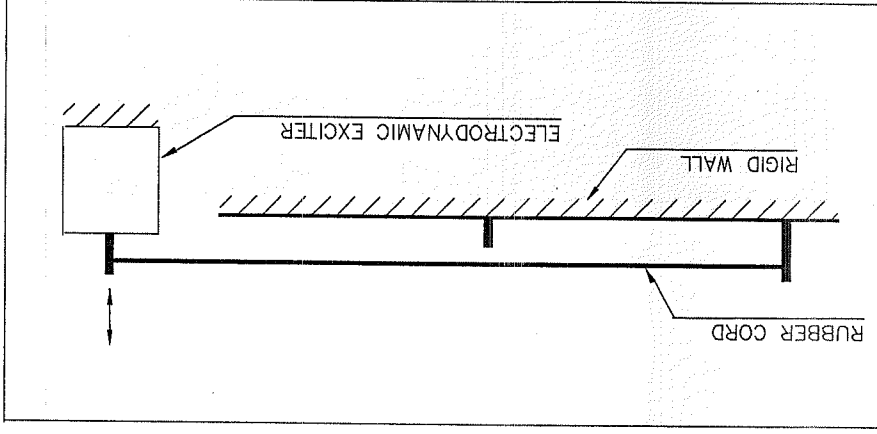


Fig. AIII 14

At the instant of contact of the short base with the limiter, the point of the contact stops its motion. The trapezoid splits itself into the symmetrical double trapezoid (Fig. AIII 15c) of a smaller scale. In the following motion the trapezoidal forms change themselves gradually into the symmetrical angular stationary forms at the extreme position of the cord (Fig. AIII 5d). The adjacent sides of the angular forms are connected at the point of the limiter disposition. A shift in a position of the limiter from the axis of symmetry leads to the gradual degradation of the piece-linear form of the wave profile but preserve an angular structure in adjacent pieces of profile. At the most distant position of wave from the limiter, the profile is close to triangular with a vertex being shifted in the direction opposite to the offset of the limiter. On the motion to the limiter, the profile transforms itself into trapezoidal with a small base, which is gradually extending and approaches the limiter with symmetrical disposition against it. On the further move of a cord after the contact, the split trapezoidal waves form the vertexes simultaneously in the extreme position of the cord. The experiments conducted with two point limiters revealed a similar behaviour of the steady state waves (Fig. AIII 16). During a motion within the gap the profile of the cord transforms itself from triangular (Fig. AIII 16a) to trapezoidal

wave (Fig. AIII 16c). The figures correspond to the frequency of 34 Hz and both limiters were installed with the clearance of $\Delta = 20$ mm.

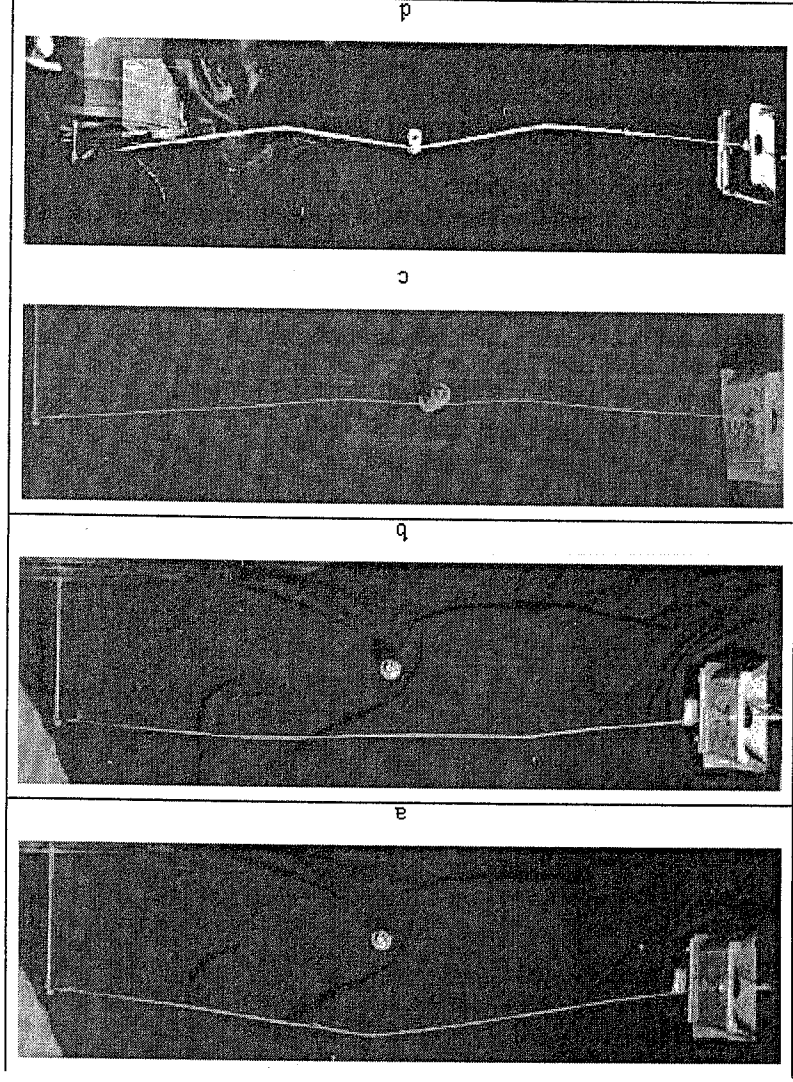


Fig. AIII 15

The amplitude response for the system in the described steady state modes is reminiscent of a response of impact oscillator with a one-sided limiter. This demonstrates the frequency and amplitude pulling and jump phenomena.

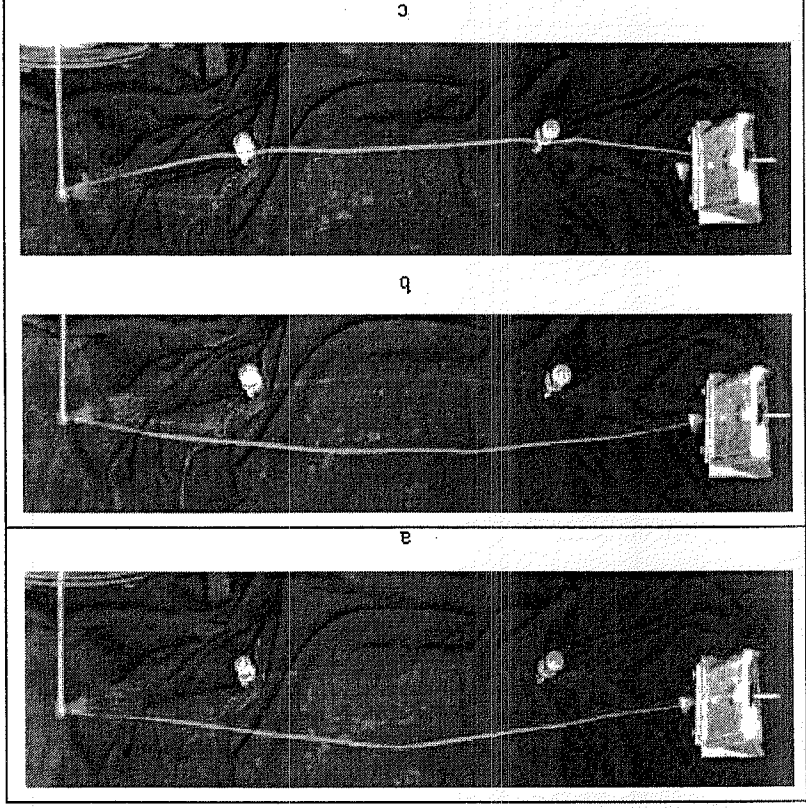


Fig. AIII 16

frequency) detract from the use of a tuned dynamic absorber in applications operating with variable excitation frequency.

The use of the optimised amount of damping in the design of a dynamic absorber results in a close control of both resonant responses which are intrinsic to a system. However, the penalty is that the positive feature of vibration suppression at the antiresonant frequency is diminished.^{1,2}

In 1928 Ormondroyd and Den Hartog (see footnote ¹) proposed an interesting approach to the vibration control of a system equipped by a lightly damped tuned dynamic absorber. In order to avoid resonances in the system containing such a dynamic absorber, they proposed to limit the relative motion of the absorber to the primary structure by means of elastic stops. They also suggested designing the sway space of the dynamic absorber in such a manner as to provide the impactless linear motions of both masses in the desired vicinity of the antiresonant frequency. Such a design allows for maintaining the desired feature of linear dynamic suppression of harmonic vibration.

As soon as the relative motion of the absorber exceeds the pre-designed clearance (as a result of approaching the excitation frequency of one of the resonant frequencies, in this instance) the stops come into play, thus limiting the relative motion of the absorber. Ormondroyd and Den Hartog believed that as the stops change the elastic properties of the entire system "no resonance can occur even if no damping exists in the system". They also indicated that an analytical description of the dynamics of such a system is extremely difficult.

Since then, numerous researchers have been attracted by the simplicity of this elegant engineering solution and have studied such an approach numerically^{3,4}. To our knowledge, the analytical solution has not yet been obtained.

It is evident that in the case mentioned, the control over the resonances relies entirely on an intensive vibro-impact interaction between the absorber and the primary structure. This may cause excessive dynamic responses and, therefore, damage to the sensitive internal components. As distinct from the model studied by Ormondroyd and Den Hartog, in the present paper the restriction of motion of a dynamic absorber takes place in its absolute motion by means of the stop (bumper) mounted upon the stationary base. Since the control of resonant responses relies now on the vibro-impact process, which takes place between the dynamic absorber and the stationary base, the primary system and its sensitive components may be significantly relieved from the impact dynamic loading.

- ¹ Ormondroyd J, Den Hartog JP, The theory of the dynamic vibration absorber, Transactions of ASME, Vol. 50, 1928.
- ² Timoshenko S, Vibration problems in engineering, D. Van Nostrand Company, NY, 1955.
- ³ Masri SF, Forced vibration of a class of nonlinear two-degree-of freedom oscillators, International Journal of Non-Linear Mechanics, Vol 7, 1972.
- ⁴ Gonsalves DH, Neilson RD, Barr ADS, The dynamics and design of a non-linear vibration absorber, Journal of Mechanical Engineering Science, Vol. 207, 1993.

Nonlinear correction of a vibration protection system containing tuned dynamic absorber¹

Appendix IV

The material of this appendix follows reference [239] where the practical problem of vibration engineering was solved using the method of PGF.

The system of vibration protection containing the tuned dynamic absorber is analysed. To control the system's extraneous resonant responses without affecting the ability of essential linear vibration suppression at antiresonant frequency, the absolute motion of the absorber is limited by the stops mounted upon the base.

The analytical solution relies on the theory of momentary impact and the technique of PGF and is obtained in the explicit closed form. The frequency responses of the vibration protection system under harmonic excitation with variable frequency are obtained in terms of impact impulses, magnitudes of fundamental harmonics of motion for the primary and secondary systems and also the forces transmitted to the base.

The results of calculations are in good agreement with a numerical simulation, which utilises the realistic model of visco-elastic collision.

By combining the results obtained using the PGF method and the theory of non-momentary viscoelastic impact, it is possible to estimate the values of peak accelerations and contact forces which are developed at impact and also the values of the peak deformations of the bumpers. This allows for the important information for engineering design to be obtained. Based on these results, some general concepts of practical design of such a vibration protection system are discussed. In particular, the influences of restitution ratio and clearance value are addressed.

Introduction

The method of dynamic absorption, in general, relies on the attachment of a linear lightly damped SDOF system to the harmonically excited primary structure.

It is the simplest case when both the primary and the secondary sub-structures are lightly damped SDOF systems. The frequency response function of such a combined TDof system typically displays two sharp resonant peaks and one deep antiresonant notch which is located exactly where the partial natural frequency of the secondary sub-structure is. The appearance of such a notch is used for the essential suppression of vibration of the primary system caused by a harmonic force with constant frequency.

However, the presence of a modified primary resonance and the occurrence of an additional resonance (typically very close to the disposition of the antiresonant

¹ Reprinted from Journal of Sound and Vibration: AM Veprik, VI Babitsky, Nonlinear correction of a vibration protection system containing tuned dynamic absorber, (2001) 239(2), 335-356. By permission of Academic Press, © 2001.

The authors utilised the theory of momentary impact and the technique of periodic Green functions to study the dynamics of the non-linear vibration protection system under harmonic excitation with variable frequency [17]. The obtained exact analytical expressions describe the time histories of the motions of the primary and secondary systems and also the forces transmitted to the base, and contain a full set of harmonics. The fundamental harmonics are then extracted from the exact solutions and used for representation of correspondent frequency responses.

The results of calculations are in good agreement with a numerical simulation, which relies on the realistic model of visco-elastic collision.

Some general concepts of practical design of such a vibration protection system are discussed. In particular, the influences of restitution ratio and clearance value are addressed. Further, it is shown how the results of the analysis, which is based on the theory of the momentary impact and the PGF method, may be combined with the theory of non-momentary visco-elastic impact. This allows for estimation of the values of peak impact forces and accelerations and also peak deflections of the bumpers.

Dynamic model and analytical solution of vibro-impact problem

Fig. AIV 1 shows the dynamic model of the system.

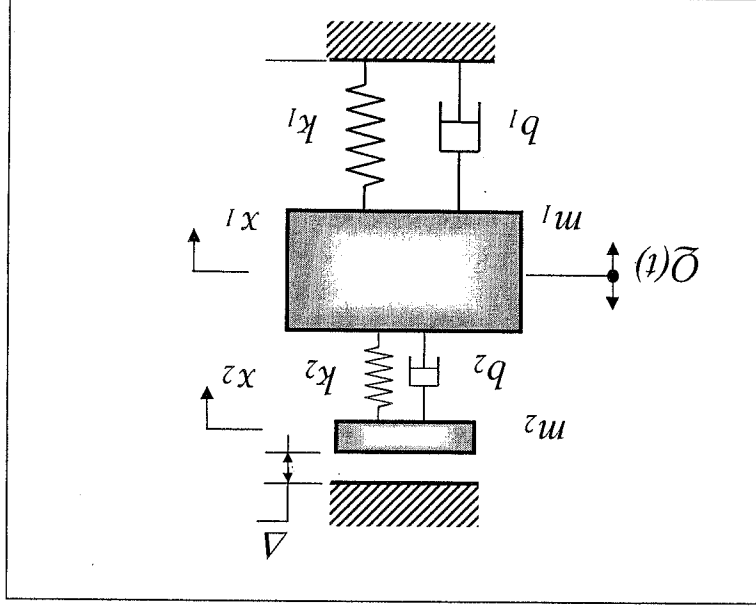


Fig. AIV 1

where

$$H(t) = \Phi(x_2, \dot{x}_2) = \sum_{m=-\infty}^{\infty} H_m \exp(im\omega t), \quad t \in [-\infty, \infty] \quad (\text{AIV 2})$$

in the form of the Fourier series

$$H_m = \frac{1}{T} \int_0^T H(t) \exp(-im\omega t) dt \quad (\text{AIV 3})$$

⁶ Ormondroyd and Den Hartog ignored absolutely this fact. To control these resonances the sufficient amount of energy must be dissipated at collisions against the bumpers.

The primary system of mass m_1 is suspended from the base by a visco-elastic flexural element which is represented by a parallel combination of spring, k_1 , and dashpot, b_1 . The absorber of the mass m_2 is attached to the primary system by means of the visco-elastic flexural element which is shown as a parallel combination of the spring, k_2 , and the dashpot, b_2 . The primary system is subjected to the action of the external force $\bar{Q}(t) = q \sin \omega t$ where q and ω are the amplitude and angular frequency, respectively. As a result of the excitation, both masses are involved in the absolute motions $x_1(t)$ and $x_2(t)$, as shown in Fig. AIV 1. The absolute motion of the secondary mass is limited by a solid stationary stop positioned with a clearance, Δ , with respect to the static equilibrium of the absorber mass.

The equations of motion which account for the collision of the secondary sub-system with the stop take the form [17]

$$m_1 \ddot{x}_1 + b_1 \dot{x}_1 + b_2 (\dot{x}_1 - \dot{x}_2) + k_1 x_1 + k_2 (x_1 - x_2) = Q(t) \\ m_2 \ddot{x}_2 + b_2 (\dot{x}_2 - \dot{x}_1) + k_2 (x_2 - x_1) + \Phi(x_2, \dot{x}_2) = 0 \quad (\text{AIV 1})$$

where the force of impact interaction $\Phi(x_2, \dot{x}_2)$ is a function of the absolute coordinate and velocity of the absorber.

Due to the special choice of the clearance and small damping which is used in the absorber design, the system vibrates linearly (without collisions) for almost the entire frequency span. The only exceptions are in the vicinities of linear resonant frequencies; the phenomenon of nonlinear (vibro-impact) resonance takes place to the right of these frequencies [17]. As a result of periodic excitation in the above frequency bands, a steady-state nonlinear resonant regime with a single collision per period arises⁶.

The method of PGF is a convenient tool, which is well suited for the analysis of such resonant regimes [17]. The force of impact interaction in this regime may be thought of as the periodic function of time $H(t)$ and, generally speaking, may be represented

and $T = \frac{\omega}{2\pi}$ is the period of the process.

By letting $\Phi(x_2, \dot{x}_2) = 0$, the complex dynamic compliances (receptances) of the linear system are obtained from (AIV 1) in the form

$$(AIV 4) \quad L_{11}(i\omega) = \frac{D(j\omega)}{(-m_1\omega^2 + ib_1\omega + K_1)}, \quad L_{12}(i\omega) = \frac{D(j\omega)}{(ib_2\omega + K_2)}, \quad L_{22}(i\omega) = \frac{D(i\omega)}{[-m_1\omega^2 + i(b_1 + b_2)\omega + K_1 + K_2]}$$

where $i = \sqrt{-1}$ and

$$(AIV 5) \quad D(i\omega) = [-m_1\omega^2 + i(b_1 + b_2)\omega + K_1 + K_2] - (ib_2\omega + K_2)^2$$

For the periodic single-impact process (single impact over the period T), we find by superposition

$$(AIV 6) \quad \begin{aligned} x_1(t) &= q \left[L_{11}(i\omega) \cos[\omega t + \psi_{11}(\omega) + \phi] - \sum_{m=-\infty}^{\infty} L_{12}(im\omega) N^m \exp(im\omega t) \right] \\ x_2(t) &= q \left[L_{12}(i\omega) \cos[\omega t + \psi_{12}(\omega) + \phi] - \sum_{m=-\infty}^{\infty} L_{22}(im\omega) N^m \exp(im\omega t) \right] \end{aligned} \quad t \in [-\infty, \infty]$$

The "first terms" on the right-hand side of both equations (AIV 6) represent the steady-state solution to a linear problem - impactless motion, where $|L_{11}(i\omega)|$ and $|L_{12}(i\omega)|$ are the moduli of dynamic compliance at the frequency ω , $\psi_{11}(\omega)$, $\psi_{12}(\omega)$ are the arguments, ϕ is the phase of impact relative to the external force.

The summation terms in both equations (AIV 6) represent the "vibro-impact portion" of the solution which is the steady-state response of the system to the periodic sequence of impact impulses.

Substitution of (AIV 3) into the second addenda of both equations (AIV 6), and changing the sequence of summation and integration yields the non-linear integro-differential equations in the form

$$\begin{aligned} x_1(t) &= q \left[L_{11}(i\omega) \cos[\omega t + \psi_{11}(\omega) + \phi] - \int_{-T}^0 \chi_{12}(t-t) \Phi[x_2(t), \dot{x}_2(t)] dt \right] \\ x_2(t) &= q \left[L_{12}(i\omega) \cos[\omega t + \psi_{12}(\omega) + \phi] - \int_{-T}^0 \chi_{22}(t-t) \Phi[x_2(t), \dot{x}_2(t)] dt \right] \end{aligned}$$

where the PGFs $\chi_{12}(t)$, $\chi_{22}(t)$ are defined as [5]

$$(AIV 7) \quad \chi_{12}(t) = \frac{1}{T} \sum_{m=-\infty}^{\infty} L_{12}(im\omega) \exp(im\omega t), \quad \chi_{22}(t) = \frac{1}{T} \sum_{m=-\infty}^{\infty} L_{22}(im\omega) \exp(im\omega t) \quad t \in [-\infty, \infty]$$

The duration of impact is typically very small as compared with the period T . Therefore, with the use of the mean-value theorem, we obtain the two-parametric representation of the desired solution in terms of the unknown impulse of impact force J and phase ϕ ,

$$(AIV 8) \quad \begin{aligned} x_1(t) &= q \left[L_{11}(j\omega) \cos[\omega t + \psi_{11}(\omega) + \phi] - J \chi_{12}(t) \right] \\ x_2(t) &= q \left[L_{12}(j\omega) \cos[\omega t + \psi_{12}(\omega) + \phi] - J \chi_{22}(t) \right] \end{aligned} \quad \text{where } t \in [-\infty, \infty] \text{ and also}$$

$$J = \int_{-T}^0 \Phi[x_2(\tau), \dot{x}_2(\tau)] d\tau$$

For the momentary impact [17], the impulse of impact force may be expressed through the restitution ratio, R , and the pre-impact velocity, $\dot{x}_2(-0)$,

$$(AIV 9) \quad J = m_2(1 + R)\dot{x}_2(-0)$$

Series (AIV 7) is defined in the infinite time interval. To simplify the solution, it may be summed and corresponding PGFs may be found in an explicit form for the finite time interval $[0, T]$ of periodicity [17].

The complex dynamic compliance, $L(i\omega)$, of a lightly damped system with different and well distanced natural frequencies may be approximated by the eigenform decomposition

$$(AIV 10) \quad L(j\omega) \approx \sum_{i=1}^N \frac{A_i \Omega_i^2}{\Omega_i^2 - \omega^2 + 2j\zeta_i \Omega_i \omega}$$

where n is the number of relevant degrees of freedom ($N = 2$, in this case), Ω_i , ζ_i , A_i are undamped natural frequencies, loss factors and form factors, respectively. From (5), by letting $b_{1,2} = 0$, the natural undamped frequencies $\Omega_{1,2}$ are found as the roots of the equation $D(i\omega) = 0$,

$$(AIV 11) \quad \Omega_{1,2} = \sqrt{\frac{2m_1m_2}{(k_1m_1 + k_1m_2 + k_2m_1) \pm \sqrt{(k_1m_1 + k_1m_2 + k_2m_1)^2 - 4k_1k_2m_1m_2}}}$$

The loss factors and form factors can not be estimated explicitly for the damped system. In this appendix these parameters are obtained approximately from a curve-fitting procedure with expressions (AIV 4) and (AIV 10) involved.

The complex dynamic compliance in the form (AIV10) yields the closed form representation of the PGF in the interval of periodicity [17],

$$\chi(t) = \sum_{i=1}^N \frac{A_i e^{-\xi_i \Omega_i t} \left[\sin \Theta_i t + e^{-\xi_i \Omega_i T} \sin \Theta_i (T-t) \right]}{\Theta_i \left(1 + e^{-2\xi_i \Omega_i T} - 2e^{-\xi_i \Omega_i T} \cos \Theta_i T \right)}, \quad 0 \leq t \leq T \quad (\text{AIV12})$$

where the damped natural frequencies are $\Theta_i = \Omega_i \sqrt{1 - \xi_i^2}$, $i = 1, 2$.

The unknown phase and impulse may be obtained from the impact conditions [17],

$$x_2^*(0) = \Delta, \quad J = m_2(1+R)x_2^*(-0), \quad J \geq 0 \quad (\text{AIV13})$$

Substitution of (AIV8) into (AIV13) yields

$$J = (1+R)m_2 \left[-\omega q |L_{12}(j\omega)| \sin[\phi + \psi_{12}(\omega)] + J\chi_{22}(-0) \right] \\ q |L_{12}(j\omega)| \cos[\phi + \psi_{12}(\omega)] - J\chi_{22}(0) = \Delta$$

where for lightly damped systems [17]

$$\chi_{22}(-0) \approx \frac{1}{2m_2}$$

In accordance with (AIV12) we find

$$\chi_{22}(0) = \sum_{i=1}^N \frac{A_i e^{-\xi_i \Omega_i T} \sin \Theta_i T}{\Theta_i \left(1 + e^{-2\xi_i \Omega_i T} - 2e^{-\xi_i \Omega_i T} \cos \Theta_i T \right)}, \quad 0 \leq t \leq T \quad (\text{AIV14})$$

where the natural undamped frequencies $\Omega_{1,2}$ are derived from (AIV11) and modal parameters A_i, ξ_i are obtained from eigenform decomposition of complex compliance $L_{22}(j\omega)$ (see (AIV4)) in the form (AIV10).

From the impact conditions, we have two equations in the unknown phase $\phi = \phi + \psi_{12}(\omega)$ and impulse J ,

$$\sin \phi = -\frac{J(1-R)}{2(1+R)m_2 \omega q |L_{12}(j\omega)|}, \quad \cos \phi = \frac{\Delta + J\chi_{22}(0)}{q |L_{12}(j\omega)|} \quad (\text{AIV15})$$

Eliminating the phase ϕ , we find for the impact impulse

$$J(\omega) = \frac{-\Delta \chi_{22}(0) \pm \sqrt{\Delta^2 \chi_{22}^2(0) - \left[\frac{2(1+R)m_2 \omega}{(1-R)} + \chi_{22}^2(0) \right] \left[\frac{2(1+R)m_2 \omega}{(1-R)} + \chi_{22}^2(0) \right]}}{\left[\frac{2(1+R)m_2 \omega}{(1-R)} + \chi_{22}^2(0) \right]} \quad (\text{AIV16})$$

Analysis of fundamental harmonics and impact impulses

The substitution of the estimated values for the impulse J and phase ϕ back into equations (AIV8) yields the closed form explicit solution describing the periodic forced vibration of the TDOF system with collisions under the harmonic excitation of arbitrary frequency.

The stability analysis which was conducted in [17] indicates that a stable regime corresponds to the positive sign in (AIV16). From [17], the obvious vibroimpact regime existence conditions are

$$|\sin \phi| \leq 1, \quad J \geq 0$$

- the force transmitted to the base,
- accelerations and deflections of the primary and secondary systems,

Since we assumed the model of momentary impact, it is impossible to estimate the peak values of impact forces and accelerations (these take infinite values at the moment of collision).

As the systems under consideration are well pronounced low-pass filters, the first harmonics give good representation of the entire process [17]. Therefore, we can perform an analysis of the corresponding frequency responses in terms of the fundamental harmonics, which are extracted from the above exact solutions.

Numerical simulations validate this approach. Since the values of phase and impulse have already been obtained from equations (AIV15), (AIV16) we can calculate readily the fundamental harmonics $X_1(t)$ and $X_2(t)$ of the corresponding processes (AIV8)

$$X_1(t) = q |L_{11}(j\omega)| \cos[\omega t + \psi_{11}(\omega) + \phi] - \frac{J}{2L} |L_{12}(j\omega)| \cos[\omega t + \psi_{12}(\omega)] \quad (\text{AIV17}) \\ X_2(t) = q |L_{12}(j\omega)| \cos[\omega t + \psi_{12}(\omega) + \phi] - \frac{J}{2L} |L_{22}(j\omega)| \cos[\omega t + \psi_{22}(\omega)] \quad (\text{AIV18})$$

The magnitudes of the fundamental harmonics $|X_1(t)|$ and $|X_2(t)|$ may be calculated readily from (AIV17) and (AIV18).

The force which is transmitted to the base may be represented in the form

$$f(t) = f_s(t) + f_{im}(t)$$

where

$$f_s(t) = k_1 x_1(t) + b_1 \dot{x}_1(t) \quad (\text{AIV19})$$

is the force transmitted through the flexural suspension and

$$f_{im}(t) = J \delta^T(t) \quad (\text{AIV20})$$

is the force transmitted through impacts.

The fundamental harmonic of the force transmitted through flexural suspension is calculated using (AIV17) and (AIV19),

$$F_s(t) = k_1 X_1(t) + b_1 \dot{X}_1(t) \quad (\text{AIV21})$$

The fundamental harmonic of the force transmitted through impact is obtained using (AIV20) is

$$F_m(t) = \frac{T}{2J} \cos \omega t \quad (\text{AIV22})$$

Using (AIV17), (AIV18), (AIV20) and (AIV22), the fundamental harmonic of the resultant force is

$$F(t) = k_1 \left\{ q |L_{11}(i\omega)| \cos[\omega t + \psi_{11}(\omega) + \phi] - \frac{T}{2J} |L_{12}(i\omega)| \cos[\omega t + \psi_{12}(\omega)] \right\} + b_1 \left\{ -q\omega |L_{11}(i\omega)| \sin[\omega t + \psi_{11}(\omega) + \phi] + \frac{T}{2J} \omega |L_{12}(i\omega)| \sin[\omega t + \psi_{12}(\omega)] \right\} + \frac{T}{2J} \cos \omega t \quad (\text{AIV23})$$

From (AIV23), the magnitude of the fundamental harmonic of the force transmitted to the base $|F(t)|$ may be estimated readily.

Numerical example and discussion

The parameters for the numerical example were taken to be

$$m_1 = 1 \text{ kg}; m_2 = 0.1 \text{ kg}; k_1 = 25000 \frac{\text{N}}{\text{m}}; k_2 = 4300 \frac{\text{N}}{\text{m}}; b_1 = 0.8 \frac{\text{kg}}{\text{s}}; b_2 = 0.08 \frac{\text{kg}}{\text{s}}; q = 10 \text{ N}; \Delta = 3 \text{ mm}$$

First, we consider as a reference the linear case as $\Delta \rightarrow \infty$.

Fig. AIV 2 shows the superimposed magnitudes of the fundamental harmonics of displacement of the primary and secondary mass against frequency. It is seen that linear resonances take place at the frequencies 23Hz and 36Hz, where the magnitudes attain large values. The antiresonant frequency of the primary mass is 33Hz, where the dynamic response becomes negligible.

Fig. AIV 3 shows the magnitude of the force transmitted to the base through the suspension of the primary system (impacts are absent in this case) against frequency. It is seen that in the vicinities of linear resonances of 23Hz and 36Hz essential force transmission takes place, and in the vicinity of linear antiresonance (33Hz) the force transmission becomes negligible.

It is important to remind to reader that it is desirable to control the resonant responses of the system and to keep the linear anti-resonance response unaffected. From Fig. AIV2, the response of the secondary system at anti-resonance is 2.27mm. The choice

of the clearance $\Delta = 3\text{mm}$ (see Fig. AIV3) allows the impactless motion of the secondary system in a sufficient frequency band, which is 27-34Hz.

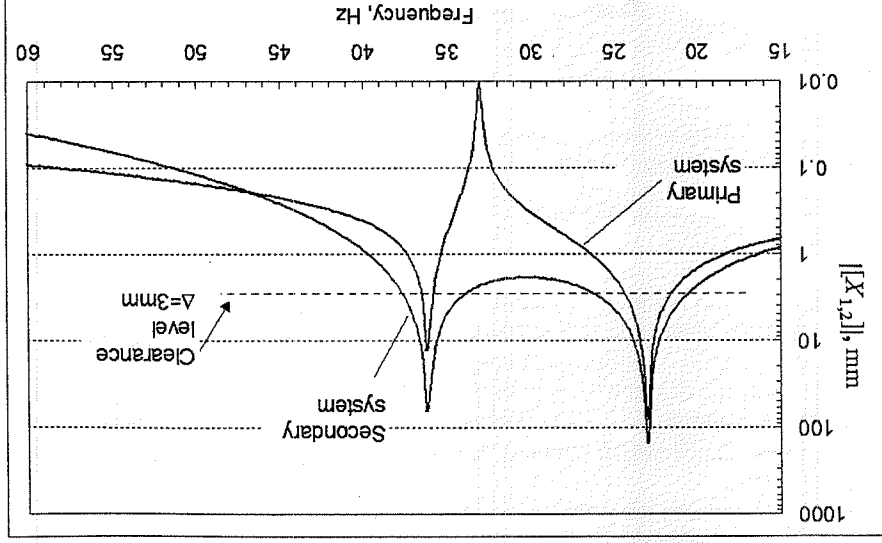


Fig. AIV 2

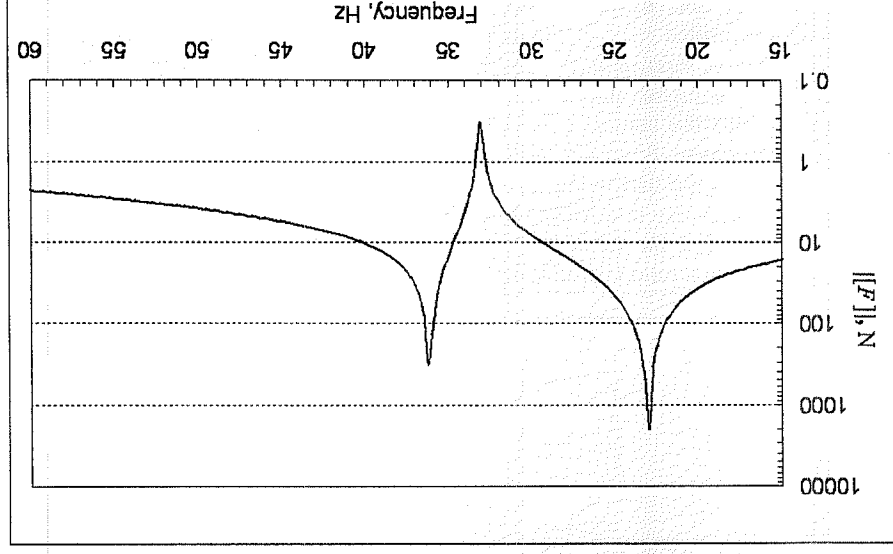


Fig. AIV 3

The modal parameters Ω_1 , ξ_1 , A_1 are required now for calculation of $X_{22}(0)$ in accordance with expression (AIV14). The undamped natural frequencies $\Omega_{1,2}$ are obtained for numerical values (AIV24) from equation (AIV11),

$$\frac{\Omega_1^2}{2\pi} = 23\text{Hz}; \frac{\Omega_2^2}{2\pi} = 36\text{Hz} \quad (\text{AIV25})$$

The complex compliance $L_{22}(i\omega)$ from (AIV4) was approximated by eigenform decomposition (AIV10). The following modal parameters were estimated from curvfitting with the numerical values (AIV24) used,

$$A_1 = 2.75; A_2 = 7.25; \xi_1 = 0.0022; \xi_2 = 0.0023 \quad (\text{AIV26})$$

Fig. AIV4 shows the module of complex compliance $|L_{22}(i\omega)|$ which is calculated firstly in accordance with (AIV4) with numerical parameters (AIV24) (solid line) and secondly in accordance with (AIV10) with numerical values (AIV25) and (AIV26).

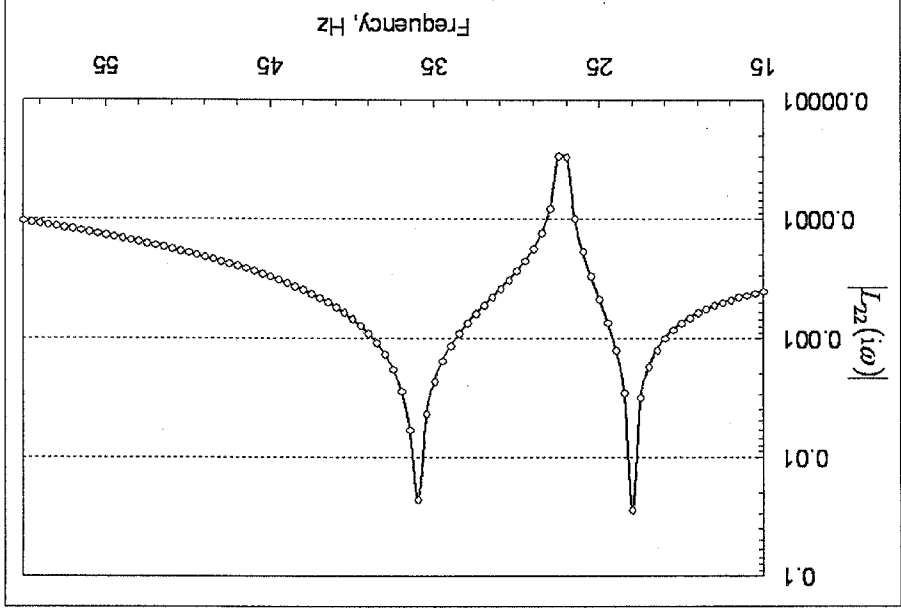


Fig. AIV4

Fig. AIV5,6 show the magnitudes of the fundamental harmonics (formulae (AIV17), (AIV18)) of the secondary and primary masses against excitation frequency at different restitution ratios R (see legends). The corresponding linear response of the system (labelled as LIN, see legend) is superimposed for reference. The dynamic responses of both secondary and primary systems show the distortion and essential

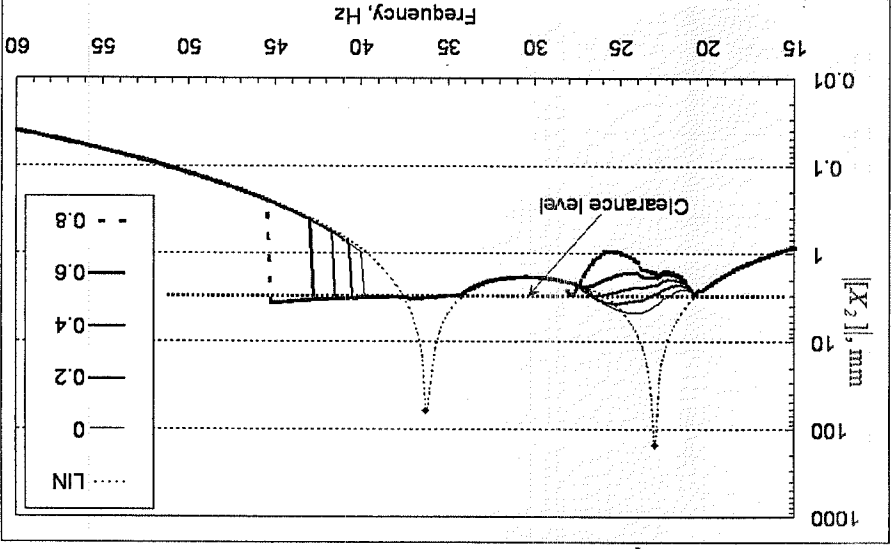


Fig. AIV 5

The latter is of particular importance for the primary system, since vibration control of the primary system is our main objective. For example, from Fig. AIV 7, for the restitution ratio $R = 0.2$ we obtain the suppression of the dynamic responses in the vicinity of the first and second resonances by factors of 28 and 23, respectively. At the same time, the portion of the frequency response in the frequency band 27-34Hz (see Fig. AIV 2,5,6) which contains the deep linear antiresonant notch which remains unaffected. Fig. AIV 7 shows the magnitude of the fundamental harmonic of the force transmitted to the base through the primary suspension and impact (formula (AIV21) was used), against excitation frequency at different restitution ratios R (see legend). The

suppression (as compared with the linear case) in the vicinities of the linear resonances. This is due to the violation of the conditions of linear resonance caused by the interaction of the secondary system with the obstacle. At the same time, the presence of the obstacle causes the development of nonlinear (vibro-impact) resonances pulled well to the right of the frequencies of linear resonance. At the frequencies of vibro-impact resonance, the magnitudes of the fundamental harmonic may be even higher than that in the linear case. An implementation of sufficient damping into the contact zone is, therefore, a must for the close control of vibro-impact resonance. From Fig. AIV 5,6, the decrease in the value of the restitution ratio, R , results in the narrowing of the frequency band of the existence of vibro-impact resonance and also in a reduction of the peak value of response. It is obvious from these figures that for the better correction of the frequency response, the lowest value of the restitution ratio is preferable.

corresponding graph representing the force transmission in the linear case (labelled as LIN, see legend) is superimposed for reference.

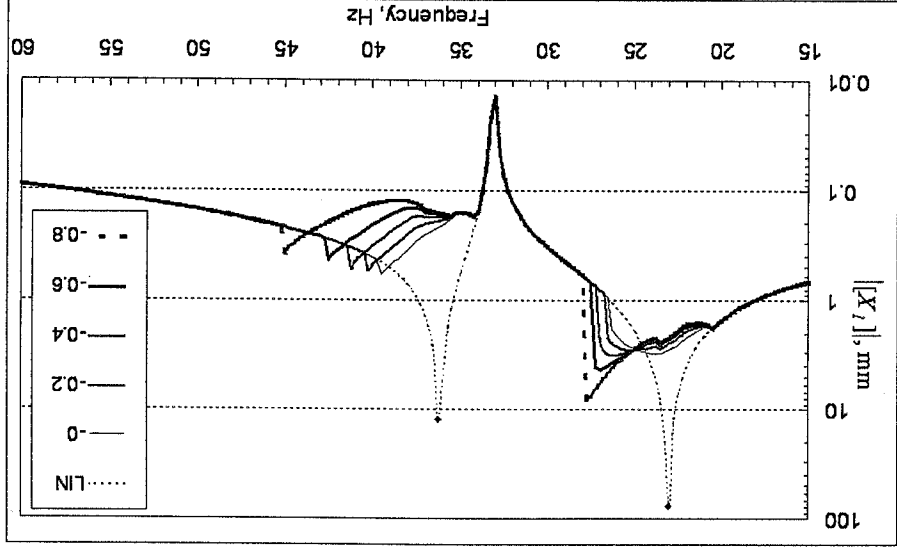


Fig. AIV 6

Fig. AIV 7 shows the magnitude of the fundamental harmonic of the force transmitted to the base through the primary suspension and impact (formula (AIV21) was used), against excitation frequency at different restitution ratios R (see legend). The corresponding graph representing the force transmission in the linear case (labelled as LIN, see legend) is superimposed for reference. From Fig. AIV 7, the essential control of the force transmitted to the base takes place in spite of the impacts. The force transmission at the first and second resonances was decreased in the case of $R = 0.2$ by factors of 25 and 30, respectively. It is of particular interest to analyse in Fig. AIV 8 the dependencies of the impact impulse against the frequency at different restitution ratios (see legend). This graph shows the frequency bands of existence of vibro-impact regimes, the character of frequency pulling and change in intensity of vibro-impact processes at different restitution ratios. The calculations were performed in accordance with formula (AIV16). The values of full impact impulses are used for the estimation of the peak accelerations.

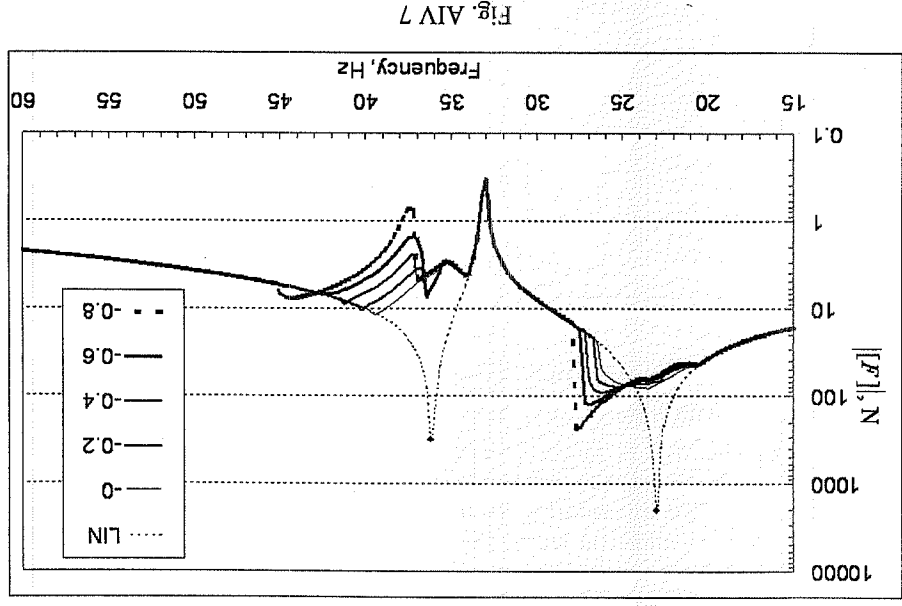


Fig. AIV 7

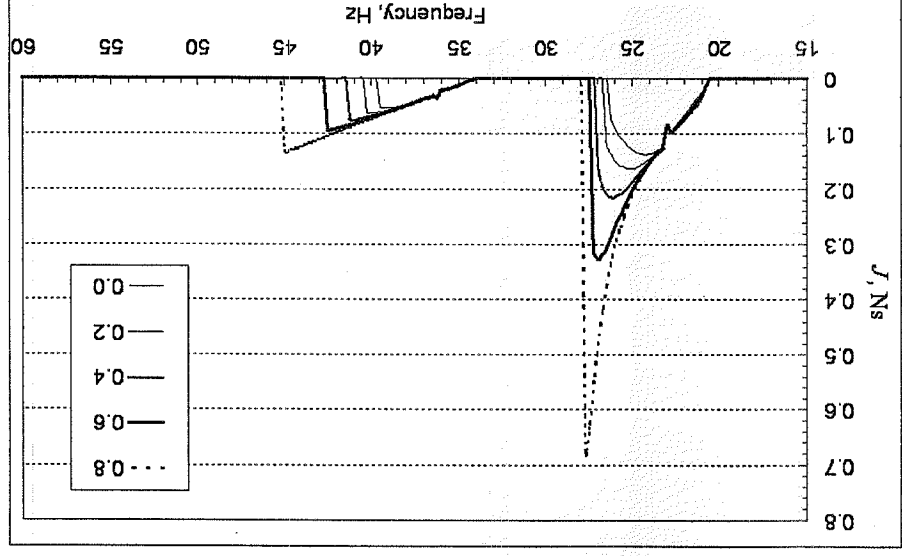


Fig. AIV 8

The present simulation is based on the realistic model of non-momentary visco-elastic collision suggested by the authors in [17, 240]. Such a model is required to produce a reference solution containing the peak values of impact forces and impulses. Fig. AIV 9 shows the dynamic model of the system. This model is identical to that shown in Fig. AIV 1 with the exception that the motion of the secondary mass is restricted by the compliant visco-elastic bumper, which is modelled as a parallel combination of the linear spring, k , and the dashpot, b .

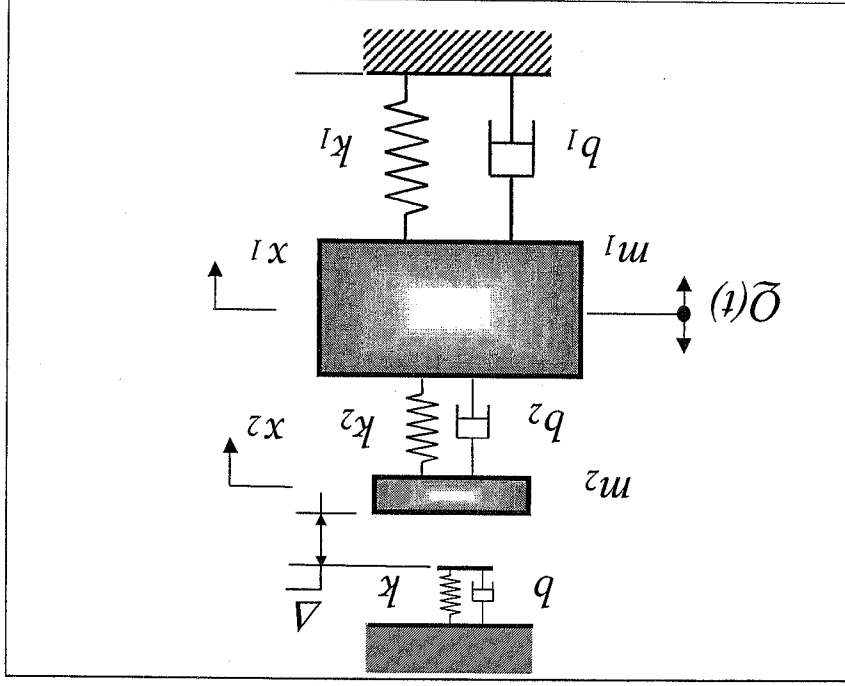


Fig. AIV 9

The equations of motion accounting for the collisions of the secondary system against the bumper take the form of equations (AIV 1), where, as above, the threshold force of impact interaction $\Phi(x_2, \dot{x}_2)$ is a function of coordinate and velocity and takes the form [17, 240]

$$\Phi(x_2, \dot{x}_2) = \begin{cases} k(x_2 - \Delta) + b\dot{x}_2 & \text{if } x_2 \geq \Delta \text{ and } \Phi(x_2, \dot{x}_2) > 0 \\ 0 & \text{if } x_2 \geq \Delta \text{ and } \Phi(x_2, \dot{x}_2) < 0 \\ 0 & \text{if } x_2 < \Delta \end{cases}$$

Simulink diagram

The system in Fig. AIV 9 is "subjected" to the direct swept sine test. Fig. AIV 10 portrays the corresponding Simulink diagram.

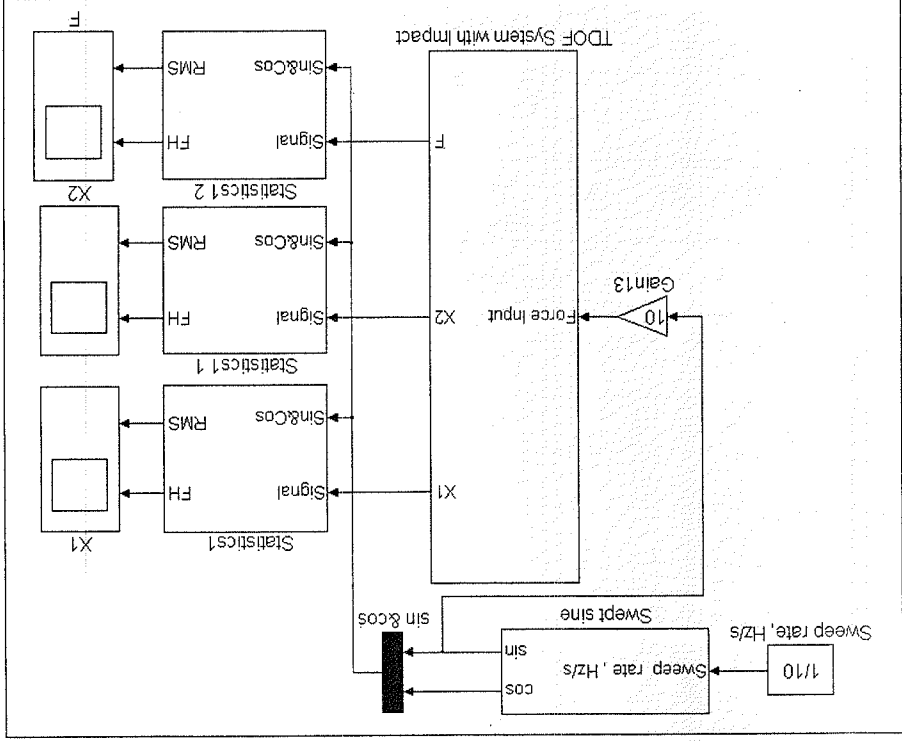


Fig. AIV 10

In Fig. AIV 10, the subsystem "Swept sine" produces two signals, these are $\sin \omega t$ and $\cos \omega t$ with linearly varying frequency $\omega = \omega(t) = 2\pi \alpha t$, where α is a sweep rate, Hz/s. Fig. AIV 11 shows the internal structure of the above subsystem. These signals are required for the current calculation of the fundamental harmonics of the processes. Simultaneously, one of these signals is used to evaluate the excitation signal $q \sin \omega t$, which is fed to the input of the subsystem "TDOF System with Impact".

"Impact" performing a calculation of the displacements of the primary and secondary system and of the force transmitted to the base (X1, X2, F, as shown in Fig. AIV 10).

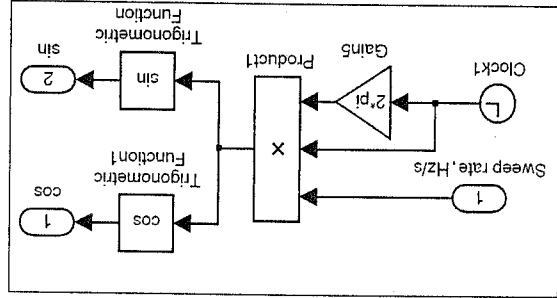


Fig. AIV 11

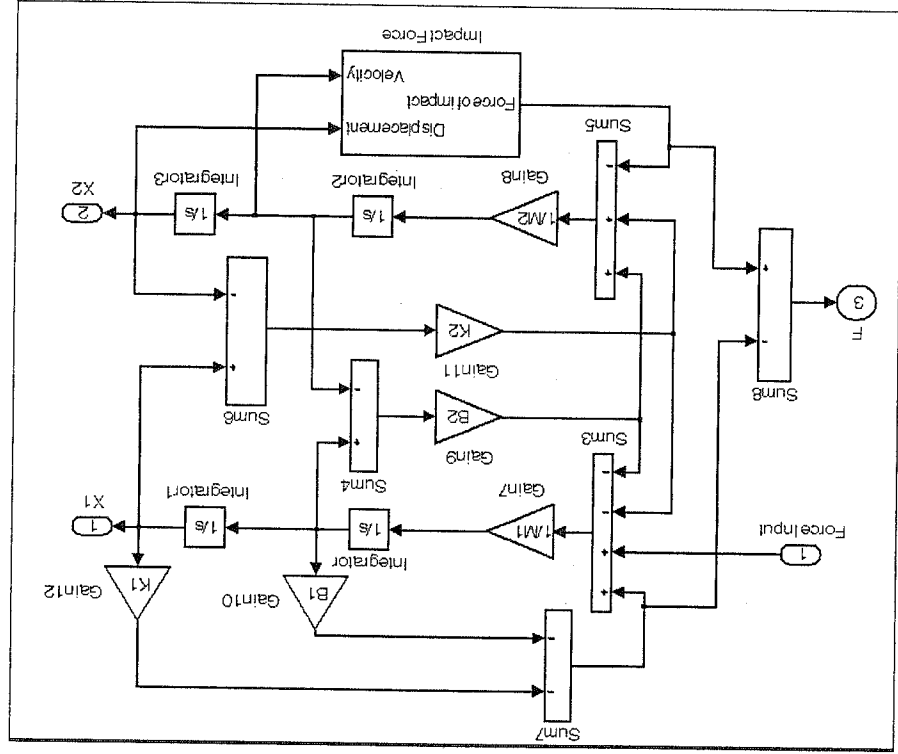


Fig AIV 12

Fig. AIV 12 shows the internal structure of the above subsystem. It contains the diagram for simulation of the linear part of the TDOF system and nonlinear block "Impact Force" (see Fig. AIV 13) which performs the transformation of the displacement and velocity of the secondary system into the impact force in accordance with formula (AIV 25). In this diagram, the lower limit in the "Dead Zone" block is set at $-\infty$. The block "Relational Operator" produces unity if displacement and impact force are of the same polarity and zero otherwise.

The obtained signals (displacements and force transmitted to the base) (see Fig. AIV 10), are then fed to the "Statistics" blocks which calculate the current value of the fundamental harmonic (FH) and root mean square (RMS) of the corresponding signals. The internal structure of these blocks is shown in Fig AIV 13, 14.

By using this diagram, we can simulate the dynamic response of the mechanical system in terms of fundamental harmonics and root mean square levels. This gives us the opportunity to make a general assessment of the accuracy of theoretical analysis which relies on the method of PGFs.

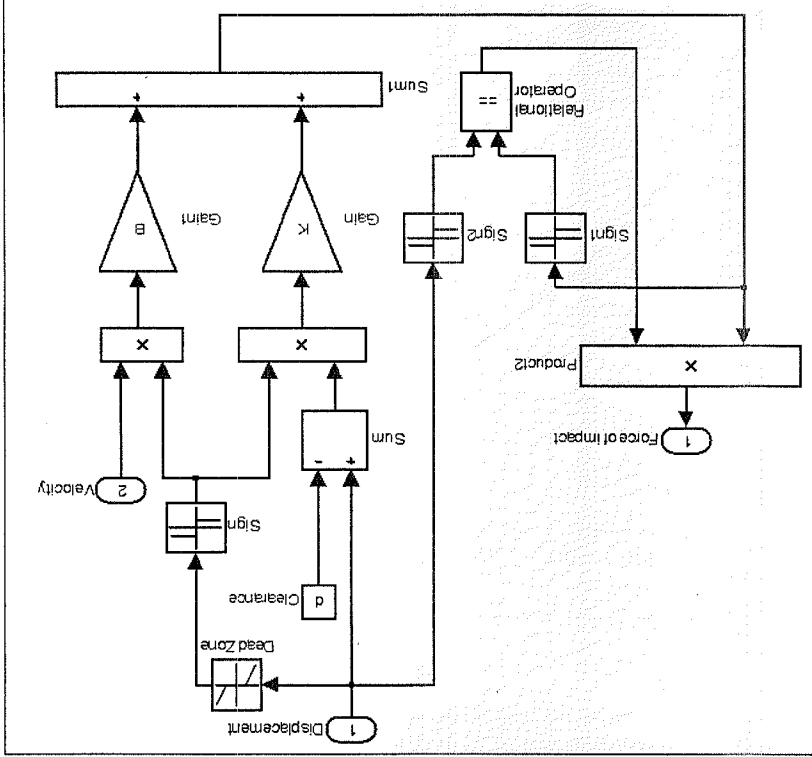


Fig. AIV 13

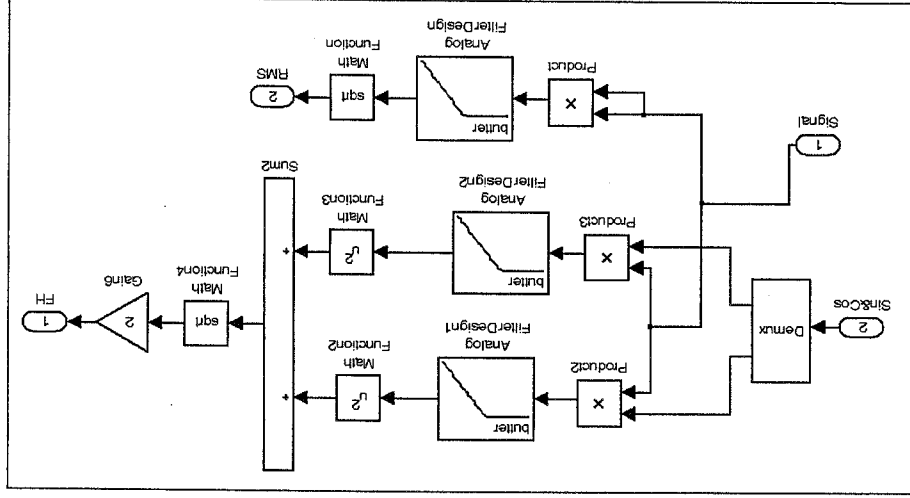


Fig. AIV 14

Results of simulation and discussion

We will perform the comparison of the analytical solution obtained using the PGF method in the case of the restitution ratio $R = 0.2$. For this purpose we consider, as above, the mechanical system with parameters (AIV 24).

It is convenient [17,240] to express the elastic and damping properties of the bumper in terms of the apparent natural frequency $\Omega_b = \sqrt{\frac{k}{m_2}}$ and the apparent loss factor

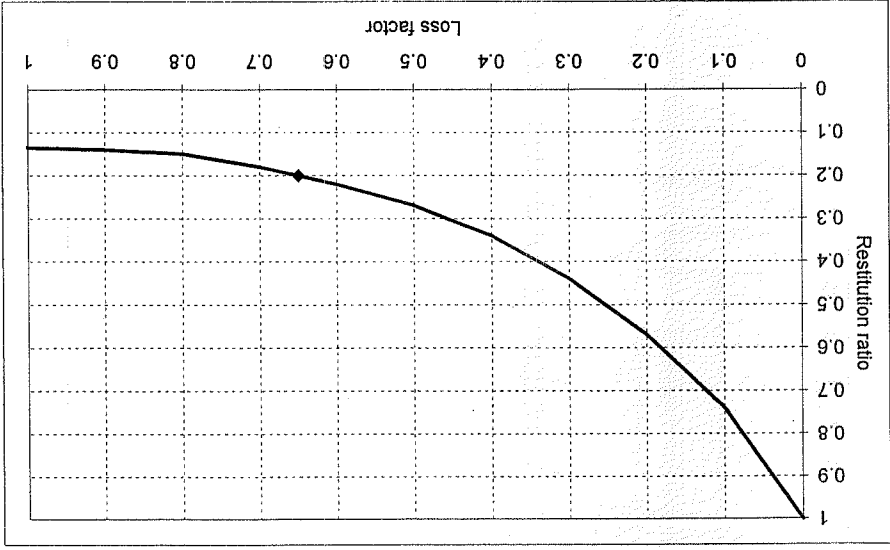
$$\xi_b = \frac{2m_2\Omega_b}{b}. \text{ It is important to note that the dynamic system with such parameters exists during the short time of impact only.}$$

In these notations the expression of the bumper stiffness and damping take the form $k = m_2\Omega_b^2$, $b = 2m_2\xi_b\Omega_b$.

From [17, 240], at visco-elastic collision the restitution ratio is a function of the loss factor only

$$R(\xi_b) = -\exp\left[-\frac{\xi_b}{\sqrt{1-\xi_b^2}} \operatorname{atan}\left(\frac{2\xi_b\sqrt{1-\xi_b^2}}{1-2\xi_b^2}\right) \sin\left[\operatorname{atan}\left(\frac{2\xi_b\sqrt{1-\xi_b^2}}{1-2\xi_b^2}\right)\right]\right] \quad (\text{AIV 26})$$

Fig. AIV 15



as Fig. AIV 15 shows. From Fig. AIV 15, the loss factor $\xi_b = 0.65$ corresponds to the restitution ratio $R = 0.2$ (this point is labelled as ♦). For the purpose of numerical simulation $\frac{\Omega_b}{2\pi} = 250\text{Hz}$. The sweep rate is taken as $\alpha = 0.1 \frac{\text{Hz}}{\text{s}}$.

Fig. AIV 16,17 show the superimposed frequency responses of the primary and of secondary systems in terms of overall RMS levels and RMS level of the fundamental harmonics. The entire process is well represented by the fundamental harmonic. This particularly holds true for the primary system. The influence of impacts resulting in a singularity which is represented in the motion of the secondary system is not seen in the motion of the primary system due to the filtering feature [17].

Fig. AIV 18,19,20 show the magnitudes of fundamental harmonics of the primary and secondary systems and the force transmitted to the base, and also compare these with the corresponding responses of the linear system (reference). These plots indicate the desired correction of the frequency responses in the vicinities of linear resonances. At the same time, the deep linear antiresonant notch remains unaffected.

Fig. AIV 21,22,23 compare the magnitudes of fundamental harmonics of the frequency responses of the primary and secondary systems and the force transmitted to the base which are obtained by means of the analytical method of PGF (labelled as (PGF)), with that obtained by numerical simulation (labelled as (simulation)). It is seen that the results obtained are in a good agreement.

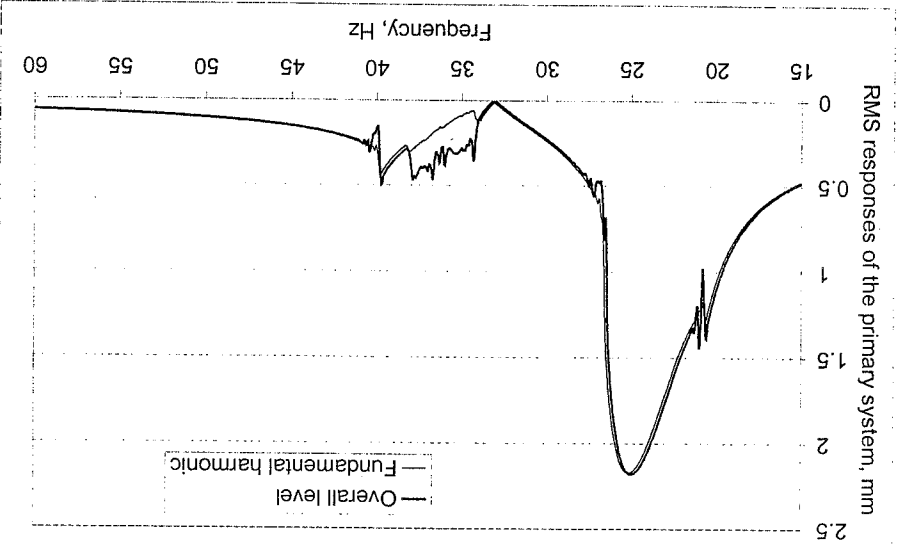


Fig. AIV 16

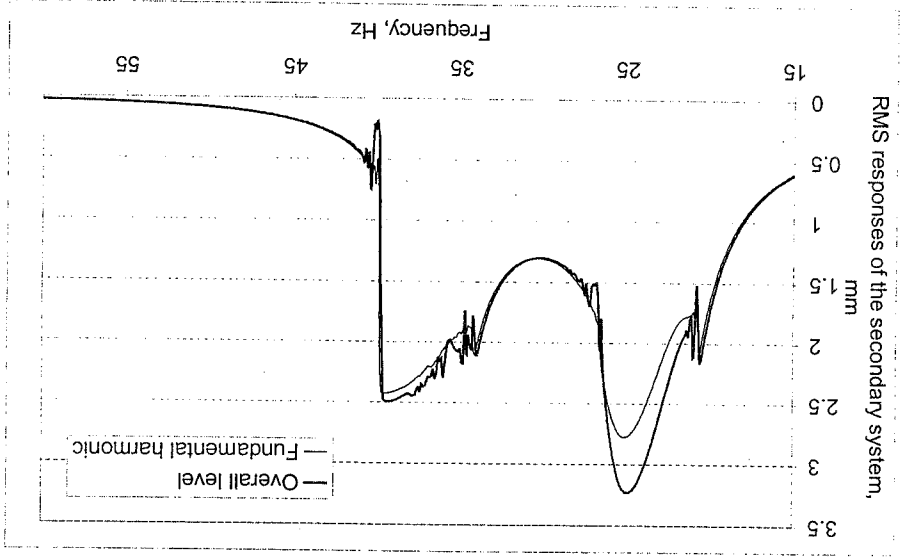


Fig. AIV 17

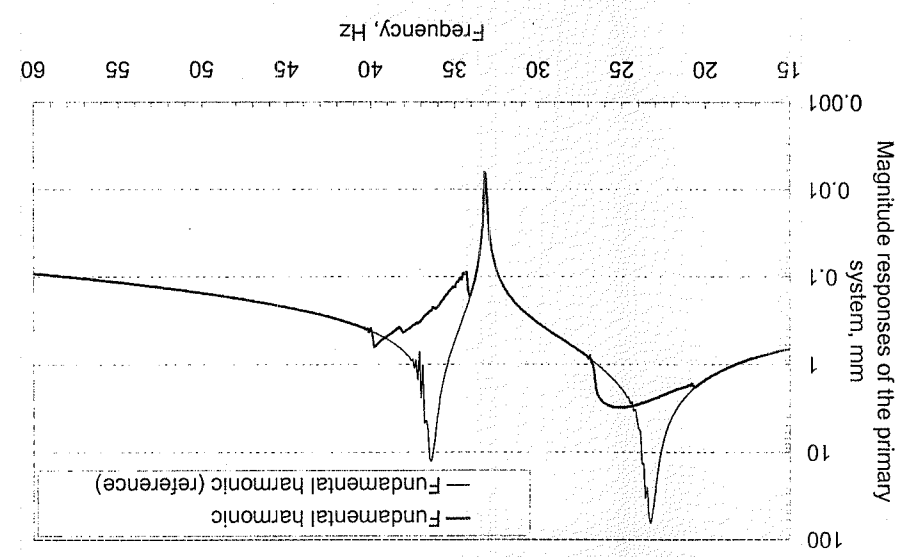


Fig. AIV 18

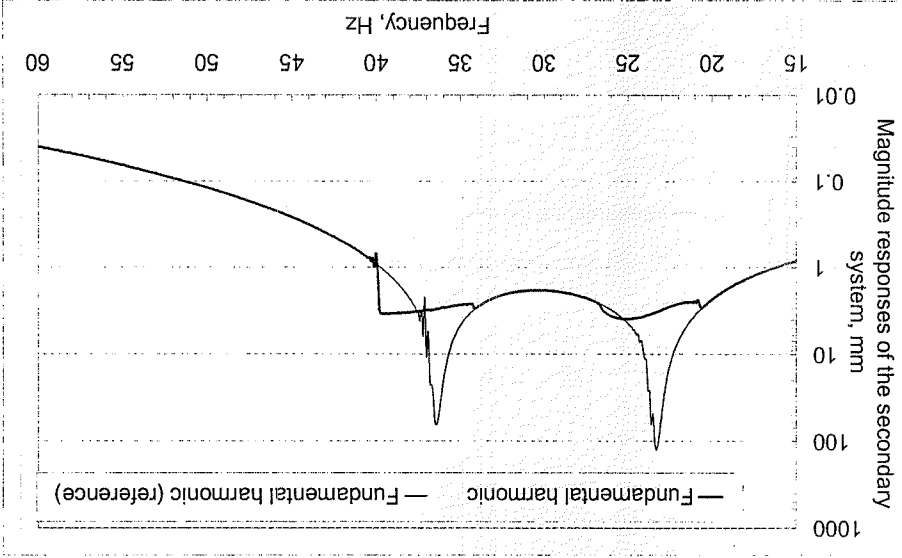


Fig. AIV 19

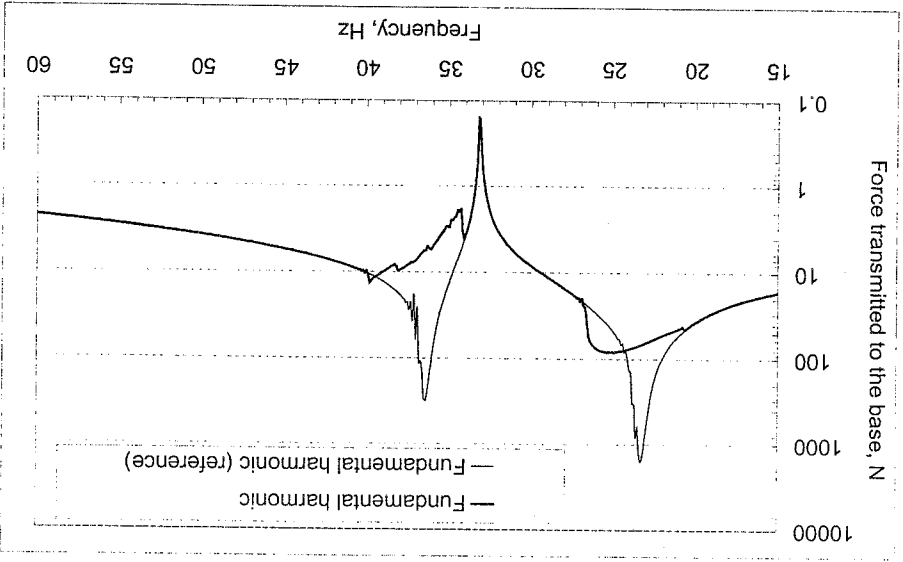


Fig. AIV 20

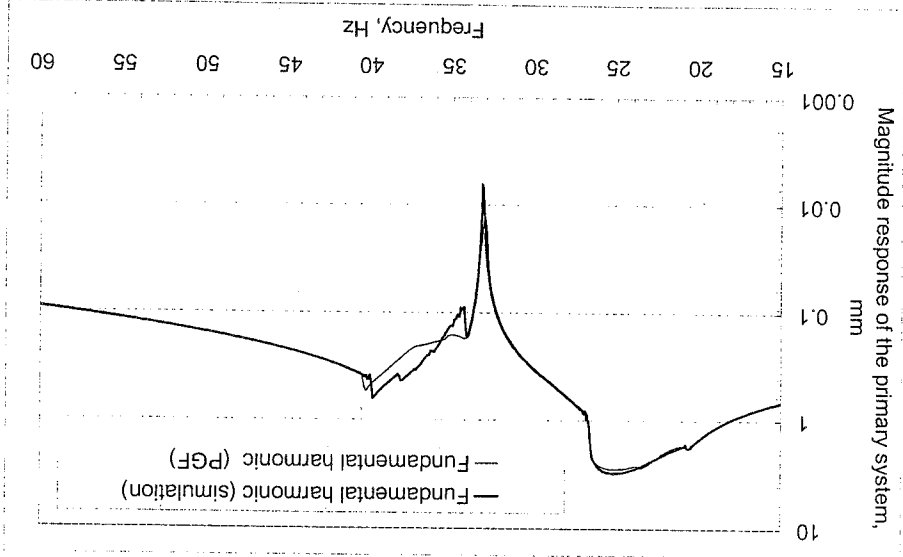


Fig. AIV 21

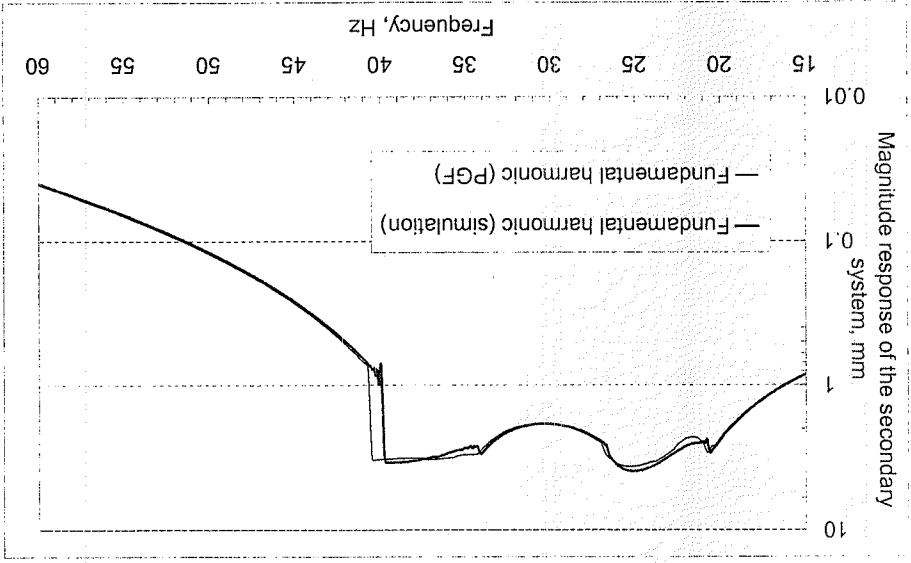


Fig. AIV 22

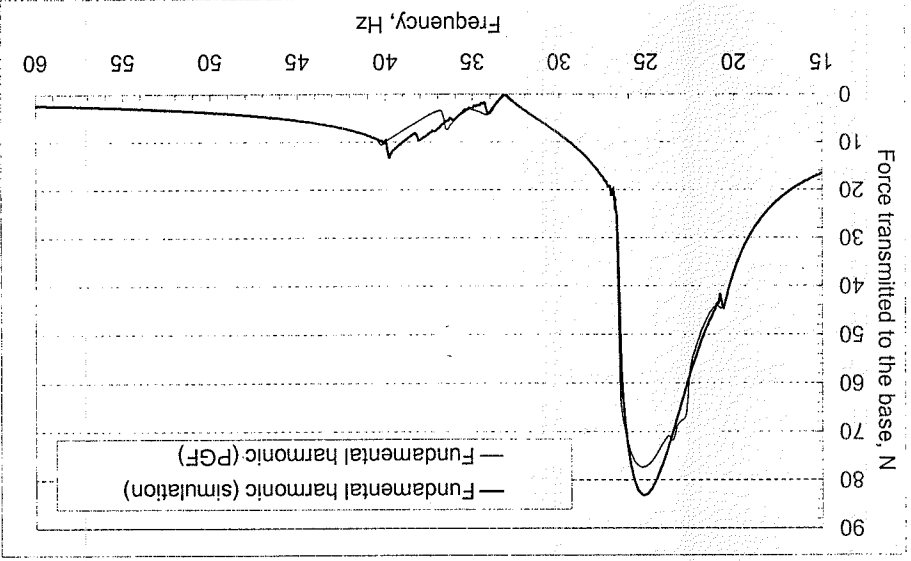


Fig. AIV 23

Fig. AIV 24 shows the comparison of the dependences of impact impulses against frequency obtained by the PGF method (labelled as PGF) with those obtained by

numerical simulation (labelled as SIMULATION). Once again, the results are in fair agreement.

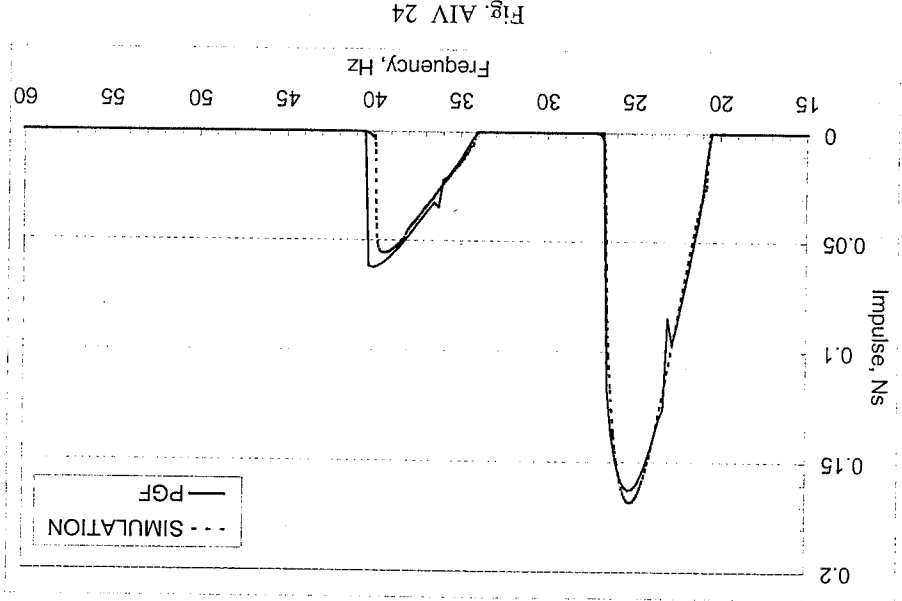


Fig. AIV 24

Design Concepts

Influence of the restitution ratio

Using the obtained analytical solution, we can estimate the influence of the restitution ratio, R , on the performance of the vibration protection system. Fig. AIV 25 shows the dependence of the peak magnitudes of the fundamental harmonic of deflection of the primary system on the restitution ratio at the constant clearance $\Delta = 3\text{mm}$. The analytical results (dashed curve, label PGF) are compared with the numerical simulation (solid curve, label SIMULATION). In the numerical simulation, the dependence $R(\xi_b)$ in Fig. AIV 14 was used for calculation of the bumper loss factor ξ_b . The apparent natural frequency of the bumper was taken as $\frac{\Omega_b}{2\pi} = 250\text{Hz}$. The results of numerical simulation and analytical calculation are in good agreement and indicate the usefulness of the low restitution ratio at impact for the quality of vibration protection systems. As seen from Fig. AIV 25, the further decrease of the restitution ratio below the value of $R = 0.2$ ($\xi_b = 0.65$) becomes ineffective.

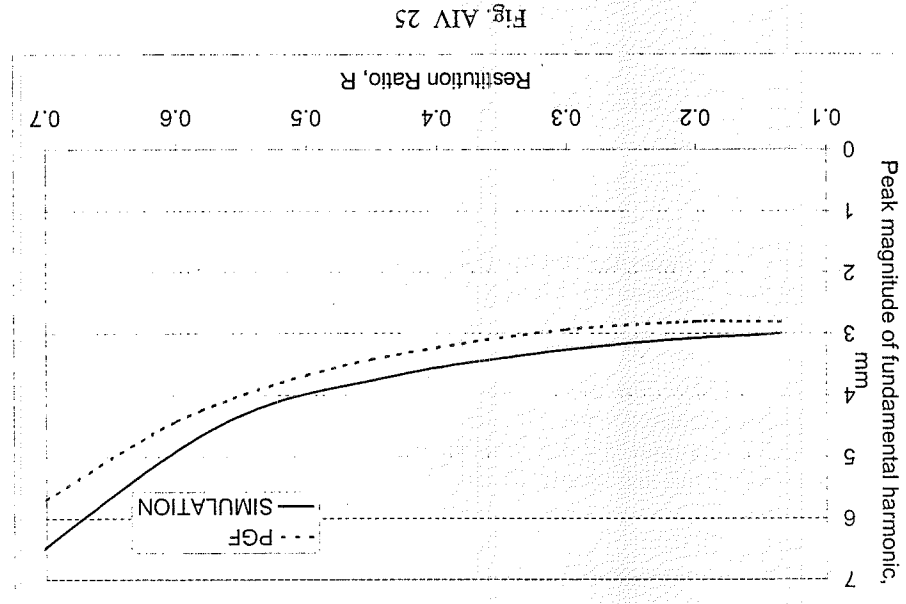


Fig. AIV 25

Influence of clearance

Following the previous analysis, it is useful to estimate the influence of the clearance Δ on the entire performance of the vibration protection system. Fig. AIV 26 shows the dependence of the peak magnitude of the fundamental harmonic against the clearance at the constant restitution ratio $R = 0.2$. The analytical results (dashed line, labelled PGF) are compared with the numerical simulation (solid line, labelled SIMULATION). The apparent natural frequency of the bumper was $\frac{\Omega_b}{2\pi} = 250\text{Hz}$. The results of numerical simulation and analytical results are in good agreement and indicate the usefulness of the low clearance for the quality of a vibration protection system. It is evident that the value of the clearance is limited from below by the value of the impactless amplitude of the secondary system at antiresonant frequency and the desirable width of the frequency span of impactless operation of the vibration protection system (see also Fig. AIV 2,4).

Calculation of peak accelerations

It was indicated above that consideration of momentary impact does not allow for the estimation of the peak values of impact accelerations. Nevertheless, for the practical design of the bumper these values are of significant importance.

The combination of the method of PGF with the theory of viscoelastic impact developed in [17,240] allows an approximate estimation of the values of impact peak accelerations.

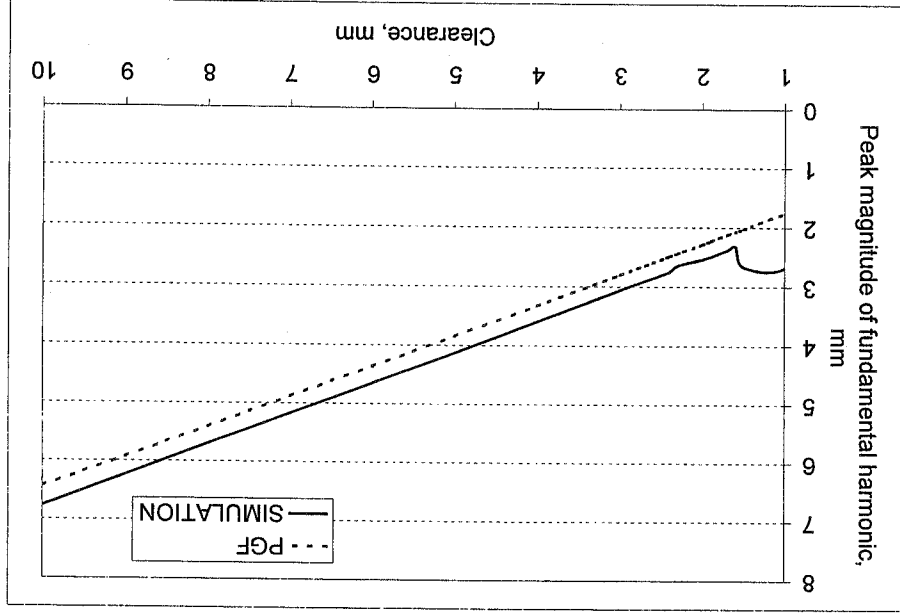


Fig. AIV 26

With reference to formula (AIV 9), the value of pre-impact velocity may be calculated as

$$\dot{x}_2(-0) = \frac{m_2(1+R)}{J}.$$

From [17,228], the peak value of acceleration at viscoelastic impact may be expressed in terms of the pre-impact velocity $\dot{x}_2(-0)$, apparent natural frequency of the bumper Ω_b and the function $\Gamma(\xi_b)$,

$$A_{peak} = \dot{x}_2(-0)\Omega_b\Gamma(\xi_b),$$

where

$$\Gamma(\xi_b) = \begin{cases} \gamma(\xi_b) & \text{if } \gamma(\xi_b) \geq 2\xi_b, \\ 2\xi_b & \text{if } \gamma(\xi_b) < 2\xi_b, \end{cases}$$

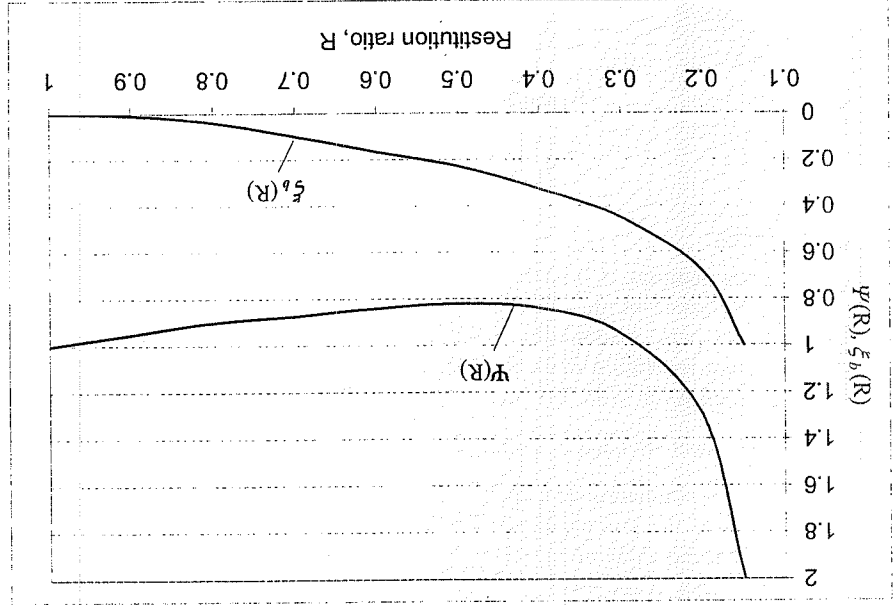
and

The agreement is evident.

$$\frac{\Omega_b}{2\pi} = 250\text{Hz}.$$

Fig. AIV 28 shows the results of direct numerical simulation of the peak values of acceleration against the excitation frequency for the system with the clearance $\Delta = 3\text{mm}$ and the bumper with $\frac{\Omega_b}{2\pi} = 250\text{Hz}$ and $\xi_b = 0.65$ (labelled SIMULATION). The peak values of accelerations against the excitation frequency were also calculated using the latter formula for the system with the same clearance and restitution ratio $R = 0.2$ (labelled PGF). The values of the impact impulses J were calculated by means of the PGF method. It was considered again that

Fig. AIV 27



$$A_{peak} = \frac{J\Omega_b\Psi(R)}{m_2(1+R)}$$

By using the dependencies $R = R(\xi_b)$ from (AIV 26) and $\Gamma = \Gamma(\xi_b)$ we construct the dependencies $\xi_b(R)$ and $\Psi(R) = \Gamma[\xi_b(R)]$ (see Fig. AIV 27). Finally we have

$$\gamma(\xi_b) = \frac{1}{\xi_b(3-4\xi_b^2)} \exp \left[-\frac{\sqrt{1-\xi_b^2}}{\xi_b} \operatorname{atan} \frac{(4\xi_b^2-1)\sqrt{1-\xi_b^2}}{\xi_b(3-4\xi_b^2)} \right] \cos \left[\operatorname{atan} \frac{(4\xi_b^2-1)\sqrt{1-\xi_b^2}}{\xi_b(3-4\xi_b^2)} \right]$$

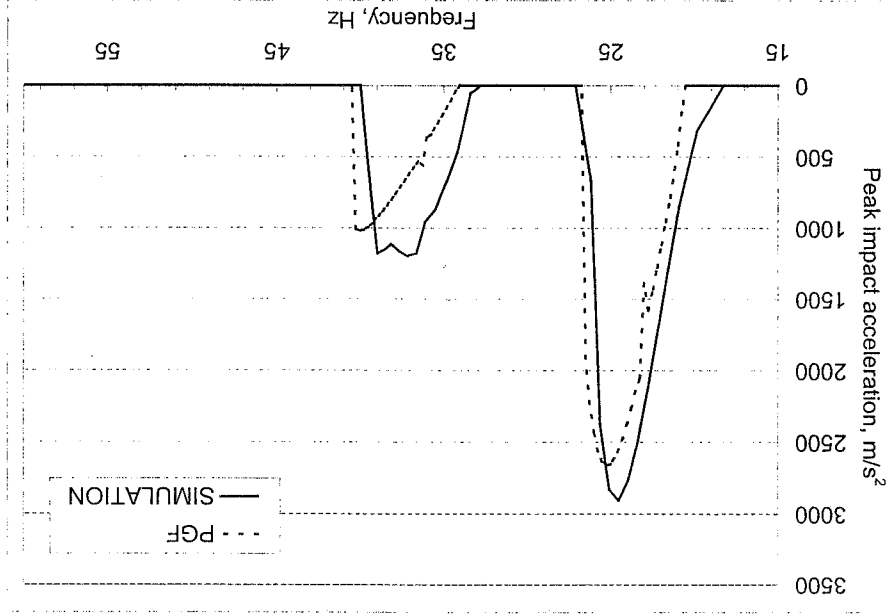


Fig. AIV 28

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