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Yury A. Rossikhin · Marina V. Shitikova

Dynamic Response of Pre-Stressed Spatially Curved Thin-Walled Beams of Open Profile

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Preface

This book is written by the experts in the transient wave propagation first of all for all those who are interested in this field: graduate students, PhD candidates, professors, scientists, researchers in various industrial and government institutes, and engineers. The book could be considered not only as a graduate textbook, but also as a guide for those working or interested in solving dynamic problems of Applied Mathematics, Continuum Mechanics, Stress Analysis, Civil Engineering, Mechanical Design, and Bridge Design resulting in the propagation of the transient waves.

The book presents the theory of discontinuities and the theory of ray expansions as applied to pre-stressed spatially curved thin-walled beams of open profile, because they are extensively used as structural components in different structures in civil, mechanical and aeronautical engineering fields. These structures have to resist dynamic loads such as wind, traffic and earthquake loadings, so that the understanding of the dynamic behavior of the structures becomes increasingly important.

Moreover, the increasing use of curved thin-walled beams in highway bridges and aircraft has resulted in considerable effort being directed toward developing accurate methods for analyzing the dynamic behaviour of such structures. Curved members in modern bridges and architectural structures continue to predominate because of emphasis on aesthetics and transportation alignment restrictions in metropolitan areas.

The reader is assumed to be familiar with the classical dynamic theory of elasticity, and a preliminary elemental knowledge in wave surfaces of discontinuity and conditions of compatibility on these surfaces is helpful. In one of the universities, where we were delivering a presentation about the dynamic response of thin-walled beams of open profile, we were asked if the conditions of compatibility adopted in our approach were the Beltrami–Michell compatibility conditions in strains. We did not answer this 'dilettante question' last time, but further, after some conversation, we have decided to write a book which could be a guide in the field. That is why at the end of this book we place in Chap. 6 brief but main data about the surfaces of strong discontinuity and the compatibility conditions.

However, the readers, who want to study the theory of discontinuities in more detail, are referred to the book by Thomas (1961) *Plastic Flow and Fracture in Solids*, Academic Press, New York.

The second stumbling block for some readers could be the theory of thin-walled beams of open section which abounds with such new terms as ‘flexural-torsional’ moment, ‘bimoment’, ‘warping of the cross section’, ‘sectorial area’, and others. This theory was pioneered by Vlasov in his *Thin-Walled Elastic Beams* published first in Russian in 1940 and then in English in 1961, where he explained in detail all these new phenomena which are characteristic of thin-walled beams of open profile as compared with simple beams. It seems likely that for a present-day young Western reader the Vlasov theory is known after the book by Gjelsvik (1981) *Theory of Thin Walled Bars*, Wiley, New York.

This book comprises six chapters. The motivation of its writing is discussed in Introduction. Chapter 2 presents the analytical review of the existing dynamic technical theories of thin-walled beams of open profile. The new theory dealing with the transient wave propagation in pre-stressed spatially curved thin-walled beams of open profile is suggested in Chap. 3. Chapter 4 treats the impact response of this walled beams, since the proposed theory is well suited for solving the problems of shock interactions.

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Voronezh, March 2011

Yury Rossikhin
Marina Shitikova

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Chapter 1
Introduction

The Vlasov theory [1] was elaborated more than 70-years ago for static analysis of straight thin-walled beams of open profile. However among a large body of sets of equations, which appeared during these years and which are intended to describe the dynamic response of thin-walled beams of open section, only a small percent fits the following simple requirements: hyperbolicity, and physical justification. At first glance it would seem that it is easy to match these requirements, but this is not the case. The authors of this book have suggested a special criterion which allows one to analyze quickly the sets of equations describing the dynamic behaviour of thin-walled beams of open profile, as well as to reveal their merits and demerits.

The first set of equations describing the dynamic response of straight thin-walled beams of open section was suggested by Vlasov [1]. In these equations, the rotary inertia was taken into account in a similar way as it was done by Rayleigh for a simple beam. However, as it would be shown below, such a set of equation falls into contradiction with the physical meaning. Dynamic equations for thin-walled beams proposed later consider both the rotary inertia and transverse shear deformation. Although for the beams of solid section, as it was shown by Timoshenko [2], the consideration for these two factors is essential for creating the physically justified hyperbolic dynamic equations, it turns out that it is not sufficiently advanced for describing the dynamic behaviour of thin-walled beams.

The third significant step on the road to developing the physically justified hyperbolic set of equations for thin-walled beams of open section was the paper by Gol'denveizer [3], wherein he suggested, instead of the Vlasov hypothesis about the warping of the cross section as the first derivative in the longitudinal coordinate of the rotation angle around the centroidal axis, to consider the warping as the independent function.

As it is shown in this book, only the consideration of three enumerated hypotheses at a time, which were proposed by four coryphaei in mechanics of solids, namely: Rayleigh, Timoshenko, Vlasov, and Gol'denveizer, allows one to derive the physically justified hyperbolic set of equations for thin-walled beams of

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open section. Such a set of equations was obtained in 1974 by Korbut and Lazarev [4]. Their paper has gone astray in the Lethean streams, and nobody of present-day researchers has cited it although some of them utilize the similar equations.

This story could be finished at this point if it were not for one circumstance. The sets of physically justified and hyperbolic equations determine one longitudinal wave propagating with the velocity of the longitudinal wave in a thin rod and three or four transverse waves, the velocities of which depend on the geometrical characteristics of thin-walled beams of open profile. This brings up two questions. The first one is how determine experimentally the velocities of the transverse waves which depend essentially on the choice of the tested beams. The second question bears some philosophical nature: whether it is possible to develop such a theory of dynamic behaviour of thin-walled beams which admits the propagation only of two waves with the velocities depending only on elastic constants.

As an answer to the second question, we want the suggest to our readers a novel theory, which is based on the three-dimensional equations of elasticity and on the theory of discontinuities and which satisfies the three hypotheses formulated above. To the authors' knowledge, this theory is unique and has no analogs.

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Chapter 2
Engineering Theories of Thin-Walled Beams of Open Section

Abstract The analytical review of the existing dynamic technical theories of thin-walled beams of open profile is carried out, from which it follows that all papers in the field can be divided into three groups. The papers, wherein the governing set of equations is both hyperbolic and correct from the viewpoint of the physically admissible magnitudes of the velocities of the transient waves resulting from these equations, fall into the first category. The second category involves the articles presenting hyperbolic but incorrect equations from the above mentioned viewpoint, i.e. resulting in incorrect magnitudes of the transient waves. The papers providing the governing system of equations which are not hyperbolic fall into the third group. The simple but effective procedure for checking for the category, within which this or that paper falls in, has been proposed and illustrated by several examples. It has been shown that only the theories of the first group, such as the Korbut–Lazarev theory, could be used for solving the problems dealing with transient wave propagation, while the theories belonging to the second and third group could be adopted for static problems only.

Keywords Thin-walled beam of open section · Vlasov theory · Gol'denveizer approach · Korbut–Lazarev equations

2.1 Review of Dynamic Engineering Theories of Thin-Walled Beams of Open Section

Thin-walled beams of open section are extensively used as structural components in different structures in civil, mechanical and aeronautical engineering fields. These structures have to resist dynamic loads such as wind, traffic and earthquake loadings, so that the understanding of the dynamic behavior of the structures becomes increasingly important. Ship hulls are also can be modelled as

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thin-walled girders during investigation of hydroelastic response of large container ships in waves [1].

The classical engineering theory of thin-walled uniform straight and horizontally curved beams of open cross-section was developed by Vlasov [2] in the early 1960s without due account for rotational inertia and transverse shear deformations [3]. The Vlasov theory is the generalization of the Bernoulli—Navier law to the thin-walled open section beams by including the sectorial warping of the section into account by the law of sectorial areas, providing that the first derivative of the torsion angle with respect to the longitudinal axis serves as a measure of the warping of the section. Thus, this theory results in the four differential equations of free vibrations of a thin-walled beam with an open inflexible section contour of arbitrary shape. For the case of a straight beam, the first second-order equation determines, independently of the other three and together with the initial and boundary conditions, the longitudinal vibrations of the beam. The remaining three fourth-order differential equations form a symmetrical system which, together with the initial and boundary conditions determines the transverse flexural-torsional vibrations of the beam (see page 388 in [2]). In the case of a curved beam, all four equations are coupled. However, as it will be shown later on following [4], Vlasov's equations are inappropriate for use in the problems dealing with the transient wave propagation.

Many researchers have tried to modify the Vlasov theory for dynamic analysis of elastic isotropic thin-walled beams with uniform cross-section by including into consideration the rotary inertia and transverse shear deformations [5–22], and/or by taking into consideration thin-walled beams of variable open section [23, 24], and/or investigating composite beams [25–27], and/or considering coupled bending–torsional vibration of axially loaded thin-walled beams [7, 14, 28, 29].

The increasing use of curved thin-walled beams in highway bridges and aircraft has resulted in considerable effort being directed toward developing accurate methods for analyzing the dynamic behaviour of such structures [30–36]. Curved members in modern bridges and architectural structures continue to predominate because of emphasis on aesthetics and transportation alignment restrictions in metropolitan areas.

It is well known that Timoshenko [37] in order to generalize the Bernoulli–Euler beam model has introduced two distinct functions, namely: the deflection of the centroid of the cross-section and the rotation of the normal to the cross-section through the centroid, i.e., he considered the transverse shear angle to be the independent variable. This starting point was the basis for the derivation of a set of two hyperbolic differential equations describing the dynamic behavior of a beam, resulting in the fact that two transient waves propagate in the Timoshenko beam with finite velocities: the longitudinal wave with the velocity equal to $G_L = \sqrt{E/\rho}$, and the wave of transverse shear with the velocity equal to $G_T = \sqrt{K\mu/\rho}$, where E and μ are the elastic moduli, ρ is the density, and K is the shear coefficient which is weakly dependent on the geometry of the beam [38].

Many of the up-to-date technical articles involve the derivation of the equations which, from the authors viewpoint, should describe the dynamic behavior of *thin-walled beams of the Timoshenko type* [6–10, 13–15, 19, 21]. Moreover, practically in each such paper it is written that such equations are novel, and no analogs were available previously in scientific literature [6, 8, 9, 14, 19].

All papers in the field can be divided into three groups. The papers, wherein the governing set of equations is both hyperbolic and correct from the viewpoint of the physically admissible magnitudes of the velocities of the transient waves resulting from these equations, fall into the first category, i.e. the velocity of the longitudinal wave is $G_L = \sqrt{E/\rho}$, while the velocities of the three transverse shear waves, in the general case of arbitrary cross sections of thin-walled beams with open profile, depend essentially of the geometry of the open section beam [9, 12 , 18]. There are seven independent unknowns in the displacement field in the general case if only primary warping is included into consideration [12], or with additional three generalized displacements describing the variation of the secondary warping due to non-uniform bending and torsion [9], or with additional three variables describing a "complete homogeneous deformation of the microstructure" [18]. As this takes place, different authors obtain different magnitudes for the velocities of transverse shear waves.

The second category involves the articles presenting hyperbolic but incorrect equations from the above mentioned viewpoint, i.e. resulting in incorrect magnitudes of the transient waves. This concerns, first of all, the velocity of the longitudinal waves which should not deviate from $G_L = \sqrt{E/\rho}$, nevertheless, there are some examples [16] where such a situation takes place. Secondly, in some papers one can find equations looking like hyperbolic ones [7, 19, 21, 39] but from which it is impossible to obtain the velocity, at least, of one transient wave at all. In such papers, usually six generalized displacements are independent (for monosymmetric cross sections they are four, and two in the case of bisymmetric profiles) while warping is assumed to be dependent on the derivative of the torsional rotation with respect to the beam axial coordinate [19, 21], or is neglected in the analysis [7, 39]. In other words, there is a hybrid of two approaches: Timoshenko's beam theory [37] and Vlasov's thin-walled beam theory [2], some times resulting in a set of equations wherein some of them are hyperbolic, while others are not. Thirdly, not all inertia terms are included into consideration.

The papers providing the governing system of equations which are not hyperbolic belong to the third group [6, 8, 10]. In such papers, the waves of transverse shear are the diffusion waves possessing infinitely large velocities, and therefore, from our point of view, the dynamic equations presented in [6, 8, 10] cannot be named as the Timoshenko type equations.

2.2 Procedure for Defining the Correctness of Engineering Approaches

Checking for the category, within which this or that paper falls in, is carried out rather easily if one uses the following reasoning proposed by Rossikhin and Shitikova [4].

Suppose that the given governing set of equations is the hyperbolic one. Then as a result of non-stationary excitations on a beam, transient waves in the form of surfaces of strong or weak discontinuity are generated in this beam (see Chap. 6 for details). We shall interpret the wave surface as a limiting layer with the thickness h, inside of which the desired field Z changes monotonically and continuously from the magnitude Z_+ to the magnitude Z_-. Now we can differentiate the set of equations n times with respect to time t, then rewrite it inside the layer, and change all time-derivatives by the derivatives with respect to the axial coordinate z using the one-dimensional condition of compatibility (see Sect. 6.2)

$$(-1)^n Z_{,(n)} = G^n \frac{\partial^n Z}{\partial z^n} + \sum_{m=0}^{n-1} (-1)^{m+1} \frac{n!}{m!(n-m)!} \frac{\delta^{n-m} Z_{,(m)}}{\delta t^{n-m}}, \tag{2.1}$$

where G is the normal velocity of the limiting layer, $\delta/\delta t$ is the Thomas δ-derivative [40], and $Z_{,(k)} = \partial^k Z/\partial t^k$.

Integrating the resulting equations n times with respect to z, where n is the order of the highest z-derivative, writing the net equations at $z = -h/2$ and $z = h/2$, and taking their difference, we are led at $h \to 0$ to the relationships which involve the discontinuities in the desired field $[Z] = Z_+ - Z_-$ and which are used for determining the velocities of the transient waves, i.e. the magnitude of G, what allows one to clarify the type of the given equations.

If the values entering in the governing equations could not experience the discontinuity during transition through the wave surface, generalized displacements as an example, then in this case the governing set of equations should be differentiated one time with respect to time in order to substitute the generalized displacements by their velocities. Thus, after the procedure described above, the governing equations will involve not the discontinuities in the desired values Z but the discontinuities in their time-derivatives, i.e. $[\dot{Z}] = (\partial Z/\partial t)_+ - (\partial Z/\partial t)_-$.

Using the procedure described above, it can be shown that the correct hyperbolic set of equations taking shear deformation due to bending and coupled bending torsion was suggested by Aggarwal and Cranch [5], but their theory is strictly applied only to a channel-section beam.

It seems likely that for a straight elastic thin-walled beam with a generic open section this problem was pioneered in 1974 by Korbut and Lazarev [12], who generalized the Vlasov theory by adopting the assumptions proposed in 1949 by Gol'denveizer [41] that the angles of in-plane rotation do not coincide with the first derivatives of the lateral displacement components and, analogously, warping does not coincide with the first derivative of the torsional rotation. It should be emphasized that it was precisely Gol'denveizer [41] who pioneered in combining Timoshenko's beam theory [37] and Vlasov thin-walled beam theory [2] (note that the first edition of Vlasov's book was published in Moscow in 1940) and who suggested to characterize the displacements of the thin-walled beam's cross-section by seven generalized displacements. It is interesting to note that the approach proposed by Gol'denveizer [41] for solving static problems (which has being widely used by Russian researchers and engineers since 1949) was re-discovered approximately 50 years later by Back and Will [42], who have inserted it in finite element codes.

The set of seven second-order differential equations with due account for rotational inertia and transverse shear deformations derived in [12] using the Reissner's variational principle really describes the dynamic behavior of a straight beam of the Timoshenko type and has the following form:

the equations of motion

$$\begin{aligned} \rho I_x \dot{\mathrm{B}}_x - M_{x,z} + Q_{y\omega} &= 0, \\ \rho I_y \dot{\mathrm{B}}_y - M_{y,z} - Q_{x\omega} &= 0, \\ \rho I_\omega \dot{\Psi} - \mathrm{B}_{,z} - Q_{xy} &= 0, \\ \rho F \dot{v}_z - N_{,z} &= 0, \end{aligned} \tag{2.2}$$

$$\begin{aligned} &\rho F \dot{v}_x + \rho a_y F \dot{\Phi} - Q_{x\omega,z} = 0, \\ &\rho F \dot{v}_y - \rho a_x F \dot{\Phi} - Q_{y\omega,z} = 0, \\ &\rho I_p \dot{\Phi} + \rho a_y F_x - \rho a_x F \dot{v}_y - (Q_{xy} + H)_{,z} = 0; \end{aligned} \tag{2.3}$$

the generalized Hook's law

$$\begin{aligned} \dot{M}_x &= EI_x \mathrm{B}_{x,z}, \\ \dot{M}_y &= EI_y \mathrm{B}_{y,z}, \\ \dot{\mathrm{B}} &= EI_\omega \Psi_{,z}, \\ \dot{N} &= EF v_{z,z}, \end{aligned} \tag{2.4}$$

$$\begin{aligned} \mu(v_{x,z} - \mathrm{B}_y) &= k_y \dot{Q}_{x\omega} + k_{xy} \dot{Q}_{y\omega} + k_{y\omega} \dot{Q}_{xy}, \\ \mu(v_{y,z} + \mathrm{B}_x) &= k_{xy} \dot{Q}_{x\omega} + k_x \dot{Q}_{y\omega} + k_{x\omega} \dot{Q}_{xy}, \\ \mu(\Phi_{,z} - \Psi) &= k_{y\omega} \dot{Q}_{x\omega} + k_{x\omega} \dot{Q}_{y\omega} + k_\omega \dot{Q}_{xy}, \\ \dot{H} &= \mu I_k \Phi_{,z} \ , \end{aligned} \tag{2.5}$$

where ρ is the beam's material density, F is the cross-section area, ω is the sectorial coordinate, I_x and I_y are centroidal moments of inertia, I_ω is the sectorial moment of inertia, I_p is the polar moment of inertia about the flexure center A, I_k is the moment of inertia due to pure torsion, a_x and a_y are the coordinates of the flexural center, E and μ are the Young's and shear moduli, respectively, $\mathrm{B}_x = \dot{\beta}_x$, $\mathrm{B}_y = \dot{\beta}_y$, $\Phi = \dot{\varphi}$, β_x, β_y and φ are the angles of rotation of the cross section about x-, y- and z-axes, respectively, $\Psi = \dot{\psi}$, ψ is the warping function, v_x, v_y, v_z are the velocities of displacements of the flexural center, u, v, and w, along the central principal axes x and y and the longitudinal z-axis, respectively, M_x and M_y are the bending moments, B is the bimoment, N is the longitudinal (membrane) force, $Q_{x\omega}$ and $Q_{y\omega}$ are the transverse forces, H is the moment of pure torsion, Q_{xy} is the bending-torsional moment from the axial shear forces acting at a tangent to the contour of the cross section about the flexural center, overdots denote the time derivatives, and the index z after a point defines the derivative with respect to the z-coordinate.

In (2.5), k_x, k_y, k_ω, $k_{x\omega}$, $k_{y\omega}$, and k_{xy} are the cross-sectional geometrical characteristics which take shears into consideration:

$$\begin{aligned} k_x &= \frac{1}{I_x^2}\int\limits_F \frac{S_x^2}{\delta_s^2}\mathrm{d}F, \quad k_y = \frac{1}{I_y^2}\int\limits_F \frac{S_y^2}{\delta_s^2}\mathrm{d}F, \quad k_\omega = \frac{1}{I_\omega^2}\int\limits_F \frac{S_\omega^2}{\delta_s^2}\mathrm{d}F, \\ k_{x\omega} &= \frac{1}{I_x I_\omega}\int\limits_F \frac{S_x S_\omega}{\delta_s^2}\mathrm{d}F, \quad k_{y\omega} = \frac{1}{I_y I_\omega}\int\limits_F \frac{S_y S_\omega}{\delta_s^2}\mathrm{d}F, \quad k_{xy} = \frac{1}{I_x I_y}\int\limits_F \frac{S_x S_y}{\delta_s^2}\mathrm{d}F, \end{aligned} \tag{2.6}$$

where S_x, S_y, and S_ω are the axial and sectorial static moments of the intercepted part of the cross section, and δ_s is the width of the web of the beam.

Note that 25 years later the shear coefficients (2.6) were re-derived by means of the Reissner principle in [10].

2.2.1 *Velocities of Transient Waves in Thin-Walled Beams of Open Section due to the Korbut–Lazarev Theory and Its Generalizations*

To show that the set of Eqs. 2.2–2.5 governs three transient shear waves which propagate with the finite velocities depending on the geometrical characteristics of the thin-walled beam (2.6), we can use the approach suggested above. If we write (2.2)–(2.5) inside the layer and apply the condition of compatibility (2.1) at $n = 1$, as a result, we find [20]

$$\begin{aligned} -\rho I_x G[\mathrm{B}_x] - [M_x] &= 0, \\ -\rho I_y G[\mathrm{B}_y] - [M_y] &= 0, \\ -\rho I_\omega G[\Psi] - [\mathrm{B}] &= 0, \\ -\rho F G[v_z] - [N] &= 0, \end{aligned} \tag{2.7}$$

$$\begin{aligned} -\rho F G[v_x] - \rho F G a_y[\Phi] - [Q_{x\omega}] &= 0, \\ -\rho F G[v_y] + \rho F G a_x[\Phi] - [Q_{y\omega}] &= 0, \\ -\rho I_p G[\Phi] - \rho F G a_y[v_x] + \rho F G a_x[v_y] - [Q_{xy}] - [H] &= 0, \end{aligned} \tag{2.8}$$

$$
\begin{aligned}
-G[M_x] &= EI_x[\mathrm{B}_x],\\
-G[M_y] &= EI_y[\mathrm{B}_y],\\
-G[\mathrm{B}] &= EI_\omega[\Psi],\\
-G[N] &= EF[v_z],
\end{aligned}
\tag{2.9}
$$

$$
\begin{aligned}
\mu[v_x] &= -Gk_y[Q_{x\omega}] - Gk_{xy}[Q_{y\omega}] - Gk_{y\omega}[Q_{xy}],\\
\mu[v_y] &= -Gk_{xy}[Q_{x\omega}] - Gk_x[Q_{y\omega}] - Gk_{x\omega}[Q_{xy}],\\
\mu[\Phi] &= -Gk_{y\omega}[Q_{x\omega}] - Gk_{x\omega}[Q_{y\omega}] - Gk_\omega[Q_{xy}],\\
-G[H] &= \mu I_k[\Phi].
\end{aligned}
\tag{2.10}
$$

Eliminating the values $[M_x]$, $[M_y]$, [B] and $[N]$ from (2.7) and (2.9), we obtain the velocity of the longitudinal-flexural-warping wave

$$G_4 = \sqrt{E\rho^{-1}}, \tag{2.11}$$

on which $[\mathrm{B}_x] \neq 0$, $[\mathrm{B}_y] \neq 0$, $[\Psi] \neq 0$, and $[v_z] \neq 0$, while $[v_x] = [v_y] = [\Phi] = 0$.

Eliminating the values $[Q_{x\omega}]$, $[Q_{y\omega}]$, $[Q_{xy}]$, and [H] from (2.8) and (2.10), we arrive at the system of three linear homogeneous equations:

$$\sum_{j=1}^{3} a_{ij}[v_j] = 0 \quad (i,j = 1,2,3), \tag{2.12}$$

where $[v_1] = [v_x]$, $[v_2] = [v_y]$, $[v_3] = [\Phi]$,

$$
\begin{aligned}
a_{11} &= \rho FG^2(k_y + a_y k_{y\omega}) - \mu, \quad a_{12} = \rho FG^2(k_{xy} - a_x k_{y\omega}),\\
a_{13} &= \rho FG^2(a_y k_y - a_x k_{xy}) + k_{y\omega}(\rho I_p G^2 - \mu I_k),\\
a_{21} &= \rho FG^2(k_{xy} + a_y k_{x\omega}), \quad a_{22} = \rho FG^2(k_x - a_x k_{x\omega}) - \mu,\\
a_{23} &= \rho FG^2(a_y k_{xy} - a_x k_x) + k_{x\omega}(\rho I_p G^2 - \mu I_k),\\
a_{31} &= G^2(k_{y\omega} + a_y k_\omega), \quad a_{32} = \rho FG^2(k_{x\omega} - a_x k_\omega),\\
a_{33} &= \rho FG^2(a_y k_{y\omega} - a_x k_{x\omega}) + k_\omega(\rho I_p G^2 -_k) - \mu.
\end{aligned}
$$

Setting determinant of the set of Eq. 2.12 equal to zero

$$|a_{ij}| = 0, \tag{2.13}$$

we are led to the cubic equation governing the velocities G_1, G_2, and G_3 of three twisting-shear waves, on which only the values $[v_x]$, $[v_y]$ and $[\Phi]$ are nonzero such that

$$[v_x] = \gamma[\Phi], \quad [v_y] = \delta[\Phi], \tag{2.14}$$

where

$$\gamma = \frac{a_{23}a_{12} - a_{13}a_{22}}{a_{11}a_{22} - a_{12}a_{21}}, \quad \delta = \frac{a_{13}a_{21} - a_{23}a_{11}}{a_{11}a_{22} - a_{12}a_{21}}.$$

If the beam has the central symmetry, then the flexural center coincides with the center of symmetry, i.e. $a_x = a_y = 0$, and the parameters $k_{x\omega}$ and $k_{y\omega}$ vanish. In this case, the third equation from the system of Eq. 2.12 determines the velocity of the twisting wave [20]

$$G_1 = \sqrt{\mu(1 + I_k k_\omega)(\rho I_p k_\omega)^{-1}}, \tag{2.15}$$

on which only the value $[\Phi] \neq 0$, but the first and second equations of (2.12) have the form

$$\begin{aligned} a_{11}^0[v_x] + a_{12}^0[v_y] = 0, \\ a_{21}^0[v_x] + a_{22}^0[v_y] = 0, \end{aligned} \tag{2.16}$$

where $a_{ij}^0 = a_{ij}|_{a_x=a_y=0} \quad (i = 1, 2)$.

Setting determinant of the set of Eq. 2.16 equal to zero,

$$\begin{vmatrix} a_{11}^0 & a_{12}^0 \\ a_{21}^0 & a_{22}^0 \end{vmatrix} = 0$$

we are led to the biquadratic equation governing the velocities G_2 and G_3 of two shear waves

$$G_{2,3} = \sqrt{\frac{\mu}{2\rho F} \frac{k_x + k_y \pm \sqrt{(k_x - k_y)^2 + 4k_{xy}^2}}{q_{xy}^2}}, \tag{2.17}$$

where $q_{xy} = k_x k_y - k_{xy}^2$. On these waves only the values $[v_x]$ and $[v_y]$ are nonzero such that $[v_x] = \theta[v_y]$, where $\theta = -a_{12}^0/a_{11}^0$.

If the beam has the plane of symmetry containing, for example, the y-axis, then the parameters a_x, k_{xy} and $k_{x\omega}$ vanish. In this case, the second equation from the system of Eq. 2.12 determines the velocity of the shear wave

$$G_3 = \sqrt{\mu(k_x \rho F)^{-1}}, \tag{2.18}$$

on which only the value $[v_y] \neq 0$, but the first and third equations of (2.12) have the form

$$\begin{aligned} a_{11}[v_x] + a_{13}^0[\Phi] = 0, \\ a_{31}[v_x] + a_{33}^0[\Phi] = 0, \end{aligned} \tag{2.19}$$

where $a_{13}^0 = a_{13}|_{a_x=0}$, and $a_{33}^0 = a_{33}|_{a_x=0}$.

Setting determinant of the set of Eq. 2.19 equal to zero,

$$\begin{vmatrix} a_{11} & a_{13}^0 \\ a_{31} & a_{33}^0 \end{vmatrix} = 0$$

we are led to the biquadratic equation governing the velocities G_1 and G_2 of two twisting-shear waves

$$G_{1,2} = \sqrt{\frac{\mu}{2\rho F I_z q_{y\omega}}\left(q_1 \mp \sqrt{q_1^2 - q_2}\right)}, \tag{2.20}$$

where $q_{y\omega} = k_y k_\omega - k_{y\omega}^2$, $q_1 = k_\omega I_p + I_k F q_{y\omega} + F(k_y + 2a_y k_{y\omega})$, $q_2 = 4FI_z(1 + k_\omega I_k)$, $I_z = I_x + I_y$. On these waves only the values $[v_x]$ and $[\Phi]$ are nonzero such that $[v_x] = \chi[\Phi]$, where $\chi = -a_{13}^0/a_{11}$.

For the bisymmetrical beam, the values a_x, a_y, k_{xy}, $k_{x\omega}$, and $k_{y\omega}$ vanish. In this case, the set of (2.12) becomes the three independent equations defining the velocities of two shear waves

$$G_2 = \sqrt{\frac{\mu}{\rho F k_y}}, \quad G_3 = \sqrt{\frac{\mu}{\rho F k_x}}, \tag{2.21}$$

and one twisting wave

$$G_1 = \sqrt{\frac{\mu(1 + k_\omega I_k)}{\rho I_p k_\omega}}, \tag{2.22}$$

on which $[v_x]$, $[v_y]$, and $[\Phi]$ are nonzero, respectively.

It is strange to these authors that the Korbut and Lazarev theory [12] appeared in 1974 is absolutely unaware to the international mechanics community, in spite of the fact that it was published in the Soviet academic journal which is available in English due to translation made by Springer.

The Korbut–Lazarev theory [12], which provides the physically admissible velocities of propagation of transient waves, was generalized in [20] taking the extension of the thin-walled beam's middle surface into account.

Nine years later after the appearance of [12], Muller [18] suggested the theory (which generalized the Korbut–Lazarev approach [12]), wherein the additional deformations of two lateral contractions and the so-called effect of distortion shear were taken into consideration. This allowed the author to receive correctly the velocity of the longitudinal-flexural-warping wave (2.11), three velocities of the transverse shear waves due to coupled flexural translational-torsional motions, which strongly depend of the geometry of the beam's cross section as in the case of (2.13) defined by the Korbut−Lazarev theory [12], and the wave of pure shear due to lateral distortion deformation, which propagates with the velocity $G_T = \sqrt{\mu/\rho}$.

One more example of the correct generalization of the Timoshenko beam model to an open section thin-walled beam is the approach proposed in [9, 13] in the early 1990s. Once again it is the generalization of the Korbut−Lazarev theory [12], since three additional deformations describing the secondary warping due to non-uniform bending and torsion are taken into account. The hyperbolic set of ten equations presented in [13] allows one to obtain the velocity of longitudinal-flexural-warping wave (2.11), and three velocities of the transverse shear waves due to coupled flexural translational-torsional motions similar to (2.13). As this takes place, the found shear constants (see relationships (45) in [9]) coincide completely with those of (2.6).

The presence of three [9, 12, 20] or four [18] transverse shear waves, which propagate with different velocities dependent strongly on geometric characteristics of the thin-walled beam, severely limits the application of such theories in solving engineering problems. As for the experimental verification of the existence of the three shear waves in thin-walled beams of open section, then it appears to be hampered by the fact that the velocities of these waves depend on the choice of the beam's cross section.

2.2.2 *Velocities of Transient Waves in Thin-Walled Beams of Open Section due to the Vlasov Theory and Its Modifications*

Note that only inclusion into consideration of three factors, namely: shear deformations, rotary inertia, and warping deformations as the independent field—could lead to the correct system of hyperbolic equations of the Timoshenko type for describing the dynamic behaviour of thin bodies. Ignoring one of the factors or its incomplete account immediately results in an incorrect set of governing equations.

Let us consider, as an example, the dynamic equations suggested by Vlasov (see (1.8) in page 388 in [2]) to describe the behaviour of thin-walled straight beams of open profile:

$$
\begin{aligned}
&EF\frac{\partial^2\zeta}{\partial z^2} - \rho F\frac{\partial^2\zeta}{\partial t^2} = 0,\\
&EI_y\frac{\partial^4\xi}{\partial z^4} - \rho I_y\frac{\partial^4\xi}{\partial z^2\partial t^2} + \rho F\frac{\partial^2\xi}{\partial t^2} + a_y\rho F\frac{\partial^2\theta}{2} = 0,\\
&EI_x\frac{\partial^4\eta}{\partial z^4} - \rho I_x\frac{\partial^4\eta}{\partial z^2\partial t^2} + \rho F\frac{\partial^2\eta}{\partial t^2} - a_x\rho F\frac{\partial^2\theta}{\partial t^2} = 0,\\
&EI_\omega\frac{\partial^4\theta}{\partial z^4} - \mu I_k\frac{\partial^2\theta}{\partial z^2} - \rho I_\omega\frac{\partial^4\theta}{{}^2\partial t^2} + \rho I_p\frac{\partial^2\theta}{2} + a_y\rho F\frac{\partial^2\xi}{\partial t^2} - a_x\rho F\frac{\partial^2\eta}{\partial t^2} = 0,
\end{aligned}
\tag{2.23}
$$

which were obtained with due account for the rotary inertia but neglecting the shear deformations, where ζ, ξ and η are the displacements of the flexural center along the central principal axes x and y and the longitudinal z-axis, respectively, and θ is the angle of the rotation about the z-axis.

If we differentiate all equations in (2.23) one time with respect to time, and then apply to them the suggested above procedure, as a result we obtain

$$\begin{aligned}(\rho G^2 - E)[\dot{\zeta}] &= 0,\\ (\rho G^2 - E)[\dot{\xi}] &= 0,\\ (\rho G^2 - E)[\dot{\eta}] &= 0,\\ (\rho G^2 - E)[\dot{\theta}] &= 0.\end{aligned} \tag{2.24}$$

Reference to (2.24) shows that on the transient longitudinal wave of strong discontinuity propagating with the velocity $G_L = \sqrt{E/\rho}$, not only the velocity of longitudinal displacement ζ experiences discontinuity but the velocities of transverse displacements ξ and η as well, what is characteristic for the transient transverse shear wave of strong discontinuity. Therefore, the set of Eq. 2.23 could not be considered as a correct hyperbolic set of equations. In other words, the values connected with the phenomenon of shear propagate with the velocity G_L, what falls into contradiction with the physical sense, and thus the Vlasov theory is applicable only for the static problems.

Note that for a rod of a massive cross-section the account only for the rotary inertia was made for the first time by Lord Rayleigh in his *Theory of Sound* in the form of a mixed derivative of the displacement with respect to time and coordinate.

If we exclude from (2.23) the terms responsible for the rotary inertia, i.e. $\rho I_y \frac{\partial^4 \xi}{\partial z^2 \partial t^2}$, $\rho I_x \frac{\partial^4 \eta}{\partial z^2 \partial t^2}$, and $\rho I_\omega \frac{\partial^4 \theta}{\partial z^2 \partial t^2}$, then we obtain the equations describing the dynamic behaviour of the Bernoulli-Euler beams. In such beams, the velocity of the propagation of the transient transverse shear wave of strong discontinuity is equal to infinity.

The second example is not mere expressive. Let us consider the set of equations suggested by Meshcheriakov [16] for describing the straight thin-walled beam of open bisymmetric profile

$$\begin{aligned}&EI_y \frac{\partial^4 \beta_y}{4} - \rho I_y \frac{\partial^4 \beta_y}{2\partial t^2} + \rho F \frac{\partial^2 \beta_y}{2} + 2(1+\nu)\frac{S_{xx}}{I_y}\rho F \frac{\partial^4 \beta_y}{\partial z^2 \partial t^2} = 0,\\ &EI_x \frac{\partial^4 \beta_x}{\partial z^4} - \rho I_x \frac{\partial^4 \beta_x}{\partial z^2 \partial t^2} + \rho F \frac{\partial^2 \beta_x}{\partial t^2} + 2(1+\nu)\frac{S_{yy}}{I_x}\rho F \frac{\partial^4 \beta_x}{\partial z^2 \partial t^2} = 0,\\ &EI_\omega \frac{\partial^4 \psi}{\partial z^4} - \rho I_\omega \frac{\partial^4 \psi}{\partial z^2 \partial t^2} - \rho I_k \frac{\partial^2 \psi}{\partial z^2} + \rho I_p \left[\frac{\partial^2 \psi}{\partial t^2} + 2(1+\nu)\frac{S_{\omega\omega}}{I_\omega}\frac{\partial^4 \psi}{\partial z^2 \partial t^2}\right] = 0,\end{aligned} \tag{2.25}$$

where β_x and β_y are the angles of rotation around the main transverse axes, ψ is the warping of the cross section, S_{xx}, S_{yy}, and $S_{\omega\omega}$ are shear coefficients [16], and ν is the Poisson's ratio.

If we differentiate all equations from (2.25) one time with respect to time t and then apply to them the procedure described above, as a result we obtain

$$
\begin{aligned}
I_y\left\{E-\rho G^2\left[1-2(1+v)\frac{S_{xx}}{I_y^2}\,F\right]\right\}[\dot{\beta}_y]&=0,\\
I_x\left\{E-\rho G^2\left[1-2(1+v)\frac{S_{yy}}{I_x^2}\,F\right]\right\}[\dot{\beta}_x]&=0,\\
I_\omega\left\{E-\rho G^2\left[1-2(1+v)\frac{S_{\omega\omega}}{I_\omega^2}\,I_p\right]\right\}[\dot{\psi}]&=0.
\end{aligned}
\tag{2.26}
$$

Reference to (2.26) shows that absolutely absurd velocities of three transient longitudinal waves of strong discontinuity

$$
\begin{aligned}
G_1&=\sqrt{E\rho^{-1}\left[1-2(1+v)\frac{S_{xx}}{I_y^2}\,F\right]^{-1}},\\
G_2&=\sqrt{E\rho^{-1}\left[1-2(1+v)\frac{S_{yy}}{I_x^2}\,F\right]^{-1}},\\
G_3&=\sqrt{E\rho^{-1}\left[1-2(1+v)\frac{S_{\omega\omega}}{I_\omega^2}\,I_p\right]^{-1}}
\end{aligned}
\tag{2.27}
$$

are obtained.

However if the rotary inertia is considered sequentially, as it was done by Timoshenko in *Vibration Problems in Engineering* [37], then the additional terms

$$
\begin{aligned}
&-2(1+v)\frac{S_{xx}}{I_y}\rho F\;\rho E^{-1}\;\frac{\partial^4\beta_y}{\partial t^4},\\
&-2(1+v)\frac{S_{yy}}{I_x}\rho F\;\rho E^{-1}\;\frac{\partial^4\beta_x}{\partial t^4},\\
&-2(1+v)\frac{S_{\omega\omega}}{I_\omega}\rho I_p\;\rho E^{-1}\;\frac{\partial^4\psi}{\partial t^4},
\end{aligned}
\tag{2.28}
$$

will enter in (2.25) as it is in [17], which could remedy all velocities of transient longitudinal waves, since the procedure suggested in [4] and described above in Sect. 2.2 transforms the additional terms (2.28) to the form

$$
\begin{aligned}
&-2(1+v)\frac{S_{xx}}{I_y^2}\rho F\;\rho E^{-1}\;G^4[\dot{\beta}_y],\\
&-2(1+v)\frac{S_{yy}}{I_x^2}\rho F\;\rho E^{-1}\;G^4[\dot{\beta}_x],\\
&-2(1+v)\frac{S_{\omega\omega}}{I_\omega^2}\rho I_p\;\rho E^{-1}\;G^4[\dot{\psi}].
\end{aligned}
\tag{2.29}
$$

Relationships (2.29) will be added, respectively, in Eq. 2.26, and will transform them, in their turn, to the form

$$
\begin{aligned}
&(E-\rho G^2)\left[1+2(1+v)\frac{S_{xx}}{I_y^2}\,E^{-1}F\rho G^2\right][\dot{\beta}_y]=0,\\
&(E-\rho G^2)\left[1+2(1+v)\frac{S_{yy}}{I_x^2}\,E^{-1}F\rho G^2\right][\dot{\beta}_x]=0,\\
&(E-\rho G^2)\left[1+2(1+v)\frac{S_{\omega\omega}}{I_\omega^2}\,E^{-1}I_p\rho G^2\right][\dot{\psi}]=0,
\end{aligned}
\tag{2.30}
$$

whence it follows that the velocity of the longitudinal wave of strong discontinuity is equal to $G_L=\sqrt{E/\rho}$, what matches to the reality.

It should be noted also that a reader could even find such papers in the field which are apparently false. Thus, the following set of equations is presented in [19] (it is written below in the notation adopted in Sect. 2.2.1 for convenience):

$$
\begin{aligned}
&EFw''-\rho F\ddot{w}=0,\\
&k_x\mu F(u''-\beta_y')-\rho F\ddot{u}-\rho a_yF\ddot{\varphi}=0,\\
&k_y\mu F(v''+\beta_x')-\rho F\ddot{v}+\rho a_xF\ddot{\varphi}=0,\\
&EI_x\beta_x''-k_y\mu F(v'+\beta_x)-\rho I_x\ddot{\beta}_x=0,\\
&EI_y\beta_y''+k_x\mu F(u'-\beta_y)-\rho I_y\ddot{\beta}_y=0,\\
&EI_\omega\varphi''''-\mu I_k\varphi''-\rho I_\omega\ddot{\varphi}''+\rho a_yF\ddot{u}-\rho a_xF\ddot{v}+\rho I_p\ddot{\varphi}=0,
\end{aligned}
\tag{2.31}
$$

where primes denote derivatives with respect to the coordinate z, and k_x and k_y are the shear correction factors in principal planes [19].

The author of [19] has declared that the system of (2.31) is responsible for describing the transverse shear deformations and rotary inertia in a thin-walled beam of open profile, that is to describe the dynamic response of a Timoshenko-like beam.

But this set of equations is not even correct one, and thus it could not describe the dynamic behaviour of the thin-walled Timoshenko-like beam. Really, applying the procedure proposed above, we can rewrite (2.31) in terms of discontinuities

$$
\begin{aligned}
&EF[v_z]-\rho G^2F[v_z]=0\ ,\\
&k_x\mu F[v_x]-\rho G^2F[v_x]-\rho G^2a_yF[\Phi]=0,\\
&k_y\mu F[v_y]-\rho G^2F[v_y]+\rho G^2a_xF[\Phi]=0,\\
&EI_x[B_x]-\rho G^2I_x[B_x]=0,\\
&EI_y[B_y]-\rho G^2I_y[B_y]=0,\\
&EI_\omega[\Phi]-\rho G^2I_\omega[\Phi]=0.
\end{aligned}
\tag{2.32}
$$

From (2.32) it follows that when $[v_z] \neq 0$, $[B_x] \neq 0$, and $[B_y] \neq 0$, i.e. on the longitudinal wave, the velocity G is equal to the velocity of the longitudinal wave $\sqrt{E/\rho}$. Furthermore, on the longitudinal wave, the discontinuity $[\Phi]$ is also distinct from zero, while the value $[\Phi]$ should be nonzero only on the transverse wave. Moreover, the velocity of the transverse shear wave could not be obtained from the second and third equations of (2.32) at all.

The contradiction obtained points to the fact that (2.31) is the incorrect system of equations, and nobody, including the author of [19], knows what phenomenon is described by these equations.

Thus, the examples presented above have demonstrated the effectiveness of the procedure suggested by the authors for identifying the category of the equations of motion and its applicability for using in dynamic problems dealing with the propagation of the transient waves in thin-walled structures.

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Chapter 3
Transient Dynamics of Pre-Stressed Spatially Curved Thin-Walled Beams of Open Profile

Abstract The dynamic stability with respect to small pertubations, as well as the local damage of geometrically nonlinear elastic spatially curved open section beams with axial precompression have been analyzed. Transient waves, which are the surfaces of strong discontinuity and wherein the stress and strain fields experience discontinuities, are used as small pertubations, in so doing the discontinuities are considered to be of small values. Such waves are initiated during low-velocity impacts upon thin-walled beams. The theory of discontinuities and the method of ray expansions, which allow one to find the desired fields behind the fronts of the transient waves in terms of discontinuities in time-derivatives of the values to be found, are used as the methods of solution for short-time dynamic processes.

Keywords Pre-stressed spatially curved thin-walled beam of open section · Small transient perturbations · Surfaces of strong discontinuity · Recurrent equations of the ray method · Ray expansions

3.1 Theory of Thin-Walled Beams Based on 3D Equations of the Theory of Elasticity

For investigating the dynamic behaviour of thin-walled beams of open section, we shall proceed from three-dimensional (3D) equations of isotropic elasticity written in the Cartesian system of coordinates x_1, x_2, and x_3.

3.1.1 Problem Formulation and Governing Equations

Let us consider a certain unperturbed equilibrium of an elastic body characterized by the displacement vector u_i^0, stress tensor σ_{ij}^0, and the vector of volume forces X_i^0

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(surface forces are absent). The characteristics of the unperturbed equilibrium state satisfy the following geometrically nonlinear equations and boundary conditions on the surface [1]:

$$\{\sigma_{jk}^{0}(\delta_{ik} + u_{i,k}^{0})\}_{,j} + X_i^0 = 0, \tag{3.1}$$

$$\sigma_{jk}^{0}(\delta_{ik} + u_{i,k}^{0})v_j = 0, \tag{3.2}$$

where δ_{ik} is the Kronecker's symbol, v_j are the components of the unit vector normal to the boundary surface, Latin indices take on the magnitudes 1, 2, 3, a Latin index after comma denotes the partial derivative with respect to the corresponding spatial coordinate x_1, x_2, x_3, and the summation is understood over the repeated indices.

Let us perturb the body with some small deviations from the unperturbed equilibrium: u_i, σ_{ij}, $X_i = -\rho\ddot{u}_i$, where ρ is the material density, and an overdot denotes the time-derivative. The characteristic components of the perturbed motion $\widetilde{u}_i$, $\widetilde{\sigma}_{ij}$, and $\widetilde{X}_i$ take the form

$$\widetilde{u}_i = u_i^0 + u_i, \quad \widetilde{\sigma}_{ij} = \sigma_{ij}^0 + \sigma_{ij}, \quad \widetilde{X}_i = -\rho\dot{v}_i, \tag{3.3}$$

where v_i is the displacement velocity.

The components of the perturbed state satisfy the following equations and boundary conditions:

$$\left\{\widetilde{\sigma}_{ik}(\delta_{ik} + \widetilde{u}_{i,k})\right\}_{,j} + \widetilde{X}_i = 0, \tag{3.4}$$

$$\widetilde{\sigma}_{jk}\left(\delta_{ik} + \widetilde{u}_{i,k}\right)v_j = 0. \tag{3.5}$$

Substituting (3.3) in (3.4) and (3.5), carrying out the linearization of the resulting equations, and taking (3.1) and (3.2) into account, we find

$$\left\{\sigma_{jk}\left(\delta_{ik} + u_{i,k}^0\right) + \sigma_{jk}^0 u_{i,k}\right\}_{,j} = \rho\ddot{u}_i, \tag{3.6}$$

$$\left\{\sigma_{jk}\left(\delta_{ik} + u_{i,k}^0\right) + \sigma_{jk}^0 u_{i,k}\right\}v_j = 0. \tag{3.7}$$

Suppose that the unperturbed state deviates a little from the initial undeformed state. In the majority of engineering problems the given assumption is found to be valid. In this case, the terms $u_{i,k}^0$ can be neglected, and Eqs. 3.2, 3.6 and 3.7 at $\sigma_{ij}^0 = c_{ij}$, where c_{ij} $(i,j = 1,2,3)$ are certain constant values (moreover, some of them could be vanished according to the conditions of the problem under consideration), take the following form:

$$\sigma_{ij,j} + \sigma_{jl}^0 u_{i,lj} = \rho\dot{v}_i, \tag{3.8}$$

$$\sigma_{ij}v_j = 0, \tag{3.9}$$

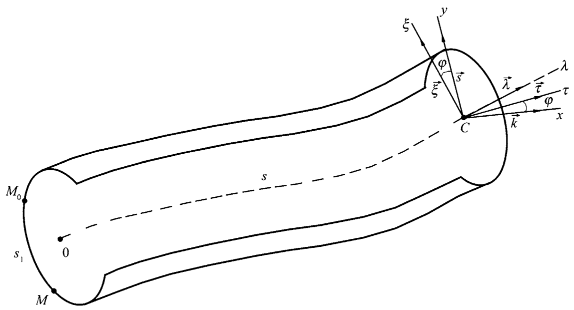

Fig. 3.1 Scheme of a spatially curved linear elastic beam of arbitrary open cross-section

$$\sigma_{ij}^{0} v_j = 0. \tag{3.10}$$

Equations 3.8–3.10 should be considered together with the relationship

$$\dot{\sigma}_{ij} = \lambda v_{l,l} \delta_{ij} + \mu \left(v_{i,j} + v_{j,i} \right), \tag{3.11}$$

where λ and μ are Lamé constants.

Suppose that the wave surface of strong discontinuity exists in a spatially curved thin-walled beam of open section. Then let us differentiate (3.8), (3.11) and (3.9) k times with respect to time, write them on the both sides of the wave surface, and take their difference. As a result we obtain

$$\left[\sigma_{ij,(k+1)}\right] = \lambda \left[v_{l,l(k)}\right] \delta_{ij} + \mu \left(\left[v_{i,j(k)}\right] + \left[v_{j,i(k)}\right] \right), \tag{3.12}$$

$$\left[\sigma_{ij,j(k)}\right] + \sigma_{jl}^{0} \left[e_{il,j(k-1)}\right] = \rho \left[v_{i,(k+1)}\right], \tag{3.13}$$

$$\left[\sigma_{ij,(k)}\right] v_j = 0, \tag{3.14}$$

where $e_{ij} = v_{i,j}$, and $\left[Z_{,(k)}\right] = \left(\partial^k Z / \partial t^k\right)^{+} - \left(\partial^k Z / \partial t^k\right)^{-}$.

Equation 3.13 lacks the discontinuities in the external forces, since they are continuous on the wave surface of strong discontinuity.

For the ease of further treatment, let us introduce two sets of coordinates: λ, τ, ξ with the unit vectors $\boldsymbol{\lambda}\{\lambda_i\}$, $\boldsymbol{\tau}\{\tau_i\}$, and $\boldsymbol{\xi}\{\xi_i\}$, and λ, x, y with the unit vectors $\boldsymbol{\lambda}$, $\mathbf{k}\{k_i\}$, and $\mathbf{s}\{s_i\}$. The axes λ, τ, ξ are the natural axes for the curved axis of the beam, in so doing the λ-axis is the tangent to the beam's axis, the τ-axis is its binormal, the ξ-axis is its main normal, s is the arc length calculated from a certain point with the coordinate s_0 along the beam axis (Fig. 3.1), while the x- and y-axes are the main central axes of the beam's normal section. The angle $\varphi(s)$ is the angle between the x- and τ-axes and the y- and ξ-axes.

Following [2], it can be shown that (for details see Appendix 1)

$$\frac{\mathrm{d}s_i}{\mathrm{d}s} = k_i(K+\tau) - \text{æ}\lambda_i\cos\varphi(s), \tag{3.15}$$

$$\frac{\mathrm{d}k_i}{\mathrm{d}s} = -s_i(K+\tau) + \text{æ}\lambda_i\sin\varphi(s), \tag{3.16}$$

$$\frac{\mathrm{d}\lambda_i}{\mathrm{d}s} = -k_i\text{æ}\sin\varphi(s) + s_i\text{æ}\cos\varphi(s), \tag{3.17}$$

where $K = \mathrm{d}\varphi/\mathrm{d}s$, while $\text{æ}(s)$ and $\tau(s)$ are the curvature and the torsion of the beam's axis, respectively. Note that it is precisely these three values, $K(s)$, $\text{æ}(s)$ and $\tau(s)$, that are of prime engineering interest in studying the spatially curved thin-walled beams of open profile. Moreover, as it will be shown in the further analysis, these values produce new features in the dynamic response of such beams as compared with straight thin-walled beams of open cross-section.

Considering the conditions of compatibility (see Sect. 6.1)

$$\left[\sigma_{ij,j(k)}\right] = -G^{-1}\left[\sigma_{ij,(k+1)}\right]\lambda_j + \frac{\mathrm{d}[\sigma_{ij,(k)}]}{\mathrm{d}s}\lambda_j + \frac{\partial\left[\sigma_{ij,(k)}\right]}{\partial x}k_j + \frac{\partial[\sigma_{ij,(k)}]}{\partial y}s_j, \tag{3.18}$$

$$\left[e_{il,j(k)}\right] = -G^{-1}\left[e_{il,(k+1)}\right]\lambda_j + \frac{\mathrm{d}[e_{il,(k)}]}{\mathrm{d}s}\lambda_j + \frac{\partial\left[e_{il,(k)}\right]}{\partial x}k_j + \frac{\partial\left[e_{il,(k)}\right]}{\partial y}s_j, \tag{3.19}$$

$$\left[e_{il,(k)}\right] = \left[v_{i,l(k)}\right] = -G^{-1}\left[v_{i,(k+1)}\right]\lambda_l + \frac{\mathrm{d}\left[v_{i,(k)}\right]}{\mathrm{d}s}\lambda_l + \frac{\partial\left[v_{i,(k)}\right]}{\partial x}k_l + \frac{\partial\left[v_{i,(k)}\right]}{\partial y}s_l, \tag{3.20}$$

as well as formulas (3.15)–(3.17), Eqs. 3.12 and 3.13 can be rewritten as

$$\begin{aligned}\left[\sigma_{ij,(k+1)}\right] = &-G^{-1}\lambda[v_{l,(k+1)}]\lambda_l\delta_{ij} - G^{-1}\mu\big([v_{i,(k+1)}]\lambda_j + [v_{j,(k+1)}]\lambda_i\big)\\ &+\lambda\left(\frac{\mathrm{d}[v_{l,(k)}]}{\mathrm{d}s}\lambda_l + \frac{\partial[v_{l,(k)}]}{\partial x}k_l + \frac{\partial[v_{l,(k)}]}{\partial y}s_l\right)\delta_{ij}\\ &+\mu\left(\frac{\mathrm{d}[v_{i,(k)}]}{\mathrm{d}s}\lambda_j + \frac{\mathrm{d}[v_{j,(k)}]}{\mathrm{d}s}\lambda_i + \frac{\partial[v_{i,(k)}]}{\partial x}k_j\right.\\ &\left.+\frac{\partial[v_{j,(k)}]}{\partial x}k_i + \frac{\partial[v_{i,(k)}]}{\partial y}s_j + \frac{\partial[v_{j,(k)}]}{\partial y}s_i\right),\end{aligned} \tag{3.21}$$

$$\begin{aligned}\rho[v_{i,(k+1)}] = &-G^{-1}[\sigma_{ij,(k+1)}]\lambda_j + \frac{\mathrm{d}[\sigma_{ij,(k)}]}{\mathrm{d}s}\lambda_j + \frac{\partial[\sigma_{ij,(k)}]}{\partial x}k_j + \frac{\partial[\sigma_{ij,(k)}]}{\partial y}s_j\\ &+G^{-2}[v_{i,(k+1)}]\sigma^0_{\lambda\lambda} - 2G^{-1}\frac{\mathrm{d}[v_{i,(k)}]}{\mathrm{d}s}\sigma^0_{\lambda\lambda} - 2G^{-1}\frac{\partial[v_{i,(k-1)}]}{\partial x}\sigma^0_{\lambda x}\\ &-2G^{-1}\frac{\partial[v_{i,(k-1)}]}{\partial y}\sigma^0_{\lambda y} + \frac{\mathrm{d}^2[v_{i,(k-1)}]}{\mathrm{d}s^2}\sigma^0_{\lambda\lambda} + 2\frac{\partial}{\partial x}\left(\frac{\mathrm{d}[v_{i,(k-1)}]}{\mathrm{d}s}\right)\sigma^0_{\lambda x}\end{aligned}$$

$$+ 2\frac{\partial}{\partial y}\left(\frac{\mathrm{d}[v_{i,(k-1)}]}{\mathrm{d}s}\right)\sigma^0_{\lambda y} + \frac{\partial[v_{i,(k-1)}]}{\partial x}\left\{-(K+\tau)\sigma^0_{\lambda y} + æ\sigma^0_{\lambda\lambda}\sin\varphi\right\}$$
$$+ \frac{\partial[v_{i,(k-1)}]}{\partial y}\{(K+\tau)\sigma^0_{\lambda x} - æ\sigma^0_{\lambda\lambda}\cos\varphi\}$$
$$+ \left(G^{-1}[v_{i,(k)}] - \frac{\mathrm{d}[v_{i,(k-1)}]}{\mathrm{d}s}\right)æ\left(\sigma^0_{\lambda x}\sin\varphi - \sigma^0_{\lambda y}\cos\varphi\right), \tag{3.22}$$

where $\sigma^0_{\lambda\lambda} = \sigma^0_{ij}\lambda_i\lambda_j$, $\sigma^0_{\lambda x} = \sigma^0_{ij}\lambda_i k_j$, and $\sigma^0_{\lambda y} = \sigma^0_{ij}\lambda_i s_j$.

Analysis of the influence of the preliminary shear stresses $\sigma^0_{\lambda x}$ and $\sigma^0_{\lambda y}$ and the axial compression or tension stresses $\sigma^0_{\lambda\lambda}$ and their different combinations on the dynamic response of thin-walled rods of open profile is a very complicated problem which is of great importance in civil engineering and bridge construction.

However, reference to Eq. 3.22 shows that the value $\sigma^0_{\lambda\lambda}$ is multiplied by the $k+1$-order jump and by the s-derivative of the k-order, i.e. it has a strong impact both on the velocities of the surfaces of strong discontinuity and on the discontinuities, in contrast to the values $\sigma^0_{\lambda x}$ and $\sigma^0_{\lambda y}$ which are multiplied only by the k- and $k-1$-order jumps, i.e. they exert weak effect only on the jumps. That is why, below in Sect. 3.1.2 , we shall neglect $\sigma^0_{\lambda x}$ and $\sigma^0_{\lambda y}$ with respect to the value $\sigma^0_{\lambda\lambda}$.

3.1.2 Dynamic Response of Axially Pre-Stressed Spatially Curved Thin-Walled Beams of Open Profile

Thus, assuming further that $\sigma^0_{\lambda x} = \sigma^0_{\lambda y} = 0$ and $\sigma^0_{\lambda\lambda} \neq 0$, from (3.22) we have

$$\rho[v_{i,(k+1)}] = -G^{-1}[\sigma_{ij,(k+1)}]\lambda_j + \frac{\mathrm{d}[\sigma_{ij,(k)}]}{\mathrm{d}s}\lambda_j + \frac{\partial[\sigma_{ij,(k)}]}{\partial x}k_j + \frac{\partial[\sigma_{ij,(k)}]}{\partial y}s_j$$
$$+ \left(G^{-2}[v_{i,(k+1)}] - 2G^{-1}\frac{\mathrm{d}[v_{i,(k)}]}{\mathrm{d}s} + \frac{\mathrm{d}^2[v_{i,(k-1)}]}{\mathrm{d}s^2} + æ\frac{\partial[v_{i,(k-1)}]}{\partial x}\sin\varphi - æ\frac{\partial[v_{i,(k-1)}]}{\partial y}\cos\varphi\right)\sigma^0_{\lambda\lambda}. \tag{3.23}$$

To satisfy Eq. 3.14, it is sufficient to put

$$[\sigma_{ij,(k)}]k_ik_j = 0, \tag{3.24}$$

$$[\sigma_{ij,(k)}]s_is_j = 0, \tag{3.25}$$

$$[\sigma_{ij,(k)}]s_ik_j = 0. \tag{3.26}$$

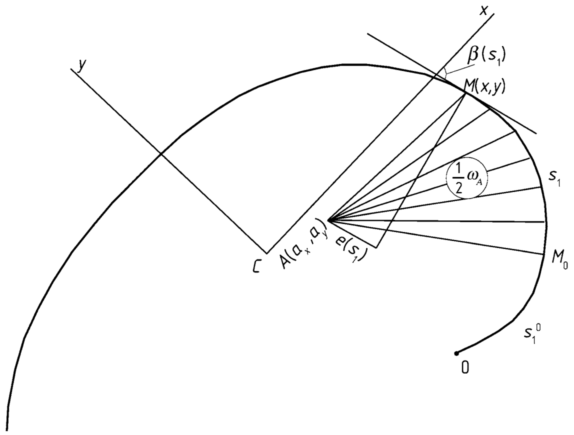

Fig. 3.2 Scheme of the cross section of a thin-walled beam with a generic open cross-section

Then the boundary surface will be free from the normal and tangential stresses.

Now we expand the value $[v_{i,(k)}]$ entering into (3.21) and (3.23)–(3.26) in terms of three mutually orthogonal vectors $\boldsymbol{\lambda}$, $\mathbf{k}$, $\mathbf{s}$. As a result we obtain

$$\left[v_{i,(k)}\right] = \omega_{(k)}\lambda_i + \theta_{(k)}k_i + \eta_{(k)}s_i, \tag{3.27}$$

where $\omega_{(k)} = [v_{i,(k)}]\lambda_i,\ \theta_{(k)} = [v_{i,(k)}]k_i$, and $\eta_{(k)} = [v_{i,(k)}]s_i$.

Suppose that for a thin-walled beam of open section the following velocity fields (involving seven independent functions at each fixed k) are fulfilled [3]:

$$\omega_{(k)} = \omega_{(k)}(s,s_1) = \omega^0_{(k)}(s) + \widetilde{\omega}^{1x}_{(k)}(s)y(s_1) + \widetilde{\omega}^{1y}_{(k)}(s)x(s_1) + \widetilde{\psi}_{(k)}\omega_A(s_1), \tag{3.28}$$

$$\theta_{(k)} = \theta_{(k)}(s,s_1) = \theta^0_{(k)}(s) - \omega^{1\lambda}_{(k)}(s)(y(s_1) - a_y), \tag{3.29}$$

$$\eta_{(k)} = \eta_{(k)}(s,s_1) = \eta^0_{(k)}(s) + \omega^{1\lambda}_{(k)}(s)(x(s_1) - a_x), \tag{3.30}$$

where s_1 is the arc length measured along the cross-section profile from the point M_0, which corresponds to the arc length of s_1^0, to the point $M[x(s_1), y(s_1)]$, which corresponds to the arc length of s_1 (Fig. 3.2).

Gol'denveizer [3] proposed that the angles of in-plane rotation do not coincide with the first derivatives of the lateral displacement components and, analogously, warping does not coincide with the first derivative of the torsional rotation. Thus Gol'denveizer [3] pioneered in combining Timoshenko's beam theory [4] and Vlasov thin-walled beam theory [5] and suggested to characterize the displacements of the thin-walled beam's cross-section by seven generalized displacements.

In order to refine the structure of formula (3.28) in the case of a spatially curved thin-walled beam, let us differentiate (3.27), which involves only the terms $\omega^0_{(k)}$, $\theta^0_{(k)}$, and $\eta^0_{(k)}$, with respect to s with due account for (3.15)–(3.17). As a result we obtain

$$\frac{\mathrm{d}\left[v_{i(k)}\right]^0}{\mathrm{d}s} = \left(\frac{\mathrm{d}\omega^0_{(k)}}{\mathrm{d}s} + \theta^0_{(k)}\text{æ}\sin\varphi - \eta^0_{(k)}\text{æ}\cos\varphi\right)\lambda_i + \left(\frac{\mathrm{d}\theta^0_{(k)}}{\mathrm{d}s} - \omega^0_{(k)}\text{æ}\sin\varphi + \eta^0_{(k)}(K+\tau)\right)k_i + \left(\frac{\mathrm{d}\eta^0_{(k)}}{\mathrm{d}s} + \omega^0_{(k)}\text{æ}\cos\varphi - \theta^0_{(k)}(K+\tau)\right)s_i. \tag{3.31}$$

Moreover, we shall use the relationship for the discontinuity in the k-order derivative with respect to time t of the angular velocity

$$\left[\Omega_{i(k)}\right] = \omega^{1\lambda}_{(k)}\lambda_i + \widetilde{\omega}^{1x}_{(k)}k_i + \widetilde{\omega}^{1y}_{(k)}s_i. \tag{3.32}$$

Differentiating (3.32) with respect to s and considering (3.15)–(3.17) yields

$$\frac{\mathrm{d}[\Omega_{i(k)}]}{\mathrm{d}s} = \left(\frac{\mathrm{d}\omega^{1\lambda}_{(k)}}{\mathrm{d}s} + \widetilde{\omega}^{1x}_{(k)}\text{æ}\sin\varphi - \widetilde{\omega}^{1y}_{(k)}\text{æ}\cos\varphi\right)\lambda_i + \left(\frac{\mathrm{d}\widetilde{\omega}^{1x}_{(k)}}{\mathrm{d}s} - \omega^{1\lambda}_{(k)}\text{æ}\sin\varphi + \widetilde{\omega}^{1y}_{(k)}(K+\tau)\right)k_i + \left(\frac{\mathrm{d}\widetilde{\omega}^{1y}_{(k)}}{\mathrm{d}s} + \omega^{1\lambda}_{(k)}\text{æ}\cos\varphi - \widetilde{\omega}^{1x}_{(k)}(K+\tau)\right)s_i. \tag{3.33}$$

If we suppose that transverse shear strains are absent, then we are led to the following relationships, which are in compliance with the Vlasov theory:

$$\widetilde{\omega}^{1x}_{(k)} = -\left(\frac{\mathrm{d}\eta^0_{(k)}}{\mathrm{d}s} + \omega^0_{(k)}\text{æ}\cos\varphi - \theta^0_{(k)}(K+\tau)\right), \tag{3.34}$$

$$\widetilde{\omega}^{1y}_{(k)} = -\left(\frac{\mathrm{d}\theta^0_{(k)}}{\mathrm{d}s} - \omega^0_{(k)}\text{æ}\sin\varphi + \eta^0_{(k)}(K+\tau)\right), \tag{3.35}$$

$$\begin{aligned}\widetilde{\psi}_{(k)} &= -\left(\frac{\mathrm{d}\omega^{1\lambda}_{(k)}}{\mathrm{d}s} + \widetilde{\omega}^{1x}_{(k)}\text{æ}\sin\varphi - \widetilde{\omega}^{1y}_{(k)}\text{æ}\cos\varphi\right) \\ &= -\frac{\mathrm{d}\omega^{1\lambda}_{(k)}}{\mathrm{d}s} + \frac{\mathrm{d}\eta^0_{(k)}}{\mathrm{d}s}\text{æ}\sin\varphi - \frac{\mathrm{d}\theta^0_{(k)}}{\mathrm{d}s}\text{æ}\cos\varphi \\ &\quad + \omega^0_{(k)}\text{æ}^2\sin\varphi - \left(\theta^0_{(k)}\sin\varphi + \eta^0_{(k)}\cos\varphi\right)\text{æ}(K+\tau).\end{aligned} \tag{3.36}$$

If we now substitute (3.34)–(3.36) in (3.28), then we obtain a formula for defining the k-order time-derivative in the discontinuity in the velocity of translation along the λ-axis without account for transverse shear deformations.

Following Gol'denveizer [3], in order to take the transverse shear deformations into account, it is essential to substitute the derivatives $\mathrm{d}\omega^{1\lambda}_{(k)}/\mathrm{d}s$, $\mathrm{d}\eta^{0}_{(k)}/\mathrm{d}s$, and $\mathrm{d}\theta^{0}_{(k)}/\mathrm{d}s$ by the independent functions $\psi_{(k)}$, $\omega^{1x}_{(k)}$, and $\omega^{1y}_{(k)}$, respectively, i.e. to rewrite Eqs. 3.34–3.36 as

$$\widetilde{\omega}^{1x}_{(k)} = -\Big(\omega^{1x}_{(k)} + \omega^{0}_{(k)}æ\cos\varphi - \theta^{0}_{(k)}(K+\tau)\Big), \tag{3.37}$$

$$\widetilde{\omega}^{1y}_{(k)} = -\Big(\omega^{1y}_{(k)} - \omega^{0}_{(k)}æ\sin\varphi + \eta^{0}_{(k)}(K+\tau)\Big), \tag{3.38}$$

$$\begin{aligned}\widetilde{\psi}_{(k)} &= -\Big(\psi_{(k)} + \widetilde{\omega}^{1x}_{(k)}æ\sin\varphi - \widetilde{\omega}^{1y}_{(k)}æ\cos\varphi\Big)\\ &= -\psi_{(k)} + \omega^{1x}_{(k)}æ\sin\varphi - \omega^{1y}_{(k)}æ\cos\varphi\\ &\quad + \omega^{0}_{(k)}æ^2\sin\varphi - \Big(\theta^{0}_{(k)}\sin\varphi + \eta^{0}_{(k)}\cos\varphi\Big)æ(K+\tau).\end{aligned} \tag{3.39}$$

As a result, instead of formula (3.28), we obtain

$$\begin{aligned}\omega_{(k)}(s,s_1) &= \omega^{0}_{(k)}(s) - \Big[\omega^{1x}_{(k)}(s) + \omega^{0}_{(k)}æ\cos\varphi - \theta^{0}_{(k)}(K+\tau)\Big]y(s_1)\\ &\quad - \Big[\omega^{1y}_{(k)}(s) - \omega^{0}_{(k)}æ\sin\varphi + \eta^{0}_{(k)}(K+\tau)\Big]x(s_1)\\ &\quad + \Big[-\psi_{(k)}(s) + \omega^{1x}_{(k)}(s)æ\sin\varphi - \omega^{1y}_{(k)}(s)æ\cos\varphi + \omega^{0}_{(k)}æ^2\sin 2\varphi\\ &\quad - \Big(\theta^{0}_{(k)}\sin\varphi + \eta^{0}_{(k)}\cos\varphi\Big)æ(K+\tau)\Big]\omega_A(s_1).\end{aligned} \tag{3.40}$$

Formulas (3.29) and (3.30) remain unchanged.

If we put $K = æ = \tau = 0$ in (3.40), then we are led to the case of a straight thin-walled rod of open profile.

In further treatment we will use formulas (3.28)–(3.30), but in the final relationships we will carry out the substitution of the values $\widetilde{\psi}_{(k)}$, $\widetilde{\omega}^{1x}_{(k)}$, and $\widetilde{\omega}^{1y}_{(k)}$ by their magnitudes defined by formulas (3.37)–(3.39).

At $k = 0$, the values entering in (3.29), (3.30) and (3.40) have the following physical meaning: $\omega^{0}_{(0)}(s)$, $\theta^{0}_{(0)}(s)$ and $\eta^{0}_{(0)}(s)$ are the discontinuities in the velocities of translatory motion of the section as a rigid body together with the point C (the center of gravity of the cross-section) along the λ-, x- and y-axes, respectively, $\omega^{1x}_{(0)}(s)$, $\omega^{1y}_{(0)}(s)$ and $\omega^{1\lambda}_{(0)}(s)$ are the discontinuities in the angular velocities of the cross-section's rotation as a rigid whole around the x-, y- and λ-axes, respectively, $\psi_{(0)}(s)$ is the discontinuity in the velocity of warping of the cross section, $\omega_A(s_1)$ is twice the area of the sector AM_0M, the point A is the center of bending (Fig. 3.2) with the coordinates [5]

$$a_x = \frac{1}{I_x}\int\limits_F \omega_C(s_1)y(s_1)\mathrm{d}F, \quad a_y = -\frac{1}{I_y}\int\limits_F \omega_C(s_1)x(s_1)\mathrm{d}F. \tag{3.41}$$

Since the section is referred to the main central axes, then

$$I_{xy} = \int\limits_F x(s_1)y(s_1)\mathrm{d}F = 0, \quad S_y = \int\limits_F x(s_1)\mathrm{d}F = 0, \quad S_x = \int\limits_F y(s_1)\mathrm{d}F = 0. \tag{3.42}$$

Moreover, the choice of the point A with the coordinates defined by formulas (3.41) results in the fulfillment of the following relationships:

$$I_{\omega x} = \int\limits_F x(s_1)\omega_A(s_1)\mathrm{d}F = 0, \quad I_{\omega y} = \int\limits_F y(s_1)\omega_A(s_1)\mathrm{d}F = 0. \tag{3.43}$$

Note that the sectorial area, generally speaking, depends on two coordinates, namely: the initial point s_1^0 and the terminal point s_1^1, i.e. $\omega_A(s_1^0, s_1^1)$. Let us choose s_1^1 in such a way that the relationship

$$\omega_A(s_1^0, s_1^1) = \frac{1}{F}\int\limits_F \omega_A(s_1^0, s_1)\mathrm{d}F \tag{3.44}$$

will be valid. Then, as it is shown in [5], taking the point with the coordinate s_1^1 as the origin of the arc length measuring, we obtain

$$S_\omega = \int\limits_F \omega_A(s_1^1, s_1)\mathrm{d}F = 0. \tag{3.45}$$

Let us name this point as the null sectorial point [5]. Since there may exist several such points, the null sectorial point nearest to the point A can be named as the main null sectorial point, and we shall take it as the initial point of reading.

Thus, using the above mentioned choice of the coordinates and the initial points of measuring, the functions 1, $x(s_1)$, $y(s_1)$, and $\omega_A(s_1^1, s_1)$, as formulas (3.42)–(3.45) show, occur to be orthogonal.

If we substitute (3.21) in (3.24) and (3.25) and consider (3.27), then we are led to the relationships

$$[\varepsilon_{x,(k)}] = [\varepsilon_{y,(k)}] = \frac{\lambda}{2(\lambda+\mu)}\left(G^{-1}\omega_{(k)} - \frac{\mathrm{d}\omega_{(k-1)}}{\mathrm{d}s} + æ\eta_{(k-1)}\cos\varphi - æ\theta_{(k-1)}\sin\varphi\right), \tag{3.46}$$

where

$$[\varepsilon_{x,(k)}] = \frac{\partial\left([v_{i,(k-1)}]k_i\right)}{\partial x}, \quad [\varepsilon_{y,(k)}] = \frac{\partial\left([v_{i,(k-1)}]s_i\right)}{\partial y}.$$

Equation 3.26, after the substitution of (3.21), (3.27), (3.29), and (3.30) in it, is fulfilled unconditionally.

Multiplying (3.21) by $\lambda_i \lambda_j$ and considering (3.15)–(3.17) and (3.46) yields

$$\left[\sigma_{ij,(k+1)}\right]\lambda_i\lambda_j = -G^{-1}E\omega_{(k+1)} + E\frac{\mathrm{d}\omega_{(k)}}{\mathrm{d}s} - E\text{æ}\Big(\eta_{(k)}\cos\varphi - \theta_{(k)}\sin\varphi\Big). \quad (3.47)$$

Multiplying (3.21) successively by $\lambda_i k_j$ and $\lambda_i s_j$ and considering (3.15)–(3.17), we obtain

$$\left[\sigma_{ij,(k+1)}\right]\lambda_i k_j = -G^{-1}\mu\theta_{(k+1)} + \mu\frac{\mathrm{d}\theta_{(k)}}{\mathrm{d}s} + \mu\frac{\partial\omega_{(k)}}{\partial x} + \mu(K+\tau)\eta_{(k)} - \mu\text{æ}\omega_{(k)}\sin\varphi, \quad (3.48)$$

$$\left[\sigma_{ij,(k+1)}\right]\lambda_i s_j = -G^{-1}\mu\eta_{(k+1)} + \mu\frac{\mathrm{d}\eta_{(k)}}{\mathrm{d}s} + \mu\frac{\partial\omega_{(k)}}{\partial y} - \mu(K+\tau)\theta_{(k)} + \mu\text{æ}\omega_{(k)}\cos\varphi. \quad (3.49)$$

Multiplying (3.23) successively by $\lambda_i,\ k_i$, and s_i and considering (3.15)–(3.17), we find

$$\begin{aligned}\rho\omega_{(k+1)} = &-G^{-1}[\sigma_{ij,(k+1)}]\lambda_i\lambda_j + \frac{\mathrm{d}([\sigma_{ij,(k)}]\lambda_i\lambda_j)}{\mathrm{d}s} + \frac{\partial([\sigma_{ij,(k)}]\lambda_i k_j)}{\partial x}\\ &+ \frac{\partial([\sigma_{ij,(k)}]\lambda_i s_j)}{\partial y} + 2\text{æ}\big([\sigma_{ij,(k)}]\lambda_i k_j\sin\varphi - [\sigma_{ij,(k)}]\lambda_i s_j\cos\varphi\big)\\ &+ G^{-1}\left\{G^{-1}\omega_{(k+1)} - 2\frac{\mathrm{d}\omega_{(k)}}{\mathrm{d}s} - 2\text{æ}\Big(\theta_{(k)}\sin\varphi - \eta_{(k)}\cos\varphi\Big)\right\}\sigma^0_{\lambda\lambda}\\ &+ f_{i(k-1)}\lambda_i\sigma^0_{\lambda\lambda},\end{aligned} \quad (3.50)$$

$$\begin{aligned}\rho\theta_{(k+1)} = &-G^{-1}[\sigma_{ij,(k+1)}]\lambda_i k_j + \frac{\mathrm{d}([\sigma_{ij,(k)}]\lambda_i k_j)}{\mathrm{d}s} + (K+\tau)[\sigma_{ij,(k)}]\lambda_i s_j\\ &+ G^{-1}\left\{G^{-1}\theta_{(k+1)} - 2\,\frac{\mathrm{d}\theta_{(k)}}{\mathrm{d}s} - 2\Big(\eta_{(k)}(K+\tau) - \text{æ}\omega_{(k)}\sin\varphi\Big)\right\}\sigma^0_{\lambda\lambda}\\ &- \text{æ}[\sigma_{ij,(k)}]\lambda_i\lambda_j\sin\varphi + f_{i(k-1)}k_i\sigma^0_{\lambda\lambda},\end{aligned} \quad (3.51)$$

$$\begin{aligned}\rho\eta_{(k+1)} = &-G^{-1}[\sigma_{ij,(k+1)}]\lambda_i s_j + \frac{\mathrm{d}([\sigma_{ij,(k)}]\lambda_i s_j)}{\mathrm{d}s} - (K+\tau)[\sigma_{ij,(k)}]\lambda_i k_j\\ &+ G^{-1}\left\{G^{-1}\eta_{(k+1)} - 2\frac{\mathrm{d}\eta_{(k)}}{\mathrm{d}s} + 2\big(\theta_{(k)}(K+\tau) - \text{æ}\omega_{(k)}\cos\varphi\big)\right\}\sigma^0_{\lambda\lambda}\\ &+ \text{æ}[\sigma_{ij,(k)}]\lambda_i\lambda_j\cos\varphi + f_{i(k-1)}s_i\sigma^0_{\lambda\lambda},\end{aligned} \quad (3.52)$$

where functions $f_{i(k-1)}$ are presented in Appendix 2.

Considering (3.48), (3.49) and (3.28)–(3.30), we have

$$\frac{\partial([\sigma_{ij,(k)}]\lambda_i k_j)}{\partial x} = \mu(K+\tau)\omega^{1\lambda}_{(k-1)} - \mu æ \widetilde{\omega}^{1y}_{(k-1)} \sin\varphi, \tag{3.53}$$

$$\frac{\partial([\sigma_{ij,(k)}]\lambda_i s_j)}{\partial y} = \mu(K+\tau)\omega^{1\lambda}_{(k-1)} + \mu æ \widetilde{\omega}^{1x}_{(k-1)} \cos\varphi. \tag{3.54}$$

Substituting (3.28)–(3.30) in (3.47)–(3.49) and integrating the final equations over the cross-sectional area of the beam, we obtain

$$N_{\lambda(k+1)} = \int_F [\sigma_{ij,(k+1)}]\lambda_i\lambda_j \mathrm{d}F = -G^{-1}EF\omega^0_{(k+1)} + EF\frac{\mathrm{d}\omega^0_{(k)}}{\mathrm{d}s} - EFæ\left(\eta^0_{(k)}\cos\varphi - \theta^0_{(k)}\sin\varphi\right) + EFæ\left(a_x\cos\varphi + a_y\sin\varphi\right)\omega^{1\lambda}_{(k)}, \tag{3.55}$$

$$Q_{\lambda x(k+1)} = \int_F [\sigma_{ij,(k+1)}]\lambda_i k_j \mathrm{d}F = -G^{-1}\mu F\theta^0_{(k+1)} + \mu F\frac{\mathrm{d}\theta^0_{(k)}}{\mathrm{d}s} + \mu F\widetilde{\omega}^{1y}_{(k)} + \mu F(K+\tau)\eta^0_{(k)} - \mu Fæ\omega^0_{(k)}\sin\varphi - G^{-1}\mu F\omega^{1\lambda}_{(k+1)}a_y + \mu F\frac{\mathrm{d}\omega^{1\lambda}_{(k)}}{\mathrm{d}s}a_y - \mu F(K+\tau)\omega^{1\lambda}_{(k)}a_x, \tag{3.56}$$

$$Q_{\lambda y(k+1)} = \int_F [\sigma_{ij,(k+1)}]\lambda_i s_j \mathrm{d}F = -G^{-1}\mu F\eta^0_{(k+1)} + \mu F\frac{\mathrm{d}\eta^0_{(k)}}{\mathrm{d}s} + \mu F\widetilde{\omega}^{1x}_{(k)} - \mu F(K+\tau)\theta^0_{(k)} + \mu Fæ\omega^0_{(k)}\cos\varphi + G^{-1}\mu F\omega^{1\lambda}_{(k+1)}a_x - \mu F\frac{\mathrm{d}\omega^{1\lambda}_{(k)}}{\mathrm{d}s}a_x - \mu F(K+\tau)\omega^{1\lambda}_{(k)}a_y. \tag{3.57}$$

Substituting (3.28)–(3.30) in (3.50)–(3.52) and then integrating over the cross-sectional area, we find

$$\rho F\omega^0_{(k+1)} = -G^{-1}N_{\lambda(k+1)} + \frac{\mathrm{d}N_{\lambda(k)}}{\mathrm{d}s} + 2æQ_{\lambda x(k)}\sin\varphi - 2æQ_{\lambda y(k)}\cos\varphi + G^{-1}F\left\{G^{-1}\omega^0_{(k+1)} - 2\frac{\mathrm{d}\omega^0_{(k)}}{\mathrm{d}s} - 2\left(\theta^0_{(k)}æ\sin\varphi - \eta^0_{(k)}æ\cos\varphi\right)\right\}\sigma^0_{\lambda\lambda} + \mu Fæ\left(\widetilde{\omega}^{1x}_{(k-1)}\cos\varphi - \widetilde{\omega}^{1y}_{(k-1)}\sin\varphi\right) + 2\mu F(K+\tau)\omega^{1\lambda}_{(k-1)} - 2G^{-1}Fæ\left(a_y\sin\varphi + a_x\cos\varphi\right)\omega^{1\lambda}_{(k)}\sigma^0_{\lambda\lambda} + Ff^0_{i(k-1)}\lambda_i\sigma^0_{\lambda\lambda}, \tag{3.58}$$

$$\begin{aligned}\rho F\theta^0_{(k+1)} &= -G^{-1}Q_{\lambda x(k+1)} + \frac{\mathrm{d}Q_{\lambda x(k)}}{\mathrm{d}s} + (K+\tau)Q_{\lambda y(k)} - æN_{\lambda(k)}\sin\varphi \\ &\quad + G^{-1}F\left\{G^{-1}\theta^0_{(k+1)} - 2\frac{\mathrm{d}\theta^0_{(k)}}{\mathrm{d}s} - 2\left(\eta^0_{(k)}(K+\tau) - \omega^0_{(k)}æ\sin\varphi\right)\right\}\sigma^0_{\lambda\lambda} \\ &\quad - \rho F\omega^{1\lambda}_{(k+1)}a_y + G^{-2}F\omega^{1\lambda}_{(k+1)}a_y\sigma^0_{\lambda\lambda} - 2G^{-1}F\frac{\mathrm{d}\omega^{1\lambda}_{(k)}}{\mathrm{d}s}a_y\sigma^0_{\lambda\lambda} \\ &\quad + 2G^{-1}Fa_x(K+\tau)\omega^{1\lambda}_{(k)}\sigma^0_{\lambda\lambda} + Ff^0_{i(k-1)}k_i\sigma^0_{\lambda\lambda},\end{aligned} \tag{3.59}$$

$$\begin{aligned}\rho F\eta^0_{(k+1)} &= -G^{-1}Q_{\lambda y(k+1)} + \frac{\mathrm{d}Q_{\lambda y(k)}}{\mathrm{d}s} - (K+\tau)Q_{\lambda x(k)} + æN_{\lambda(k)}\cos\varphi \\ &\quad + G^{-1}F\left\{G^{-1}\eta^0_{(k+1)} - 2\frac{\mathrm{d}\eta^0_{(k)}}{\mathrm{d}s} + 2\left(\theta^0_{(k)}(K+\tau) - \omega^0_{(k)}æ\cos\varphi\right)\right\}\sigma^0_{\lambda\lambda} \\ &\quad + \rho F\omega^{1\lambda}_{(k+1)}a_x - G^{-2}F\omega^{1\lambda}_{(k+1)}a_x\sigma^0_{\lambda\lambda} + 2G^{-1}F\frac{\mathrm{d}\omega^{1\lambda}_{(k)}}{\mathrm{d}s}a_x\sigma^0_{\lambda\lambda} \\ &\quad + 2G^{-1}Fa_y(K+\tau)\omega^{1\lambda}_{(k)}\sigma^0_{\lambda\lambda} + Ff^0_{i(k-1)}s_i\sigma^0_{\lambda\lambda}.\end{aligned} \tag{3.60}$$

Substituting (3.55)–(3.57) in (3.58)–(3.60) with due account for (3.37)–(3.39) yields

$$\begin{aligned}G^{-2}(\rho G^2 - \rho G_1^2 - \sigma^0_{\lambda\lambda})\omega^0_{(k+1)} &= -2G^{-1}\left(\rho G_1^2 + \sigma^0_{\lambda\lambda}\right)\frac{\mathrm{d}\omega^0_{(k)}}{\mathrm{d}s} \\ &\quad - G^{-1}\left(\rho G_1^2 + 2\rho G_2^2 + 2\sigma^0_{\lambda\lambda}\right)æ\Big[\left(a_x\cos\varphi + a_y\sin\varphi\right)\omega^{1\lambda}_{(k)} \\ &\quad - \eta^0_{(k)}\cos\varphi + \theta^0_{(k)}\sin\varphi\Big] + F_{1(k-1)},\end{aligned} \tag{3.61}$$

$$\begin{aligned}&G^{-2}(\rho G^2 - \rho G_2^2 - \sigma^0_{\lambda\lambda})(\theta^0_{(k+1)} + a_y\omega^{1\lambda}_{(k+1)}) \\ &\quad = -2G^{-1}(\rho G_2^2 + \sigma^0_{\lambda\lambda}) \\ &\qquad \times \left\{\frac{\mathrm{d}}{\mathrm{d}s}\left(\theta^0_{(k)} + a_y\omega^{1\lambda}_{(k)}\right) + (K+\tau)\left(\eta^0_{(k)} - a_x\omega^{1\lambda}_{(k)}\right)\right\} \\ &\qquad + G^{-1}\left(\rho G_1^2 + \rho G_2^2 + 2\sigma^0_{\lambda\lambda}\right)æ\omega^0_{(k)}\sin\varphi \\ &\qquad + G^{-1}\rho G_2^2\left\{\omega^{1y}_{(k)} - \omega^0_{(k)}æ\sin\phi + \eta^0_{(k)}(K+\tau)\right\} + F_{2(k-1)},\end{aligned} \tag{3.62}$$

$$G^{-2}(\rho G^2 - \rho G_2^2 - \sigma_{\lambda\lambda}^0)(\eta_{(k+1)}^0 - a_x\omega_{(k+1)}^{1\lambda})$$
$$= -2G^{-1}\left(\rho G_2^2 + \sigma_{\lambda\lambda}^0\right)\left\{\frac{\mathrm{d}}{\mathrm{d}s}\left(\eta_{(k)}^0 - a_x\omega_{(k)}^{1\lambda}\right) - (K+\tau)\left(\theta_{(k)}^0 + a_y\omega_{(k)}^{1\lambda}\right)\right\}$$
$$- G^{-1}\left(\rho G_1^2 + \rho G_2^2 + 2\sigma_{\lambda\lambda}^0\right)æ\omega_{(k)}^0\cos\varphi$$
$$+ G^{-1}\rho G_2^2\left\{\omega_{(k)}^{1x} + \omega_{(k)}^0 æ\cos\phi - \theta_{(k)}^0(K+\tau)\right\} + F_{3(k-1)}, \tag{3.63}$$

where $G_1^2 = E\rho^{-1},\ G_2^2 = \mu\rho^{-1}$, and functions $F_{j(k-1)}$ $(j = 1,2,3)$ are presented in Appendix 2.

Let us substitute formulas (3.28)–(3.30) in (3.47)–(3.49), then multiply (3.47) successively by x, y, and ω_A, and (3.48) by $(y - a_y)$ and (3.49) by $(x - a_x)$, and integrate the obtained equations over the cross-sectional area of the beam. As a result we obtain

$$M_{x(k+1)} = \int_F [\sigma_{ij,(k+1)}]\lambda_i\lambda_j y\mathrm{d}F = -G^{-1}\rho G_1^2 I_x\widetilde{\omega}_{(k+1)}^{1x}$$
$$+ \rho G_1^2 I_x\frac{\mathrm{d}\widetilde{\omega}_{(k)}^{1x}}{\mathrm{d}s} - æI_x\rho G_1^2\omega_{(k)}^{1\lambda}\sin\varphi, \tag{3.64}$$

$$M_{y(k+1)} = \int_F [\sigma_{ij,(k+1)}]\lambda_i\lambda_j x\mathrm{d}F = -G^{-1}\rho G_1^2 I_y\widetilde{\omega}_{(k+1)}^{1y}$$
$$+ \rho G_1^2 I_y\frac{\mathrm{d}\widetilde{\omega}_{(k)}^{1y}}{\mathrm{d}s} - æ\rho G_1^2 I_y\omega_{(k)}^{1\lambda}\cos\varphi, \tag{3.65}$$

$$B_{(k+1)} = \int_F [\sigma_{ij,(k+1)}]\lambda_i\lambda_j\omega_A\mathrm{d}F = -G^{-1}\rho G_1^2 I_\omega\widetilde{\psi}_{(k+1)} + \rho G_1^2 I_\omega\frac{\mathrm{d}\widetilde{\psi}_{(k)}}{\mathrm{d}s}, \tag{3.66}$$

$$M_{A(k+1)} = \int_F [\sigma_{ij,(k+1)}]\lambda_i s_j(x - a_x)\mathrm{d}F - \int_F [\sigma_{ij,(k+1)}]\lambda_i k_j(y - a_y)\mathrm{d}F$$
$$= \int_F [\sigma_{ij,(k+1)}]\lambda_i s_j x\mathrm{d}F - \int_F [\sigma_{ij,(k+1)}]\lambda_i k_j y\mathrm{d}F - a_x\int_F [\sigma_{ij,(k+1)}]\lambda_i s_j\mathrm{d}F$$
$$+ a_y\int_F [\sigma_{ij,(k+1)}]\lambda_i k_j\mathrm{d}F = M_{C(k+1)} - a_x Q_{\lambda y(k+1)} + a_y Q_{\lambda x(k+1)}$$
$$= -G^{-1}\rho G_2^2 I_p^A\omega_{(k+1)}^{1\lambda} + \rho G_2^2 I_p^A\frac{\mathrm{d}\omega_{(k)}^{1\lambda}}{\mathrm{d}s} - \rho G_2^2 Fæ\omega_{(k)}^0\left(a_x\cos\varphi + a_y\sin\varphi\right)$$

$$+ a_x F \rho G_2^2 \left\{ G^{-1} \eta^0_{(k+1)} - \frac{d\eta^0_{(k)}}{ds} + (K + \tau)\theta^0_{(k)} \right\}$$
$$+ a_y F \rho G_2^2 \left\{ -G^{-1} \theta^0_{(k+1)} + \frac{d\theta^0_{(k)}}{ds} + (K + \tau)\eta^0_{(k)} \right\}$$
$$+ \rho G_2^2 æ \left(\widetilde{\omega}^{1y}_{(k)} I_y \cos\varphi + \widetilde{\omega}^{1x}_{(k)} I_x \sin\varphi \right) + \rho G_2^2 F \left(a_y \widetilde{\omega}^{1y}_{(k)} - a_x \widetilde{\omega}^{1x}_{(k)} \right), \qquad (3.67)$$

where $M_{x(k+1)}$ and $M_{y(k+1)}$ are the discontinuities in the derivatives of the bending moments with respect to the x- and y-axes, $B_{(k+1)}$ are the discontinuities in the derivatives of the bimoment, $M_{A(k+1)}$ are the discontinuities in the derivatives of the bending-torsional moment, and $I_p^A = I_x + I_y + (a_x^2 + a_y^2)F$.

Now we substitute formulas (3.28)–(3.30) in (3.50)–(3.52), then multiply (3.50) by x, y, and ω_A, while (3.51) and (3.52) by $(y - a_y)$ and $(x - a_x)$, respectively, and integrate the obtained equations over the cross section of the beam. As a result, we obtain

$$\rho \widetilde{\omega}^{1y}_{(k+1)} = - G^{-1} I_y^{-1} M_{y(k+1)} + I_y^{-1} \frac{dM_{y(k)}}{ds} + 2æ \sin\varphi \mu (K + \tau) \omega^{1\lambda}_{(k-1)}$$
$$+ 2æ \cos\varphi \mu \left(G^{-1} \omega^{1\lambda}_{(k)} - \frac{d\omega^{1\lambda}_{(k-1)}}{ds} \right) - 2æ^2 \mu \widetilde{\omega}^{1y}_{(k-1)}$$
$$+ \sigma^0_{\lambda\lambda} \left(G^{-2} \widetilde{\omega}^{1y}_{(k+1)} - 2G^{-1} \frac{d\widetilde{\omega}^{1y}_{(k)}}{ds} + 2G^{-1} æ \cos\varphi \omega^{1\lambda}_{(k)} \right)$$
$$+ I_y^{-1} \sigma^0_{\lambda\lambda} \int_F f_{i(k-1)} \lambda_i x \, dF, \qquad (3.68)$$

$$\rho \widetilde{\omega}^{1x}_{(k+1)} = - G^{-1} I_x^{-1} M_{x(k+1)} + I_x^{-1} \frac{dM_{x(k)}}{ds} - 2æ \cos\varphi \mu (K + \tau) \omega^{1\lambda}_{(k-1)}$$
$$+ 2æ \sin\varphi \mu \left(G^{-1} \omega^{1\lambda}_{(k)} - \frac{d\omega^{1\lambda}_{(k-1)}}{ds} \right) - 2æ^2 \mu \widetilde{\omega}^{1x}_{(k-1)}$$
$$+ \sigma^0_{\lambda\lambda} \left(G^{-2} \widetilde{\omega}^{1x}_{(k+1)} - 2G^{-1} \frac{d\widetilde{\omega}^{1x}_{(k)}}{ds} + 2G^{-1} æ \sin\varphi \omega^{1\lambda}_{(k)} \right)$$
$$+ I_x^{-1} \sigma^0_{\lambda\lambda} \int_F f_{i(k-1)} \lambda_i y \, dF, \qquad (3.69)$$

$$\rho \widetilde{\psi}_{(k+1)} = -G^{-1} I_\omega^{-1} B_{(k+1)} + I_\omega^{-1} \frac{dB_{(k)}}{ds} - 2æ^2 \mu \widetilde{\psi}_{(k-1)}$$
$$+ \sigma^0_{\lambda\lambda} \left(G^{-2} \widetilde{\psi}_{(k+1)} - 2G^{-1} \frac{d\widetilde{\psi}_{(k)}}{ds} \right) + I_\omega^{-1} \sigma^0_{\lambda\lambda} \int_F f_{i(k-1)} \lambda_i \omega_A \, dF, \qquad (3.70)$$

$$
\begin{aligned}
&\rho\omega^{1\lambda}_{(k+1)}I^A_p-\rho\eta^0_{(k+1)}a_xF+\rho\theta^0_{(k+1)}a_yF\\
&=-G^{-1}M_{A(k+1)}+\frac{\mathrm{d}M_{A(k)}}{\mathrm{d}s}+æ\cos\varphi M_{y(k)}+æ\sin\varphi M_{x(k)}\\
&-æ\left(a_x\cos\varphi+a_y\sin\varphi\right)N_{\lambda(k)}-(K+\tau)\mu\Bigg\{G^{-1}F\left(a_x\theta^0_{(k)}+a_y\eta^0_{(k)}\right)\\
&-a_xF\left(\widetilde{\omega}^{1y}_{(k-1)}+\frac{\mathrm{d}\theta^0_{(k-1)}}{\mathrm{d}s}\right)-a_yF\left(\widetilde{\omega}^{1x}_{(k-1)}+\frac{\mathrm{d}\eta^0_{(k-1)}}{\mathrm{d}s}\right)\\
&+(K+\tau)\left(-a_xF\eta^0_{(k-1)}+a_yF\theta^0_{(k-1)}+I^A_p\omega^{1\lambda}_{(k-1)}\right)\\
&+æ\sin\varphi\left(a_xF\omega^0_{(k-1)}-I_y\widetilde{\omega}^{1y}_{(k-1)}\right)+æ\cos\varphi\left(-a_yF\omega^0_{(k-1)}+I_x\widetilde{\omega}^{1x}_{(k-1)}\right)\Bigg\}\\
&+\sigma^0_{\lambda\lambda}G^{-2}\left(-a_xF\eta^0_{(k+1)}+a_yF\theta^0_{(k+1)}+I^A_p\omega^{1\lambda}_{(k+1)}\right)\\
&-2\sigma^0_{\lambda\lambda}G^{-1}\Bigg\{-a_xF\frac{\mathrm{d}\eta^0_{(k)}}{\mathrm{d}s}+a_yF\frac{\mathrm{d}\theta^0_{(k)}}{\mathrm{d}s}+I^A_p\frac{\mathrm{d}\omega^{1\lambda}_{(k)}}{\mathrm{d}s}+(K+\tau)F\left(a_x\theta^0_{(k)}+a_y\eta^0_{(k)}\right)\\
&+æ\cos\varphi\left(-a_xF\omega^0_{(k)}+I_y\widetilde{\omega}^{1y}_{(k)}\right)+æ\sin\varphi\left(-a_yF\omega^0_{(k)}+I_x\widetilde{\omega}^{1x}_{(k)}\right)\Bigg\}\\
&+\sigma^0_{\lambda\lambda}\int_F\left[f_{i(k-1)}s_i(x-a_x)-f_{i(k-1)}k_i(y-a_y)\right]\mathrm{d}F. \qquad (3.71)
\end{aligned}
$$

Substituting (3.64)–(3.67) and (3.37)–(3.39) in (3.68)–(3.71) yields

$$
\begin{aligned}
&-G^{-2}(\rho G^2-\rho G_1^2-\sigma^0_{\lambda\lambda})\left[\omega^{1x}_{(k+1)}+\omega^0_{(k+1)}æ\cos\varphi-\theta^0_{(k+1)}(K+\tau)\right]\\
&\quad=2G^{-1}\left(\rho G_1^2+\sigma^0_{\lambda\lambda}\right)\frac{\mathrm{d}}{\mathrm{d}s}\left[\omega^{1x}_{(k)}+\omega^0_{(k)}æ\cos\varphi-\theta^0_{(k)}(K+\tau)\right]\\
&\quad\quad+æ\left(\rho G_1^2+2\rho G_2^2+2\sigma^0_{\lambda\lambda}\right)G^{-1}\omega^{1\lambda}_{(k)}\sin\varphi+F_{4(k-1)}, \qquad (3.72)
\end{aligned}
$$

$$
\begin{aligned}
&-G^{-2}(\rho G^2-\rho G_1^2-\sigma^0_{\lambda\lambda})\left[\omega^{1y}_{(k+1)}-\omega^0_{(k+1)}æ\sin\varphi+\eta^0_{(k+1)}(K+\tau)\right]\\
&\quad=2G^{-1}\left(\rho G_1^2+\sigma^0_{\lambda\lambda}\right)\frac{\mathrm{d}}{\mathrm{d}s}\left[\omega^{1y}_{(k)}-\omega^0_{(k)}æ\sin\varphi+\eta^0_{(k)}(K+\tau)\right]\\
&\quad\quad+æ\left(\rho G_1^2+2\rho G_2^2+2\sigma^0_{\lambda\lambda}\right)G^{-1}\omega^{1\lambda}_{(k)}\cos\varphi+F_{5(k-1)}, \qquad (3.73)
\end{aligned}
$$

$$
\begin{aligned}
&- G^{-2}\left(\rho G^2 - \rho G_1^2 - \sigma_{\lambda\lambda}^0\right)\Big[\psi_{(k+1)} - \omega_{(k+1)}^{1x}æ\sin\varphi + \omega_{(k+1)}^{1y}æ\cos\varphi \\
&\quad - \omega_{(k+1)}^0 æ^2 \sin 2\varphi + \Big(\theta_{(k+1)}^0 \sin\varphi + \eta_{(k+1)}^0 \cos\varphi\Big)æ(K+\tau)\Big] \\
&= 2G^{-1}\left(\rho G_1^2 + \sigma_{\lambda\lambda}^0\right)\frac{\mathrm{d}}{\mathrm{d}s}\Big[\psi_{(k)} - \omega_{(k)}^{1x}æ\sin\varphi + \omega_{(k)}^{1y}æ\cos\varphi \\
&\quad - \omega_{(k)}^0 æ^2 \sin 2\varphi + \Big(\theta_{(k)}^0 \sin\varphi + \eta_{(k)}^0 \cos\varphi\Big)æ(K+\tau)\Big] \\
&\quad + F_{6(k-1)},
\end{aligned}
\tag{3.74}
$$

$$
\begin{aligned}
&G^{-2}\left(\rho G^2 - \rho G_2^2 - \sigma_{\lambda\lambda}^0\right)\Big(I_P^A\omega_{(k+1)}^{1\lambda} + a_y F\theta_{(k+1)}^0 - a_x F\eta_{(k+1)}^0\Big) = -2G^{-1}\left(\rho G_2^2 + \sigma_{\lambda\lambda}^0\right) \\
&\quad \times \left\{\frac{\mathrm{d}}{\mathrm{d}s}\Big(I_P^A\omega_{(k)}^{1\lambda} + a_y F\theta_{(k)}^0 - a_x F\eta_{(k)}^0\Big) + F(K+\tau)\Big(a_x\theta_{(k)}^0 + a_y\eta_{(k)}^0\Big)\right\} \\
&\quad + G^{-1}\left[\rho G_2^2 a_y F + æI_y\cos\varphi\left(\rho G_1^2 + \rho G_2^2 + 2\sigma_{\lambda\lambda}^0\right)\right] \\
&\quad \times \Big[\omega_{(k)}^{1y} - \omega_{(k)}^0 æ\sin\varphi + \eta_{(k)}^0(K+\tau)\Big] \\
&\quad + G^{-1}\left[-\rho G_2^2 a_x F + æI_x\sin\varphi\left(\rho G_1^2 + \rho G_2^2 + 2\sigma_{\lambda\lambda}^0\right)\right] \\
&\quad \times \Big[\omega_{(k)}^{1x} + \omega_{(k)}^0 æ\cos\varphi - \theta_{(k)}^0(K+\tau)\Big] + G^{-1}æF\left(\rho G_1^2 + \rho G_2^2 + 2\sigma_{\lambda\lambda}^0\right) \\
&\quad \times \left(a_x\cos\varphi + a_y\sin\varphi\right)\omega_{(k)}^0 + F_{7(k-1)},
\end{aligned}
\tag{3.75}
$$

where functions $F_{j(k-1)}$ $(j = 4, 5, 6, 7)$ are presented in Appendix 2.

The system of seven equations, (3.61)–(3.63) and (3.72)–(3.75), involves seven unknown values: $\omega_{(k)}^0$, $\theta_{(k)}^0$, $\eta_{(k)}^0$, $\omega_{(k)}^{1x}$, $\omega_{(k)}^{1y}$, $\omega_{(k)}^{1\lambda}$, and $\psi_{(k)}$, which are defined uniquely from this set of equations.

To find these values on the quasi-longitudinal shock wave and on the quasi-transverse shock wave, we should put in all equations $G = G_I = \sqrt{G_1^2 + \rho^{-1}\sigma_{\lambda\lambda}^0}$ and $G = G_{II} = \sqrt{G_2^2 + \rho^{-1}\sigma_{\lambda\lambda}^0}$, respectively.

3.1.2.1 Solution on the Quasi-Longitudinal Wave

To define the velocity field on the quasi-longitudinal wave, first we put in all equations from (3.61)–(3.63) and (3.72)–(3.75) $G = G_I = \sqrt{G_1^2 + \rho^{-1}\sigma_{\lambda\lambda}^0}$, i.e. we write this system on the quasi-longitudinal shock wave. As a result we obtain

$$2G_I^{-1}(\rho G_1^2+\sigma^0_{\lambda\lambda})\frac{\mathrm{d}\omega^0_{(k)}}{\mathrm{d}s}=-G_I^{-1}\left(\rho G_1^2+2\rho G_2^2+2\sigma^0_{\lambda\lambda}\right)æ \\ \times\left[\left(\theta^0_{(k)}+a_y\omega^{1\lambda}_{(k)}\right)\sin\varphi-\left(\eta^0_{(k)}-a_x\omega^{1\lambda}_{(k)}\right)\cos\varphi\right] \\ +F_{1(k-1)}|_{G=G_I}, \tag{3.76}$$

$$2G_I^{-1}(\rho G_1^2+\sigma^0_{\lambda\lambda})\frac{\mathrm{d}}{\mathrm{d}s}\left[\omega^{1x}_{(k)}+\omega^0_{(k)}æ\cos\varphi-\theta^0_{(k)}(K+\tau)\right] \\ =-G_I^{-1}(\rho G_1^2+2\rho G_2^2+2\sigma^0_{\lambda\lambda})æ\omega^{1\lambda}_{(k)}\sin\varphi-F_{4(k-1)}|_{G=G_I}, \tag{3.77}$$

$$2G_I^{-1}(\rho G_1^2+\sigma^0_{\lambda\lambda})\frac{\mathrm{d}}{\mathrm{d}s}\left[\omega^{1y}_{(k)}-\omega^0_{(k)}æ\sin\varphi+\eta^0_{(k)}(K+\tau)\right] \\ =-G_I^{-1}(\rho G_1^2+2\rho G_2^2+2\sigma^0_{\lambda\lambda})æ\omega^{1\lambda}_{(k)}\cos\varphi-F_{5(k-1)}|_{G=G_I}, \tag{3.78}$$

$$2G_I^{-1}(\rho G_1^2+\sigma^0_{\lambda\lambda})\frac{\mathrm{d}}{\mathrm{d}s}\Big[\psi_{(k)}-\omega^{1x}_{(k)}æ\sin\varphi+\omega^{1y}_{(k)}æ\cos\varphi-\omega^0_{(k)}æ^2\sin 2\varphi \\ +\left(\theta^0_{(k)}\sin\varphi+\eta^0_{(k)}\cos\varphi\right)æ(K+\tau)\Big]=-F_{6(k-1)}|_{G=G_I}, \tag{3.79}$$

$$G_I^{-2}(\rho G_1^2-\rho G_2^2)\left(\theta^0_{(k)}+a_y\omega^{1\lambda}_{(k)}\right)=-2G_I^{-1}(\rho G_2^2+\sigma^0_{\lambda\lambda})\left\{\frac{\mathrm{d}}{\mathrm{d}s}\left(\theta^0_{(k-1)}+a_y\omega^{1\lambda}_{(k-1)}\right)\right. \\ \left.+(K+\tau)\left(\eta^0_{(k-1)}-a_x\omega^{1\lambda}_{(k-1)}\right)\right\}+G_I^{-1}(\rho G_1^2+\rho G_2^2+2\sigma^0_{\lambda\lambda})æ\omega^0_{(k-1)}\sin\varphi \\ +G_I^{-1}\rho G_2^2\left\{\omega^{1y}_{(k-1)}-\omega^0_{(k-1)}æ\sin\phi+\eta^0_{(k-1)}(K+\tau)\right\}+F_{2(k-2)}|_{G=G_I}, \tag{3.80}$$

$$G_I^{-2}(\rho G_1^2-\rho G_2^2)\left(\eta^0_{(k)}-a_x\omega^{1\lambda}_{(k)}\right)=-2G_I^{-1}(\rho G_2^2+\sigma^0_{\lambda\lambda})\left\{\frac{\mathrm{d}}{\mathrm{d}s}\left(\eta^0_{(k-1)}-a_x\omega^{1\lambda}_{(k-1)}\right)\right. \\ \left.-(K+\tau)\left(\theta^0_{(k-1)}+a_y\omega^{1\lambda}_{(k-1)}\right)\right\} \\ -G_I^{-1}(\rho G_1^2+\rho G_2^2+2\sigma^0_{\lambda\lambda})æ\omega^0_{(k-1)}\cos\varphi \\ +G_I^{-1}\rho G_2^2\left\{\omega^{1x}_{(k-1)}+\omega^0_{(k-1)}æ\cos\phi-\theta^0_{(k-1)}(K+\tau)\right\} \\ +F_{3(k-2)}|_{G=G_I}, \tag{3.81}$$

$$
\begin{aligned}
&G_I^{-2}(\rho G_1^2 - \rho G_2^2)\Big(I_P^A \omega_{(k)}^{1\lambda} + a_y F \theta_{(k)}^0 - a_x F \eta_{(k)}^0\Big) \\
&\quad = -2G_I^{-1}\big(\rho G_2^2 + \sigma_{\lambda\lambda}^0\big)\frac{\mathrm{d}}{\mathrm{d}s}\Big(I_P^A \omega_{(k-1)}^{1\lambda} + a_y F \theta_{(k-1)}^0 - a_x F \eta_{(k-1)}^0\Big) \\
&\qquad - 2G_I^{-1}\big(\rho G_2^2 + 2\sigma_{\lambda\lambda}^0\big)F(K+\tau)\Big(a_x \theta_{(k-1)}^0 + a_y \eta_{(k-1)}^0\Big) \\
&\qquad + G_I^{-1}\big[\rho G_2^2 a_y F + æ I_y \cos\varphi\big(\rho G_1^2 + \rho G_2^2 + 2\sigma_{\lambda\lambda}^0\big)\big] \\
&\qquad \times \Big[\omega_{(k-1)}^{1y} - \omega_{(k-1)}^0 æ \sin\varphi + \eta_{(k-1)}^0 (K+\tau)\Big] \\
&\qquad + G_I^{-1}\big[-\rho G_2^2 a_x F + æ I_x \sin\varphi\big(\rho G_1^2 + \rho G_2^2 + 2\sigma_{\lambda\lambda}^0\big)\big] \\
&\qquad \times \Big[\omega_{(k-1)}^{1x} + \omega_{(k-1)}^0 æ \cos\varphi - \theta_{(k-1)}^0 (K+\tau)\Big] \\
&\qquad + G_I^{-1} æ F\big(\rho G_1^2 + \rho G_2^2 + 2\sigma_{\lambda\lambda}^0\big)\big(a_x \cos\varphi + a_y \sin\varphi\big)\omega_{(k-1)}^0 \\
&\qquad + F_{7(k-2)}\big|_{G=G_I}.
\end{aligned}
\tag{3.82}
$$

Reference to Eqs. 3.76–3.82 shows that on the quasi-longitudinal wave the discontinuities $\omega_{(k)}^0$, $\omega_{(k)}^{1x}$, $\omega_{(k)}^{1y}$, and $\psi_{(k)}$ are defined from the differential Eqs. 3.76–3.79 within the accuracy of arbitrary constants, while the discontinuities $\theta_{(k)}^0$, $\eta_{(k)}^0$, and $\omega_{(k)}^{1\lambda}$ are found from the algebraic Eqs. 3.80–3.82, in so doing the discontinuities $\theta_{(k)}^0$, $\eta_{(k)}^0$, and $\omega_{(k)}^{1\lambda}$ have the higher order than the discontinuities $\omega_{(k)}^0$, $\omega_{(k)}^{1x}$, $\omega_{(k)}^{1y}$, and $\psi_{(k)}$.

For arbitrary magnitudes of k, the set of Eqs. 3.76–3.79 can be rewritten in the form

$$\frac{\mathrm{d}\omega_{(k)}^0}{\mathrm{d}s} = A_{0(k)}(s), \tag{3.83}$$

$$\frac{\mathrm{d}}{\mathrm{d}s}\Big(\omega_{(k)}^{1x} + \omega_{(k)}^0 æ \cos\varphi\Big) = A_{1(k)}(s), \tag{3.84}$$

$$\frac{\mathrm{d}}{\mathrm{d}s}\Big(\omega_{(k)}^{1y} - \omega_{(k)}^0 æ \sin\varphi\Big) = A_{2(k)}(s), \tag{3.85}$$

$$\frac{\mathrm{d}}{\mathrm{d}s}\Big(\psi_{(k)} - \omega_{(k)}^{1x} æ \sin\varphi + \omega_{(k)}^{1y} æ \cos\varphi - \omega_{(k)}^0 æ^2 \sin 2\varphi\Big) = A_{3(k)}(s), \tag{3.86}$$

where functions $A_{i(k)}(s)$ $(i = 0, 1, 2, 3)$ are presented in Appendix 3.

Integrating (3.83)–(3.86) yields

$$\omega_{(k)}^0 = \int_{s_0}^{s} A_{0(k)}(s)\mathrm{d}s + c_{0(k)}, \tag{3.87}$$

$$\omega^{1x}_{(k)} = -\omega^{0}_{(k)}\text{æ}\cos\varphi + c_{1(k)} + \int\limits_{s_0}^{s} A_{1(k)}(s)\mathrm{d}s, \tag{3.88}$$

$$\omega^{1y}_{(k)} = \omega^{0}_{(k)}\text{æ}\sin\varphi + c_{2(k)} + \int\limits_{s_0}^{s} A_{2(k)}(s)\mathrm{d}s, \tag{3.89}$$

$$\begin{aligned}\psi_{(k)} = &\left(c_{1(k)} + \int\limits_{s_0}^{s} A_{1(k)}(s)\mathrm{d}s\right)\text{æ}\sin\varphi \\ &- \left(c_{2(k)} + \int\limits_{s_0}^{s} A_{2(k)}(s)\mathrm{d}s\right)\text{æ}\cos\varphi + c_{3(k)} + \int\limits_{s_0}^{s} A_{3(k)}(s)\mathrm{d}s,\end{aligned} \tag{3.90}$$

where $c_{0(k)},\ c_{1(k)},\ c_{2(k)}$, and $c_{3(k)}$ are arbitrary constants to be determined from the initial conditions.

Reference to Eqs. 3.87–3.90 shows that the main values on the quasi-longitudinal wave, which define the type of this wave, i.e. $\omega^{0}_{(k)},\ \omega^{1x}_{(k)},\ \omega^{1y}_{(k)}$, and $\psi_{(k)}$, are interconnected with each other, since they are expressed in terms of $\omega^{0}_{(k)}$. This coupling is governed by the curvature æ and the angle φ between the reference systems $x - y$ and $\tau - \xi$ locating in the plane of the strong discontinuity.

Thus, for example, at $k = 0$ the values $\theta^{0}_{(0)},\ \eta^{0}_{(0)}$, and $\omega^{1\lambda}_{(0)}$ defined by the algebraic Eqs. 3.80–3.82 vanish to zero, while according to (3.87)–(3.90) the values $\omega^{0}_{(0)},\ \omega^{1x}_{(0)},\ \omega^{1y}_{(0)}$, and $\psi_{(0)}$ take the form

$$\omega^{0}_{(0)} = \text{const} = c_{0(0)}, \tag{3.91}$$

$$\omega^{1x}_{(0)} = -\omega^{0}_{(0)}\text{æ}\cos\varphi + c_{1(0)}, \tag{3.92}$$

$$\omega^{1y}_{(0)} = \omega^{0}_{(0)}\text{æ}\sin\varphi + c_{2(0)}, \tag{3.93}$$

$$\begin{aligned}\psi_{(0)} &= c_{1(0)}\text{æ}\sin\varphi - c_{2(0)}\text{æ}\cos\varphi + c_{3(0)} \\ &= \omega^{1x}_{(0)}\text{æ}\sin\varphi - \omega^{1y}_{(0)}\text{æ}\cos\varphi + \omega^{0}_{(0)}\text{æ}^2\sin 2\varphi + c_{3(0)},\end{aligned} \tag{3.94}$$

where $c_{0(0)},\ c_{1(0)},\ c_{2(0)}$, and $c_{3(0)}$ are arbitrary constants to be determined from the initial conditions.

3.1.2.2 Solution on the Quasi-Transverse Shear Wave

Now we put $G = G_{II} = \sqrt{G_2^2 + \rho^{-1}\sigma_{\lambda\lambda}^0}$ in all Eqs. 3.61–3.63 and 3.72–3.75, i.e. write them on the quasi-transverse wave. As a result we obtain

$$
\begin{aligned}
&2G_{II}^{-1}(\rho G_2^2 + \sigma_{\lambda\lambda}^0)\left[\frac{\mathrm{d}}{\mathrm{d}s}\left(\theta_{(k)}^0 + a_y\omega_{(k)}^{1\lambda}\right) + (K+\tau)\left(\eta_{(k)}^0 - a_x\omega_{(k)}^{1\lambda}\right)\right] \\
&\quad = G_{II}^{-1}æ\left(\rho G_1^2 + \rho G_2^2 + 2\sigma_{\lambda\lambda}^0\right)\omega_{(k)}^0 \sin\varphi \\
&\qquad + G_{II}^{-1}\rho G_2^2\left\{\omega_{(k)}^{1y} - \omega_{(k)}^0 æ\sin\varphi + (K+\tau)\eta_{(k)}^0\right\} + F_{2(k-1)}|_{G=G_{II}},
\end{aligned}
\tag{3.95}
$$

$$
\begin{aligned}
&2G_{II}^{-1}(\rho G_2^2 + \sigma_{\lambda\lambda}^0)\left[\frac{\mathrm{d}}{\mathrm{d}s}\left(\eta_{(k)}^0 - a_x\omega_{(k)}^{1\lambda}\right) - (K+\tau)\left(\theta_{(k)}^0 + a_y\omega_{(k)}^{1\lambda}\right)\right] \\
&\quad = -G_{II}^{-1}æ\left(\rho G_1^2 + \rho G_2^2 + 2\sigma_{\lambda\lambda}^0\right)\omega_{(k)}^0 \cos\varphi \\
&\qquad + G_{II}^{-1}\rho G_2^2\left\{\omega_{(k)}^{1x} + \omega_{(k)}^0 æ\cos\varphi - (K+\tau)\theta_{(k)}^0\right\} + F_{3(k-1)}|_{G=G_{II}},
\end{aligned}
\tag{3.96}
$$

$$
\begin{aligned}
&2G_{II}^{-1}(\rho G_2^2 + \sigma_{\lambda\lambda}^0)\left\{\frac{\mathrm{d}}{\mathrm{d}s}\left(I_p^A\omega_{(k)}^{1\lambda} - Fa_x\eta_{(k)}^0 + Fa_y\theta_{(k)}^0\right) + F(K+\tau)\left(a_y\eta_{(k)}^0 + a_x\theta_{(k)}^0\right)\right\} \\
&= G_{II}^{-1}æ\left(\rho G_1^2 + \rho G_2^2 + 2\sigma_{\lambda\lambda}^0\right)\left\{\omega_{(k)}^0 F\left(a_x\cos\varphi + a_y\sin\varphi\right) + \omega_{(k)}^{1x} I_x \sin\varphi\right. \\
&\quad \left. + \omega_{(k)}^{1y} I_y\cos\varphi + \frac{1}{2}\,\omega_{(k)}^0 æ(I_x - I_y)\sin 2\varphi - (K+\tau)\left(\theta_{(k)}^0 I_x\sin\varphi - \eta_{(k)}^0 I_y\cos\varphi\right)\right\} \\
&\quad + G_{II}^{-1}\rho G_2^2 F\left\{a_y\omega_{(k)}^{1y} - a_x\omega_{(k)}^{1x} - \omega_{(k)}^0 æ\left(a_x\cos\varphi + a_y\sin\varphi\right)\right. \\
&\quad \left. + (K+\tau)\left(a_y\eta_{(k)}^0 + a_x\theta_{(k)}^0\right)\right\} + F_{7(k-1)}|_{G=G_{II}},
\end{aligned}
\tag{3.97}
$$

$$
\begin{aligned}
&G_{II}^{-2}\rho(G_2^2 - G_1^2)\omega_{(k)}^0 = -2G_{II}^{-1}\left(\rho G_1^2 + \sigma_{\lambda\lambda}^0\right)\frac{\mathrm{d}\omega_{(k-1)}^0}{\mathrm{d}s} - G_{II}^{-1}æ\left(\rho G_1^2 + 2\rho G_2^2 + 2\sigma_{\lambda\lambda}^0\right) \\
&\quad \times\left\{\left(a_x\cos\varphi + a_y\sin\varphi\right)\omega_{(k-1)}^{1\lambda} - \eta_{(k-1)}^0\cos\varphi + \theta_{(k-1)}^0\sin\varphi\right\} \\
&\quad + F_{1(k-2)}|_{G=G_{II}},
\end{aligned}
\tag{3.98}
$$

$$
\begin{aligned}
&-G_{II}^{-2}\rho(G_2^2-G_1^2)\Big\{\omega_{(k)}^{1x}+\omega_{(k)}^{0}\text{æ}\cos\varphi-\theta_{(k)}^{0}(K+\tau)\Big\}\\
&\qquad=2G_{II}^{-1}\left(\rho G_1^2+\sigma_{\lambda\lambda}^{0}\right)\frac{\mathrm{d}}{\mathrm{d}s}\left\{\omega_{(k-1)}^{1x}+\omega_{(k-1)}^{0}\text{æ}\cos\varphi-\theta_{(k-1)}^{0}(K+\tau)\right\}\\
&\qquad\quad+\text{æ}\big(\rho G_1^2+2\rho G_2^2+2\sigma_{\lambda\lambda}^{0}\big)G_{II}^{-1}\omega_{(k-1)}^{1\lambda}\sin\varphi+F_{4(k-2)}|_{G=G_{II}},
\end{aligned}
\tag{3.99}
$$

$$
\begin{aligned}
&-G_{II}^{-2}\rho(G_2^2-G_1^2)\Big\{\omega_{(k)}^{1y}-\omega_{(k)}^{0}\text{æ}\sin\varphi+\eta_{(k)}^{0}(K+\tau)\Big\}\\
&\qquad=2G_{II}^{-1}\left(\rho G_1^2+\sigma_{\lambda\lambda}^{0}\right)\frac{\mathrm{d}}{\mathrm{d}s}\left\{\omega_{(k-1)}^{1y}-\omega_{(k-1)}^{0}\text{æ}\sin\varphi+\eta_{(k-1)}^{0}(K+\tau)\right\}\\
&\qquad\quad+\text{æ}\big(\rho G_1^2+2\rho G_2^2+2\sigma_{\lambda\lambda}^{0}\big)G_{II}^{-1}\omega_{(k-1)}^{1\lambda}\cos\varphi+F_{5(k-2)}|_{G=G_{II}},
\end{aligned}
\tag{3.100}
$$

$$
\begin{aligned}
&-G_{II}^{-2}\rho(G_2^2-G_1^2)\Big\{\psi_{(k)}-\omega_{(k)}^{1x}\text{æ}\sin\varphi+\omega_{(k)}^{1y}\text{æ}\cos\varphi\\
&\quad-\omega_{(k)}^{0}\text{æ}^2\sin 2\varphi+\text{æ}\Big(\theta_{(k)}^{0}\sin\varphi+\eta_{(k)}^{0}\cos\varphi\Big)(K+\tau)\Big\}\\
&\quad=2G_{II}^{-1}\left(\rho G_1^2+\sigma_{\lambda\lambda}^{0}\right)\frac{\mathrm{d}}{\mathrm{d}s}\Big\{\psi_{(k-1)}-\omega_{(k-1)}^{1x}\text{æ}\sin\varphi+\omega_{(k-1)}^{1y}\text{æ}\cos\varphi\\
&\quad-\omega_{(k-1)}^{0}\text{æ}^2\sin 2\varphi+\text{æ}\Big(\theta_{(k-1)}^{0}\sin\varphi+\eta_{(k-1)}^{0}\cos\varphi\Big)(K+\tau)\Big\}\\
&\quad+F_{6(k-2)}|_{G=G_{II}}.
\end{aligned}
\tag{3.101}
$$

From (3.95)–(3.101) it is seen that on the quasi-transverse wave, in contradistinction to the quasi-longitudinal wave, the discontinuities $\theta_{(k)}^{0}$, $\eta_{(k)}^{0}$, and $\omega_{(k)}^{1\lambda}$ are determined by the differential Eqs. 3.95–3.97 within the accuracy of arbitrary constants, while the discontinuities $\omega_{(k)}^{0}$, $\omega_{(k)}^{1x}$, $\omega_{(k)}^{1y}$, and $\psi_{(k)}$ are defined from the the algebraic Eqs. 3.98–3.101, in so doing the discontinuities $\omega_{(k)}^{0}$ and $\psi_{(k)}$ according to (3.98) and (3.101), respectively, have the higher order than the discontinuities $\theta_{(k)}^{0}$, $\eta_{(k)}^{0}$, and $\omega_{(k)}^{1\lambda}$, while the discontinuities $\omega_{(k)}^{1x}$ and $\omega_{(k)}^{1y}$, as it follows from (3.99) and (3.100), have the same order as the primarily components on this wave.

For arbitrary magnitudes of k, the set of Eqs. 3.95–3.97, 3.99 and 3.100 can be rewritten as

$$
\omega_{(k)}^{1x}=\theta_{(k)}^{0}(K+\tau)+B_{1(k-1)},
\tag{3.102}
$$

$$
\omega_{(k)}^{1y}=-\eta_{(k)}^{0}(K+\tau)+B_{2(k-1)},
\tag{3.103}
$$

$$
\frac{\mathrm{d}\theta_{(k)}^{0}}{\mathrm{d}s}+(K+\tau)\eta_{(k)}^{0}=(K+\tau)a_x\omega_{(k)}^{1\lambda}-a_y\frac{\mathrm{d}\omega_{(k)}^{1\lambda}}{\mathrm{d}s}+B_{3(k)},
\tag{3.104}
$$

$$
\frac{\mathrm{d}\eta_{(k)}^{0}}{\mathrm{d}s}-(K+\tau)\theta_{(k)}^{0}=(K+\tau)a_y\omega_{(k)}^{1\lambda}+a_x\frac{\mathrm{d}\omega_{(k)}^{1\lambda}}{\mathrm{d}s}+B_{4(k)},
\tag{3.105}
$$

$$I_p^C \frac{\mathrm{d}\omega_{(k)}^{1\lambda}}{\mathrm{d}s} = B_{5(k)} + F\left(a_x B_{4(k)} - a_y B_{3(k)}\right), \tag{3.106}$$

where $I_p^C = I_x + I_y$, and functions $B_{j(k)}(s)$ $(i = 1, 2, 3, 4, 5)$ are presented in Appendix 3.

The general solution of (3.104)–(3.106) has the form

$$\theta_{(k)}^0 = g_{(k)}^0 \sin\chi + h_{(k)}^0 \cos\chi - a_y\omega_{(k)}^{1\lambda} + \sin\chi \int\limits_{s_0}^{s} \left(B_{3(k)} \sin\chi + B_{4(k)} \cos\chi\right)\mathrm{d}s + \cos\chi \int\limits_{s_0}^{s} \left(B_{3(k)} \cos\chi - B_{4(k)} \sin\chi\right)\mathrm{d}s, \tag{3.107}$$

$$\eta_{(k)}^0 = g_{(k)}^0 \cos\chi - h_{(k)}^0 \sin\chi + a_x\omega_{(k)}^{1\lambda} + \cos\chi \int\limits_{s_0}^{s} \left(B_{3(k)} \sin\chi + B_{4(k)} \cos\chi\right)\mathrm{d}s - \sin\chi \int\limits_{s_0}^{s} \left(B_{3(k)} \cos\chi - B_{4(k)} \sin\chi\right)\mathrm{d}s, \tag{3.108}$$

$$\omega_{(k)}^{1\lambda} = (I_p^C)^{-1} \int\limits_{s_0}^{s} \left[B_{5(k)} + F\left(a_x B_{4(k)} - a_y B_{3(k)}\right)\right]\mathrm{d}s + k_{(k)}^0, \tag{3.109}$$

where $h_{(k)}^0$, $g_{(k)}^0$, and $k_{(k)}^0$ are arbitrary constants, and

$$\chi = -\int\limits_{s_0}^{s} (K + \tau)\mathrm{d}s.$$

Reference to Eqs. 3.107–3.109 shows that the main values on the quasi-transverse wave, which define the type of this wave, i.e. $\theta_{(k)}^0$, $\eta_{(k)}^0$, and $\omega_{(k)}^{1\lambda}$, are interconnected with each other, since they are expressed in terms of $\omega_{(k)}^{1\lambda}$ via the shear center coordinates which, in the general case of the beam's cross-section, are different from those of the center of gravity. The values $\omega_{(k)}^{1x}$ and $\omega_{(k)}^{1y}$, according to (3.102) and (3.103), ultimately depend on the value $\omega_{(k)}^{1\lambda}$ as well, and this coupling is supported by the torsion τ, the value $K = \mathrm{d}\varphi/\mathrm{d}s$, and by the angle χ. In other words, in the case of a spatially twisted thin-walled beam of open section, on the twisting-shear wave there occur flexural motions of the same order as the discontinuities in the velocities of the transverse translatory motions and in the angular velocity of rotation of the cross-section.

At $k = 0$, for example, the values $\omega^0_{(0)}$ and $\psi_{(0)}$ vanish to zero, while the values $\theta^0_{(0)}$, $\eta^0_{(0)}$, $\omega^{1\lambda}_{(0)}$, $\omega^{1x}_{(0)}$, and $\omega^{1y}_{(0)}$ take the form

$$\omega^{1\lambda}_{(0)} = \text{const}, \tag{3.110}$$

$$\theta^0_{(0)} = g^0_{(0)} \sin\chi + h^0_{(0)} \cos\chi - a_y \omega^{1\lambda}_{(0)}, \tag{3.111}$$

$$\eta^0_{(0)} = g^0_{(0)} \cos\chi - h^0_{(0)} \sin\chi + a_x \omega^{1\lambda}_{(0)}, \tag{3.112}$$

$$\omega^{1x}_{(0)} = \theta^0_{(0)}(K + \tau), \tag{3.113}$$

$$\omega^{1y}_{(0)} = -\eta^0_{(0)}(K + \tau), \tag{3.114}$$

where $g^0_{(0)}$ and $h^0_{(0)}$ are arbitrary constants.

Let us introduce into consideration two mutually orthogonal vectors: $\mathbf{b}_{(\mathbf{0})}\left\{-a_y\omega^{1\lambda}_{(0)}, a_x\omega^{1\lambda}_{(0)}\right\}$ and $\mathbf{a}\{a_x, a_y\}$, and construct the vector $\mathbf{d}_{(\mathbf{0})} = \left(\mathbf{V}_{(\mathbf{0})} - \mathbf{b}_{(\mathbf{0})}\right)$ $\left\{\theta^0_{(0)} + a_y\omega^{1\lambda}_{(0)}, \eta^0_{(0)} - a_x\omega^{1\lambda}_{(0)}\right\}$, where the vector $\mathbf{V}_{(\mathbf{0})}\left\{\theta^0_{(0)}, \eta^0_{(0)}\right\}$ is the vector of the transverse wave polarization. The vector $\mathbf{d}_{(\mathbf{0})}$ has the constant modulus

$$\left|\mathbf{d}_{(\mathbf{0})}\right| = \sqrt{\left(g^0_{(0)}\right)^2 + \left(h^0_{(0)}\right)^2} = \text{const},$$

and its angle of inclination to the x-axis is defined by the formula

$$\text{tg}\alpha_{(0)} = \frac{\eta^0_{(0)} - a_x\omega^{1\lambda}_{(0)}}{\theta^0_{(0)} + a_y\omega^{1\lambda}_{(0)}} = \frac{g^0_{(0)}\cos\chi - h^0_{(0)}\sin\chi}{g^0_{(0)}\sin\chi + h^0_{(0)}\cos\chi} = \text{tg}\left(\Phi_{(0)} - \chi\right),$$

that is

$$\alpha_{(0)} = \Phi_{(0)} - \chi,$$

and

$$\dot{\alpha}_{(0)} = -\dot{\chi} = (K + \tau)\dot{s} = (K + \tau)G_{II},$$

where $\text{tg}\Phi_{(0)} = g^0_{(0)}\left(h^0_{(0)}\right)^{-1}$.

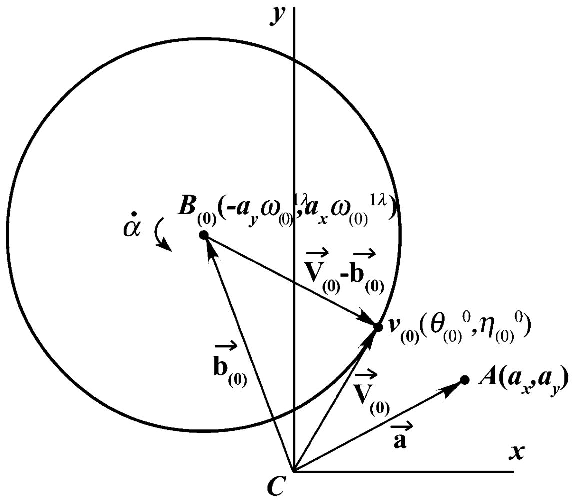

Fig. 3.3 Scheme of the location of the polarization vector of the quasi-transverse wave

In other words, on the wave surface, the vector $\mathbf{d}_{(\mathbf{0})}$ remaining constant in magnitude on the wave front rotates around the point $B_{(0)}$ with the angular velocity $(K+\tau)G_{II}$, in so doing the polarization vector during its rotation around the point C changes both in magnitude and in the direction (Fig. 3.3). We shall name the point $B_{(0)}$ as the center of rotation.

Note that for curvilinear beams of solid section the center of rotation coincides with the center of gravity of the beam's cross section.

If the cross section of the open section beam possesses two axes of symmetry, then $a_x = a_y = 0$, and the center of rotation coincides with the center of gravity.

From the comparison of Eqs. 2.14 and 3.110–3.114 it follows that three values $\theta^0_{(k)}$, $\eta^0_{(k)}$ and $\omega^{1\lambda}_{(k)}$ are related to each other both in the technical theory of thin-walled beams of open section [6] and in the theory developed here. However, in contrast to the technical theory, where these values are interrelated on three transverse waves which propagate with the velocities depending on the geometrical characteristics of the thin-walled beams of open section, in the present theory this coupling takes place only on one wave propagating with the natural velocity $G_{II} = \sqrt{\rho G_2^2 + \rho^{-1}\sigma^0_{\lambda\lambda}}$ independent of the thin-walled beam geometry.

3.1.3 *Particular Case of a Straight Thin-Walled Beam of Open Section*

In the case of a straight thin-walled beam of open section, the governing equations are considerably simplified since $æ = K = \tau = 0$ and $s = z$. Thus, for the quasi-longitudinal wave from (3.76)–(3.82) we have

$$2G_I^{-1}\frac{\mathrm{d}\omega_{(k)}^{0}}{\mathrm{d}z}=\frac{\mathrm{d}^2\omega_{(k-1)}^{0}}{\mathrm{d}z^2}, \tag{3.115}$$

$$2G_I^{-1}\frac{\mathrm{d}\omega_{(k)}^{1x}}{\mathrm{d}z}=\frac{\mathrm{d}^2\omega_{(k-1)}^{1x}}{\mathrm{d}z^2}, \tag{3.116}$$

$$2G_I^{-1}\frac{\mathrm{d}\omega_{(k)}^{1y}}{\mathrm{d}z}=\frac{\mathrm{d}^2\omega_{(k-1)}^{1y}}{\mathrm{d}z^2}, \tag{3.117}$$

$$2G_I^{-1}\frac{\mathrm{d}\psi_{(k)}}{\mathrm{d}z}=\frac{\mathrm{d}^2\psi_{(k-1)}}{\mathrm{d}z^2}, \tag{3.118}$$

$$\begin{aligned}G_I^{-2}\rho\left(G_1^2-G_2^2\right)\left(\theta_{(k)}^{0}+a_y\omega_{(k)}^{1\lambda}\right) &= -2G_I\left(\rho G_2^2+\sigma_{\lambda\lambda}^{0}\right)\frac{\mathrm{d}}{\mathrm{d}z}\left(\theta_{(k-1)}^{0}+a_y\omega_{(k-1)}^{1\lambda}\right)\\ &+G_I^{-1}\rho G_2^2\omega_{(k-1)}^{1y}+\left(\rho G_2^2+\sigma_{\lambda\lambda}^{0}\right)\frac{\mathrm{d}^2}{\mathrm{d}z^2}\left(\theta_{(k-2)}^{0}+a_y\omega_{(k-2)}^{1\lambda}\right)-\rho G_2^2\frac{\mathrm{d}\omega_{(k-2)}^{1y}}{\mathrm{d}z},\end{aligned} \tag{3.119}$$

$$\begin{aligned}G_I^{-2}\rho\left(G_1^2-G_2^2\right)\left(\eta_{(k)}^{0}-a_x\omega_{(k)}^{1\lambda}\right) &= -2G_I^{-1}\left(\rho G_2^2+\sigma_{\lambda\lambda}^{0}\right)\frac{\mathrm{d}}{\mathrm{d}z}\left(\eta_{(k-1)}^{0}-a_x\omega_{(k-1)}^{1\lambda}\right)\\ &+G_I^{-1}\rho G_2^2\omega_{(k-1)}^{1x}+\left(\rho G_2^2+\sigma_{\lambda\lambda}^{0}\right)\frac{\mathrm{d}^2}{\mathrm{d}z^2}\left(\eta_{(k-2)}^{0}-a_x\omega_{(k-2)}^{1\lambda}\right)-\rho G_2^2\frac{\mathrm{d}\omega_{(k-2)}^{1x}}{\mathrm{d}z},\end{aligned} \tag{3.120}$$

$$\begin{aligned}&G_I^{-2}\rho\left(G_1^2-G_2^2\right)\left(I_P^A\omega_{(k)}^{1\lambda}+a_yF\theta_{(k)}^{0}-a_xF\eta_{(k)}^{0}\right)\\ &\quad=-2G_I^{-1}\left(\rho G_2^2+\sigma_{\lambda\lambda}^{0}\right)\frac{\mathrm{d}}{\mathrm{d}z}\left(I_P^A\omega_{(k-1)}^{1\lambda}-Fa_x\eta_{(k-1)}^{0}+Fa_y\theta_{(k-1)}^{0}\right)\\ &\quad+G_I^{-1}\rho G_2^2F\left(a_y\omega_{(k-1)}^{1y}-a_x\omega_{(k-1)}^{1x}\right)\\ &\quad+\left(\rho G_2^2+\sigma_{\lambda\lambda}^{0}\right)\frac{\mathrm{d}^2}{\mathrm{d}z^2}\left(I_P^A\omega_{(k-2)}^{1\lambda}-Fa_x\eta_{(k-2)}^{0}+Fa_y\theta_{(k-2)}^{0}\right)\\ &\quad-\rho G_2^2F\frac{\mathrm{d}}{\mathrm{d}z}\left(a_y\omega_{(k-2)}^{1y}-a_x\omega_{(k-2)}^{1x}\right).\end{aligned} \tag{3.121}$$

For the quasi-transverse wave, from Eqs. 3.95–3.101 we have

$$\begin{aligned}&2G_{II}^{-1}\left(\rho G_2^2+\sigma_{\lambda\lambda}^{0}\right)\frac{\mathrm{d}}{\mathrm{d}z}\left(\theta_{(k)}^{0}+a_y\omega_{(k)}^{1\lambda}\right)=G_{II}^{-1}\rho G_2^2\omega_{(k)}^{1y}\\ &\quad+\left(\rho G_2^2+\sigma_{\lambda\lambda}^{0}\right)\frac{\mathrm{d}^2\theta_{(k-1)}^{0}}{\mathrm{d}z^2}-\rho G_2^2\frac{\mathrm{d}}{\mathrm{d}z}\left\{\omega_{(k-1)}^{1y}-\frac{\mathrm{d}\omega_{(k-1)}^{1\lambda}}{\mathrm{d}z}a_y\right\},\end{aligned} \tag{3.122}$$

$$2G_{II}^{-1}\left(\rho G_2^2+\sigma_{\lambda\lambda}^0\right)\frac{\mathrm{d}}{\mathrm{d}z}\left(\eta_{(k)}^0-a_x\omega_{(k)}^{1\lambda}\right)=G_{II}^{-1}\rho G_2^2\omega_{(k)}^{1x}$$
$$+\left(\rho G_2^2+\sigma_{\lambda\lambda}^0\right)\frac{\mathrm{d}^2\eta_{(k-1)}^0}{\mathrm{d}z^2}-\rho G_2^2\frac{\mathrm{d}}{\mathrm{d}z}\left\{\omega_{(k-1)}^{1x}+\frac{\mathrm{d}\omega_{(k-1)}^{1\lambda}}{\mathrm{d}z}a_x\right\}, \tag{3.123}$$

$$2G_{II}^{-1}\left(\rho G_2^2+\sigma_{\lambda\lambda}^0\right)\frac{\mathrm{d}}{\mathrm{d}z}\left(I_p^A\omega_{(k)}^{1\lambda}-Fa_x\eta_{(k)}^0+Fa_y\theta_{(k)}^0\right)$$
$$=G_{II}^{-1}\rho G_2^2F\left(a_y\omega_{(k)}^{1y}-a_x\omega_{(k)}^{1x}\right)$$
$$+\left(\rho G_2^2+\sigma_{\lambda\lambda}^0\right)\frac{\mathrm{d}^2}{\mathrm{d}z^2}\left\{I_p^A\omega_{(k-1)}^{1\lambda}-Fa_x\eta_{(k-1)}^0+Fa_y\theta_{(k-1)}^0\right\}$$
$$+\rho G_2^2F\frac{\mathrm{d}}{\mathrm{d}z}\left(a_x\omega_{(k-1)}^{1x}-a_y\omega_{(k-1)}^{1y}\right), \tag{3.124}$$

$$G_{II}^{-2}\rho\left(G_2^2-G_1^2\right)\omega_{(k)}^0=-2G_{II}^{-1}\left(\rho G_1^2+\sigma_{\lambda\lambda}^0\right)\frac{\mathrm{d}\omega_{(k-1)}^0}{\mathrm{d}z}$$
$$+\left(\rho G_1^2+\sigma_{\lambda\lambda}^0\right)\frac{\mathrm{d}^2\omega_{(k-2)}^0}{\mathrm{d}z^2}, \tag{3.125}$$

$$G_{II}^{-2}\rho\left(G_2^2-G_1^2\right)\omega_{(k)}^{1x}=-2G_{II}^{-1}\left(\rho G_1^2+\sigma_{\lambda\lambda}^0\right)\frac{\mathrm{d}\omega_{(k-1)}^{1x}}{\mathrm{d}z}$$
$$+\left(\rho G_1^2+\sigma_{\lambda\lambda}^0\right)\frac{\mathrm{d}^2\omega_{(k-2)}^{1x}}{\mathrm{d}z^2}, \tag{3.126}$$

$$G_{II}^{-2}\rho\left(G_2^2-G_1^2\right)\omega_{(k)}^{1y}=-2G_{II}^{-1}\left(\rho G_1^2+\sigma_{\lambda\lambda}^0\right)\frac{\mathrm{d}\omega_{(k-1)}^{1y}}{\mathrm{d}z}$$
$$+\left(\rho G_1^2+\sigma_{\lambda\lambda}^0\right)\frac{\mathrm{d}^2\omega_{(k-2)}^{1y}}{\mathrm{d}z^2}, \tag{3.127}$$

$$G_{II}^{-2}\rho\left(G_2^2-G_1^2\right)\psi_{(k)}=-2G_{II}^{-1}\left(\rho G_1^2+\sigma_{\lambda\lambda}^0\right)\frac{\mathrm{d}\psi_{(k-1)}}{\mathrm{d}z}$$
$$+\left(\rho G_1^2+\sigma_{\lambda\lambda}^0\right)\frac{\mathrm{d}^2\psi_{(k-2)}}{\mathrm{d}z^2}. \tag{3.128}$$

From Eqs. 3.115–3.118 it follows that on the quasi-longitudinal wave for $k \geq 0$

$$\omega_{(k)}^0=c_{(k)}^0=\text{const},\quad \omega_{(k)}^{1x}=c_{(k)}^{1x}=\text{const},$$
$$\omega_{(k)}^{1y}=c_{(k)}^{1y}=\text{const},\quad \psi_{(k)}=c_{(k)}^{\psi}=\text{const}, \tag{3.129}$$

$$\theta^0_{(k)} + a_y \omega^{1\lambda}_{(k)} = \frac{G_I G_2^2}{G_1^2 - G_2^2} c^{1y}_{(k-1)}, \quad \eta^0_{(k)} - a_x \omega^{1\lambda}_{(k)} = \frac{G_I G_2^2}{G_1^2 - G_2^2} c^{1x}_{(k-1)},$$

$$I_P^A \omega^{1\lambda}_{(k)} + a_y F \theta^0_{(k)} - a_x F \eta^0_{(k)} = \frac{G_I G_2^2}{G_1^2 - G_2^2} \left(a_y c^{1y}_{(k-1)} - a_x c^{1x}_{(k-1)} \right). \tag{3.130}$$

Solving the set of Eqs. 3.130 yields

$$\omega^{1\lambda}_{(k)} = 0, \quad \theta^0_{(k)} = \frac{G_I G_2^2}{G_1^2 - G_2^2} c^{1y}_{(k-1)}, \quad \eta^0_{(k)} = \frac{G_I G_2^2}{G_1^2 - G_2^2} c^{1x}_{(k-1)}. \tag{3.131}$$

From Eqs. 3.122–3.128 it follows that on the quasi-transverse wave for $k \geq 0$

$$\omega^0_{(k)} = \omega^{1x}_{(k)} = \omega^{1y}_{(k)} = \psi_{(k)} = 0, \tag{3.132}$$

$$\theta^0_{(k)} + a_y \omega^{1\lambda}_{(k)} = s^1_{(k)} = \text{const},$$

$$\eta^0_{(k)} - a_x \omega^{1\lambda}_{(k)} = s^2_{(k)} = \text{const}, \tag{3.133}$$

$$I_P^A \omega^{1\lambda}_{(k)} + a_y F \theta^0_{(k)} - a_x F \eta^0_{(k)} = s^3_{(k)} = \text{const}.$$

Solving (3.133) we have

$$\omega^{1\lambda}_{(k)} = \left(s^3_{(k)} + s^2_{(k)} a_x - s^1_{(k)} a_y \right) \left(I_p^C \right)^{-1} = c^{1\lambda}_{(k)} = \text{const},$$

$$\theta^0_{(k)} = s^1_{(k)} - a_y \left(s^3_{(k)} + s^2_{(k)} a_x - s^1_{(k)} a_y \right) \left(I_p^C \right)^{-1} = c^{\theta}_{(k)} = \text{const}, \tag{3.134}$$

$$\eta^0_{(k)} = s^2_{(k)} + a_x \left(s^3_{(k)} + s^2_{(k)} a_x - s^1_{(k)} a_y \right) \left(I_p^C \right)^{-1} = c^{\eta}_{(k)} = \text{const}.$$

Thus, for the particular case of a straight untwisted thin-walled beam of open profile, it has been found that the transient transverse wave is a pure shear-torsional mode as it follows from Eq. 3.132, while on the quasi-longitudinal wave the 'admixed' components of secondary shear occur of the higher order than the main values what is verified by Eq. 3.131.

If we put $a_x = a_y = 0$ in Eqs. 3.115–3.134 and neglect warping motions, then we could also obtain, as a particular case, the solution for a straight untwisted rod of a massive cross-section, which coincides with that resulting for a straight rod from Eqs. 3.43–3.48 in Rossikhin and Shitikova [2], when $æ = K = \tau = 0$. This result verifies the validity of the solution obtained for a thin-walled beam of open section.

As for the technical theory by Korbut and Lazarev [7], then there is no transition from the solution for a thin-walled beam to that for a straight beam with a massive cross-section, i.e. to the Timoshenko beam. It is well known that the

Timoshenko beam equations produce only two transient waves, longitudinal and transverse, propagating, respectively, with the velocities $G_L = \sqrt{E/\rho}$ and $G_T = \sqrt{K\mu/\rho}$. From the results presented in Sect. 2.2.1 it is evident that neglecting warping motions for a bisymmetrical beam, as it is seen from Eqs. 2.7 and 2.12, yields three transient waves: one longitudinal, which velocity (2.11) coincides with G_L, and two shear waves propagating with different velocities defined by (2.21), the magnitudes of which depend essentially on the geometry of the beams's cross-section.

3.2 Construction of the Desired Wave Fields in Terms of the Ray Series

Following the previous two papers by Rossikhin and Shitikova [8, 2] devoted to the dynamic behaviour of thin elastic bodies, where thin plates and shells have been considered in [8], and spatially curved and twisted slender rod-like solids have been studied in [2], as a method applicable for solving dynamic problems resulting in propagation of wave surfaces of strong and weak discontinuity we will use the method of ray expansions [9]. This method is one of the methods of perturbation technique, where time is used as a small parameter. The review of the papers devoted the ray method application in dynamic problems of solids and structures can be found in [8, 10].

Thus, knowing the discontinuities of the desired stress and velocity fields determined above within an accuracy of arbitrary constants on the two waves of strong discontinuity, quasi-longitudinal and quasi-transverse, propagating in the thin-walled beam of open profile, we could construct the fields of the desired functions also with an accuracy of the arbitrary constants utilizing the ray series [9], which are the power series with variable coefficients and which allow one to construct the solution behind the wave fronts of strong discontinuity [2, 8]:

$$Z(t,s) = \sum_{\alpha=I,II} \sum_{k=0}^{\infty} \frac{1}{k!} [Z^{\alpha}_{,(k)}](s) \left(t - \frac{s}{G_\alpha}\right)^k H\left(t - \frac{s}{G_\alpha}\right), \tag{3.135}$$

where Z is the desired value, $H(t - s/G_\alpha)$ is the unit Heaviside function, and the index $\alpha = I, II$ labels the ordinal number of the wave propagating with the velocity G_α.

The arbitrary constants entering into the ray expansions are determined from the initial and boundary conditions.

The example of using the ray expansions (3.135) for analyzing the impact response of spatially curved thin-walled beams of open cross-section will be demonstrated below in Sect. 4.2 by solving the problem about the normal impact of an elastic spherically-headed rod upon an elastic arch, representing itself a channel-beam curved along an arc of the circumference.

3.3 Conclusion

The theory presented in this chapter is distinct from other dynamical theories of thin-walled beams of open profile by its simplicity and physical clarity of the results obtained. The pre-stressed state in the beam has been investigated by virtue of transient waves of strong but small discontinuity. The strong or weak discontinuity of the k-order is defined by whether the value itself is discontinuous, or its k-order derivatives are discontinuous under the condition that the value itself and its $k-1$-order derivatives inclusive remain to be continuous fields.

The theory proposed admits the propagation of only two transient waves in spatially curved thin-walled beams of open section, quasi-longitudinal and quasi-transverse waves which travel with the velocities of elastic waves. We shall name the quasi-longitudinal wave as the longitudinal-flexural-warping wave, while the quasi-transverse wave will be called as torsional-shear wave. The prefix 'quasi-' points to the fact that on the longitudinal-flexural-warping wave the main values enumerated in its name experience the strong discontinuities, while the values characteristic of the quasi-transverse wave possess the weak discontinuities, and vice versa for the torsional-shear wave.

Application of any loads at the fixed instant of the time always results in the generation of transient waves (surfaces of strong or weak discontinuity). That is why the theory proposed is the general approach for solving many dynamic problems, in particular, the problems connected with impact, fracture, dynamic stability, and so on. For example, waves of small discontinuity are generated during the low-velocity impact by a falling mass. With the increase of the longitudinal compression load, which falls as the pre-stress in the expressions for defining the velocities of the waves of strong discontinuity, these velocities begin to decrease. Moreover, at a certain critical magnitude of the pre-stress the velocity of the quasi-transverse wave vanishes. In other words, the quasi-transverse wave 'locks' within the domain of the shock interaction. This leads to the fact that all its energy is concentrated in a small domain, what could result in the local damage within the contact zone.

This is the main conclusion which could be deduced from the given approach.

Appendix 1

From Fig. 3.1 it follows that the components of the vectors $\boldsymbol{\tau}\{\tau_i\}$, $\boldsymbol{\xi}\{\xi_i\}$, and $\mathbf{k}\{k_i\}$, $\mathbf{s}\{s_i\}$ are connected with each other by the relationships

$$\tau_i = k_i \cos\varphi + s_i \sin\varphi, \tag{3.136}$$

$$\xi_i = -k_i \sin\varphi + s_i \cos\varphi. \tag{3.137}$$

Substituting (3.136) and (3.137) into Frenet formulas

$$\frac{\mathrm{d}\tau_i}{\mathrm{d}s} = -\tau\xi_{,}. \tag{3.138}$$

$$\frac{\mathrm{d}\xi_i}{\mathrm{d}s} = \tau\tau_i - æ\lambda_i, \tag{3.139}$$

$$\frac{\mathrm{d}\lambda_i}{\mathrm{d}s} = æ\xi_i, \tag{3.140}$$

we obtain

$$-\frac{\mathrm{d}k_i}{\mathrm{d}s}\sin\varphi + \frac{\mathrm{d}s_i}{\mathrm{d}s}\cos\varphi = (K+\tau)k_i\cos\varphi + (K+\tau)s_i\sin\varphi - æ\lambda_i, \tag{3.141}$$

$$\frac{\mathrm{d}k_i}{\mathrm{d}s}\cos\varphi + \frac{\mathrm{d}s_i}{\mathrm{d}s}\sin\varphi = (K+\tau)k_i\sin\varphi - (K+\tau)s_i\cos\varphi, \tag{3.142}$$

$$\frac{\mathrm{d}\lambda_i}{\mathrm{d}s} = -k_i æ\sin\varphi + s_i æ\cos\varphi. \tag{3.143}$$

Multiplying (3.141) and (3.142) by $\cos\varphi$ and $\sin\varphi$, respectively, and adding the resulting equations yields

$$\frac{\mathrm{d}s_i}{\mathrm{d}s} = (K+\tau)k_i - æ\lambda_i\cos\varphi. \tag{3.144}$$

Multiplying further Eqs. 3.141 and 3.142 by $-\sin\varphi$ and $\cos\varphi$, respectively, and adding the resulting equations yields

$$\frac{\mathrm{d}k_i}{\mathrm{d}s} = -(K+\tau)s_i + æ\lambda_i\sin\varphi. \tag{3.145}$$

Formulas (3.144), (3.145) and (3.143) coincide, respectively, with Eqs. 3.15–3.17.

Appendix 2

$$f_{i(k-1)}\lambda_i = \frac{\mathrm{d}^2\omega_{(k-1)}}{\mathrm{d}s^2} - \frac{\mathrm{d}}{\mathrm{d}s}\Big(æ\eta_{(k-1)}\cos\varphi - æ\theta_{(k-1)}\sin\varphi\Big) + æ\sin\varphi\left\{\frac{\mathrm{d}\theta_{(k-1)}}{\mathrm{d}s} + \eta_{(k-1)}(K+\tau) - \omega_{(k-1)}æ\sin\varphi\right\}$$

$$- æ\cos\varphi\left\{\frac{\mathrm{d}\eta_{(k-1)}}{\mathrm{d}s} - \theta_{(k-1)}(K+\tau) + \omega_{(k-1)}æ\cos\varphi\right\}$$
$$+ \widetilde{\omega}^{1y}_{(k)}æ\sin\varphi - \widetilde{\omega}^{1y}_{(k)}æ\cos\varphi, \qquad (3.146)$$

$$f_{i(k-1)}k_i = \frac{\mathrm{d}^2\theta_{(k-1)}}{\mathrm{d}s^2} + (K+\tau)\left\{\frac{\mathrm{d}\eta_{(k-1)}}{\mathrm{d}s} - \theta_{(k-1)}(K+\tau) + \omega_{(k-1)}æ\cos\varphi\right\}$$
$$- æ\sin\varphi\left\{\frac{\mathrm{d}\omega_{(k-1)}}{\mathrm{d}s} + \theta_{(k-1)}æ\sin\varphi - \eta_{(k-1)}æ\cos\varphi\right\}$$
$$- \frac{\mathrm{d}}{\mathrm{d}s}\Big(\omega_{(k-1)}æ\sin\varphi - \eta_{(k-1)}(K+\tau)\Big) + \omega^{1\lambda}_{(k-1)}æ\cos\varphi, \qquad (3.147)$$

$$f_{i(k-1)}s_i = \frac{\mathrm{d}^2\eta_{(k-1)}}{\mathrm{d}s^2} - (K+\tau)\left\{\frac{\mathrm{d}\theta_{(k-1)}}{\mathrm{d}s} + \eta_{(k-1)}(K+\tau) - \omega_{(k-1)}æ\sin\varphi\right\}$$
$$+ æ\cos\varphi\left\{\frac{\mathrm{d}\omega_{(k-1)}}{\mathrm{d}s} + \theta_{(k-1)}æ\sin\varphi - \eta_{(k-1)}æ\cos\varphi\right\}$$
$$- \frac{\mathrm{d}}{\mathrm{d}s}\Big(\theta_{(k-1)}(K+\tau) - \omega_{(k-1)}æ\cos\varphi\Big) + \omega^{1\lambda}_{(k-1)}æ\sin\varphi, \qquad (3.148)$$

$$F_{1(k-1)} = \rho G_1^2\frac{\mathrm{d}}{\mathrm{d}s}\left\{\frac{\mathrm{d}\omega^0_{(k-1)}}{\mathrm{d}s} - æ\Big(\eta^0_{(k-1)}\cos\varphi - \theta^0_{(k-1)}\sin\varphi\Big)\right.$$
$$\left. + æ\big(a_x\cos\varphi + a_y\sin\varphi\big)\omega^{1\lambda}_{(k-1)}\right\}$$
$$+ 2æ\rho G_2^2\left\{\big(a_x\cos\varphi + a_y\sin\varphi\big)\frac{\mathrm{d}\omega^{1\lambda}_{(k-1)}}{\mathrm{d}s} + (K+\tau)\big(a_y\cos\varphi - a_x\sin\varphi\big)\omega^{1\lambda}_{(k-1)}\right\}$$
$$+ 2\rho G_2^2(K+\tau)\omega^{1\lambda}_{(k-1)}$$
$$+ 2æ\rho G_2^2\left\{\cos\varphi\left(\omega^{1x}_{(k-1)} - \frac{\mathrm{d}\eta^0_{(k-1)}}{\mathrm{d}s}\right) - \sin\varphi\left(\omega^{1y}_{(k-1)} - \frac{\mathrm{d}\theta^0_{(k-1)}}{\mathrm{d}s}\right)\right\}$$
$$+ \rho G_2^2 æ\left\{-\omega^{1x}_{(k-1)}\cos\varphi + \omega^{1y}_{(k-1)}\sin\varphi - æ\omega^0_{(k-1)}\right.$$
$$\left. + (K+\tau)\Big(\theta^0_{(k-1)}\cos\varphi + \eta^0_{(k-1)}\sin\varphi\Big)\right\} + f^0_{i(k-1)}\lambda_i\sigma^0_{\lambda\lambda}, \qquad (3.149)$$

$$F_{2(k-1)} = \rho G_2^2\frac{\mathrm{d}}{\mathrm{d}s}\left\{\frac{\mathrm{d}\theta^0_{(k-1)}}{\mathrm{d}s} - \omega^{1y}_{(k-1)} + \frac{\mathrm{d}\omega^{1\lambda}_{(k-1)}}{\mathrm{d}s}a_y - (K+\tau)a_x\omega^{1\lambda}_{(k-1)}\right\}$$
$$+ \rho G_2^2(K+\tau)\left\{\frac{\mathrm{d}\eta^0_{(k-1)}}{\mathrm{d}s} - \omega^{1x}_{(k-1)} - \frac{\mathrm{d}\omega^{1\lambda}_{(k-1)}}{\mathrm{d}s}a_x - (K+\tau)a_y\omega^{1\lambda}_{(k-1)}\right\}$$

$$- \rho G_1^2 æ \left\{ \frac{\mathrm{d}\omega^0_{(k-1)}}{\mathrm{d}s} - æ \left(\eta^0_{(k-1)} \cos\varphi - \theta^0_{(k-1)} \sin\varphi \right) \right.$$

$$\left. + æ \left(a_x \cos\varphi + a_y \sin\varphi \right) \omega^{1\lambda}_{(k-1)} \right\} \sin\varphi + f^0_{i(k-1)} k_i \sigma^0_{\lambda\lambda}, \qquad (3.150)$$

$$F_{3(k-1)} = \rho G_2^2 \frac{\mathrm{d}}{\mathrm{d}s} \left\{ \frac{\mathrm{d}\eta^0_{(k-1)}}{\mathrm{d}s} - \omega^{1x}_{(k-1)} - \frac{\mathrm{d}\omega^{1\lambda}_{(k-1)}}{\mathrm{d}s} a_x - (K+\tau) a_y \omega^{1\lambda}_{(k-1)} \right\}$$

$$- \rho G_2^2 (K+\tau) \left\{ \frac{\mathrm{d}\theta^0_{(k-1)}}{\mathrm{d}s} - \omega^{1y}_{(k-1)} + \frac{\mathrm{d}\omega^{1\lambda}_{(k-1)}}{\mathrm{d}s} a_y - (K+\tau) a_x \omega^{1\lambda}_{(k-1)} \right\}$$

$$+ \rho G_1^2 æ \left\{ \frac{\mathrm{d}\omega^0_{(k-1)}}{\mathrm{d}s} - æ \left(\eta^0_{(k-1)} \cos\varphi - \theta^0_{(k-1)} \sin\varphi \right) \right.$$

$$\left. + æ \left(a_x \cos\varphi + a_y \sin\varphi \right) \omega^{1\lambda}_{(k-1)} \right\} \cos\varphi + f^0_{i(k-1)} s_i \sigma^0_{\lambda\lambda}, \qquad (3.151)$$

$$F_{4(k-1)} = \rho G_1^2 \frac{\mathrm{d}^2}{\mathrm{d}s^2} \left[-\omega^{1x}_{(k-1)} - \omega^0_{(k-1)} æ \cos\varphi + (K+\tau) \theta^0_{(k-1)} \right]$$

$$- 2æ^2 \rho G_2^2 \left[-\omega^{1x}_{(k-1)} - \omega^0_{(k-1)} æ \cos\varphi + \theta^0_{(k-1)} (K+\tau) \right]$$

$$- 2æ \rho G_2^2 \left[\frac{\mathrm{d}\omega^{1\lambda}_{(k-1)}}{\mathrm{d}s} \sin\varphi + (K+\tau) \omega^{1\lambda}_{(k-1)} \cos\varphi \right]$$

$$- \rho G_1^2 \frac{\mathrm{d}}{\mathrm{d}s} \left(æ \sin\varphi \, \omega^{1\lambda}_{(k-1)} \right) + I_x^{-1} \sigma^0_{\lambda\lambda} \int_F f_{i(k-1)} \lambda_i y \, \mathrm{d}F, \qquad (3.152)$$

$$F_{5(k-1)} = \rho G_1^2 \frac{\mathrm{d}^2}{\mathrm{d}s^2} \left[-\omega^{1y}_{(k-1)} + \omega^0_{(k-1)} æ \sin\varphi - (K+\tau) \eta^0_{(k-1)} \right]$$

$$- 2æ^2 \rho G_2^2 \left[-\omega^{1y}_{(k-1)} + \omega^0_{(k-1)} æ \sin\varphi - \eta^0_{(k-1)} (K+\tau) \right]$$

$$- 2æ \rho G_2^2 \left[\frac{\mathrm{d}\omega^{1\lambda}_{(k-1)}}{\mathrm{d}s} \cos\varphi - (K+\tau) \omega^{1\lambda}_{(k-1)} \sin\varphi \right]$$

$$- \rho G_1^2 \frac{\mathrm{d}}{\mathrm{d}s} \left(æ \cos\varphi \, \omega^{1\lambda}_{(k-1)} \right) + I_y^{-1} \sigma^0_{\lambda\lambda} \int_F f_{i(k-1)} \lambda_i x \, \mathrm{d}F, \qquad (3.153)$$

$$F_{6(k-1)}=-æ^2(2\rho G_2^2+\sigma_{\lambda\lambda}^0)\Big\{-\psi_{(k-1)}+æ\cos\varphi\Big[-\omega_{(k-1)}^{1y}+\omega_{(k-1)}^0 æ\sin\varphi$$
$$-(K+\tau)\eta_{(k-1)}^0\Big]-æ\sin\varphi\Big[-\omega_{(k-1)}^{1x}-\omega_{(k-1)}^0 æ\cos\varphi+(K+\tau)\theta_{(k-1)}^0\Big]\Big\}$$
$$+\left(\rho G_1^2+\sigma_{\lambda\lambda}^0\right)\frac{\mathrm{d}^2}{\mathrm{d}s^2}\Big\{-\psi_{(k-1)}+æ\cos\varphi\Big[-\omega_{(k-1)}^{1y}+\omega_{(k-1)}^0 æ\sin\varphi-(K+\tau)\eta_{(k-1)}^0\Big]$$
$$-æ\sin\varphi\Big[-\omega_{(k-1)}^{1x}-\omega_{(k-1)}^0 æ\cos\varphi+(K+\tau)\theta_{(k-1)}^0\Big]\Big\}, \tag{3.154}$$

$$F_{7(k-1)}=\rho G_2^2\frac{\mathrm{d}}{\mathrm{d}s}\Big\{I_p^A\frac{\mathrm{d}\omega_{(k-1)}^{1\lambda}}{\mathrm{d}s}$$
$$+æI_y\cos\varphi\Big[-\omega_{(k-1)}^{1y}+\omega_{(k-1)}^0 æ\sin\varphi-(K+\tau)\eta_{(k-1)}^0\Big]$$
$$+æI_x\sin\varphi\Big[-\omega_{(k-1)}^{1x}-\omega_{(k-1)}^0 æ\cos\varphi+(K+\tau)\theta_{(k-1)}^0\Big]$$
$$+a_xF\left(\omega_{(k-1)}^{1x}-\frac{\mathrm{d}\eta_{(k-1)}^0}{\mathrm{d}s}\right)-a_yF\left(\omega_{(k-1)}^{1y}-\frac{\mathrm{d}\theta_{(k-1)}^0}{\mathrm{d}s}\right)\Big\}$$
$$-\rho G_2^2F(K+\tau)\Big\{a_y\left(\omega_{(k-1)}^{1x}-\frac{\mathrm{d}\eta_{(k-1)}^0}{\mathrm{d}s}\right)+a_x\left(\omega_{(k-1)}^{1y}-\frac{\mathrm{d}\theta_{(k-1)}^0}{\mathrm{d}s}\right)\Big\}$$
$$+\rho G_2^2(K+\tau)æI_y\sin\varphi\Big[-\omega_{(k-1)}^{1y}+\omega_{(k-1)}^0 æ\sin\varphi-(K+\tau)\eta_{(k-1)}^0\Big]$$
$$-\rho G_2^2(K+\tau)æI_x\cos\varphi\Big[-\omega_{(k-1)}^{1x}-\omega_{(k-1)}^0 æ\cos\varphi+(K+\tau)\theta_{(k-1)}^0\Big]$$
$$+\rho G_1^2 æF\left(a_x\cos\varphi+a_y\sin\varphi\right)\Big[-\frac{\mathrm{d}\omega_{(k-1)}^0}{\mathrm{d}s}$$
$$+æ\left(\eta_{(k-1)}^0\cos\varphi-\theta_{(k-1)}^0\sin\varphi\right)-æ(a_x\cos\varphi+a_y\sin\varphi)\omega_{(k-1)}^{1\lambda}\Big]$$
$$+\rho G_1^2 æ\Big\{I_y\cos\varphi\frac{\mathrm{d}}{\mathrm{d}s}\Big[-\omega_{(k-1)}^{1y}+\omega_{(k-1)}^0 æ\sin\varphi-(K+\tau)\eta_{(k-1)}^0\Big]$$
$$+I_x\sin\varphi\frac{\mathrm{d}}{\mathrm{d}s}\Big[-\omega_{(k-1)}^{1x}-\omega_{(k-1)}^0 æ\cos\varphi+(K+\tau)\theta_{(k-1)}^0\Big]$$
$$-æ\left(I_y\cos^2\varphi+I_x\sin^2\varphi\right)\omega_{(k-1)}^{1\lambda}\Big\}-\rho G_2^2F(K+\tau)^2I_p^A\omega_{(k-1)}^{1\lambda}$$
$$+\sigma_{\lambda\lambda}^0\int\limits_F\left[f_{i(k-1)}s_i(x-a_x)-f_{i(k-1)}k_i(y-a_y)\right]\mathrm{d}F. \tag{3.155}$$

Appendix 3

$$A_{0(k)}(s) = \Big\{G_I^{-1}\big(\rho G_1^2 + 2\rho G_2^2 + 2\sigma_{\lambda\lambda}^0\big)æ\Big[-\big(a_x\cos\varphi + a_y\sin\varphi\big)\omega_{(k)}^{1\lambda} + \eta_{(k)}^0\cos\varphi - \theta_{(k)}^0\sin\varphi\Big] + F_{1(k-1)}|_{G=G_I}\Big\}\frac{1}{2\ \rho G_I}, \quad (3.156)$$

$$A_{1(k)}(s) = \left\{2\ \rho G_I\frac{\mathrm{d}}{\mathrm{d}s}\Big[\theta_{(k)}^0(K+\tau)\Big] - æ(\rho G_1^2 + 2\rho G_2^2 + 2\sigma_{\lambda\lambda}^0)G_I^{-1}\omega_{(k)}^{1\lambda}\sin\varphi - F_{4(k-1)}|_{G=G_I}\right\}\frac{1}{2\ \rho G_I}, \quad (3.157)$$

$$A_{2(k)}(s) = \left\{-2\rho G_I\frac{\mathrm{d}}{\mathrm{d}s}\Big[\eta_{(k)}^0(K+\tau)\Big] - æ(\rho G_1^2 + 2\rho G_2^2 + 2\sigma_{\lambda\lambda}^0)G_I^{-1}\omega_{(k)}^{1\lambda}\cos\varphi - F_{5(k-1)}|_{G=G_I}\right\}\frac{1}{2\ \rho G_I}, \quad (158)$$

$$A_{3(k)}(s) = \left\{-2\rho G_I\frac{\mathrm{d}}{\mathrm{d}s}\Big[\Big(\theta_{(k)}^0\sin\varphi + \eta_{(k)}^0\cos\varphi\Big)æ(K+\tau)\Big] - F_{6(k-1)}|_{G=G_I}\right\}\frac{1}{2\ \rho G_I}, \quad (3.159)$$

$$\begin{aligned} B_{1(k-1)} = -\omega_{(k)}^0 æ\cos\varphi - \Big\{&2G_{II}^{-1}\big(\rho G_1^2 + \sigma_{\lambda\lambda}^0\big) \\ &\times\frac{\mathrm{d}}{\mathrm{d}s}\Big[\omega_{(k-1)}^{1x} + \omega_{(k-1)}^0 æ\cos\varphi - \theta_{(k-1)}^0(K+\tau)\Big] \\ &+ æ\big(\rho G_1^2 + 2\rho G_2^2 + 2\sigma_{\lambda\lambda}^0\big)G_{II}^{-1}\omega_{(k-1)}^{1\lambda}\sin\varphi \\ &+ F_{4(k-2)}|_{G=G_{II}}\Big\}G_{II}^2\rho^{-1}\big(G_2^2 - G_1^2\big)^{-1}, \end{aligned} \quad (3.160)$$

$$\begin{aligned} B_{2(k-1)} = \omega_{(k)}^0 æ\sin\varphi - \Big\{&2G_{II}^{-1}\big(\rho G_1^2 + \sigma_{\lambda\lambda}^0\big) \\ &\times\frac{\mathrm{d}}{\mathrm{d}s}\Big[\omega_{(k-1)}^{1y} - \omega_{(k-1)}^0 æ\sin\varphi + \eta_{(k-1)}^0(K+\tau)\Big] \\ &+ æ\big(\rho G_1^2 + 2\rho G_2^2 + 2\sigma_{\lambda\lambda}^0\big)G_{II}^{-1}\omega_{(k-1)}^{1\lambda}\cos\varphi \\ &+ F_{5(k-2)}|_{G=G_{II}}\Big\}G_{II}^2\rho^{-1}\big(G_2^2 - G_1^2\big)^{-1}, \end{aligned} \quad (3.161)$$

$$B_{3(k)} = \Big\{ G_{II}^{-1} æ\big(\rho G_1^2 + \rho G_2^2 + 2\sigma_{\lambda\lambda}^0\big)\omega_{(k)}^0 \sin\varphi + G_{II}^{-1}\rho G_2^2 \Big(B_{2(k-1)} - æ\omega_{(k)}^0 \sin\varphi\Big)$$
$$+ F_{2(k-1)}|_{G=G_{II}} \Big\} \frac{1}{2}\, G_{II}\big(\rho G_2^2 + \sigma_{\lambda\lambda}^0\big)^{-1}, \tag{3.162}$$

$$B_{4(k)} = \Big\{ - G_{II}^{-1} æ\big(\rho G_1^2 + \rho G_2^2 + 2\sigma_{\lambda\lambda}^0\big)\omega_{(k)}^0 \cos\varphi + G_{II}^{-1}\rho G_2^2 \Big(B_{1(k-1)} + æ\omega_{(k)}^0 \cos\varphi\Big)$$
$$+ F_{3(k-1)}|_{G=G_{II}} \Big\} \frac{1}{2}\, G_{II}\big(\rho G_2^2 + \sigma_{\lambda\lambda}^0\big)^{-1}, \tag{3.163}$$

$$B_{5(k)} = \Big\{ G_{II}^{-1} æ\big(\rho G_1^2 + \rho G_2^2 + 2\sigma_{\lambda\lambda}^0\big)\big[B_{1(k-1)} I_x \sin\varphi + B_{2(k-1)} I_y \cos\varphi$$
$$+ \omega_{(k)}^0 F\big(a_x \cos\varphi + a_y \sin\varphi\big) + \frac{1}{2}\, \omega_{(k)}^0 æ(I_x - I_y)\sin 2\varphi\Big]$$
$$+ G_{II}^{-1}\rho G_2^2 F\Big[B_{2(k-1)} a_y - B_{1(k-1)} a_x - \omega_{(k)}^0 æ\big(a_x \cos\varphi + a_y \sin\varphi\big)\Big]$$
$$+ F_{7(k-1)}|_{G=G_{II}}, \Big\} \frac{1}{2}\, G_{II}\big(\rho G_2^2 + \sigma_{\lambda\lambda}^0\big)^{-1}. \tag{3.164}$$

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Chapter 4
Impact Response of Thin-Walled Beams of Open Profile

Abstract The dynamic theory of thin-walled beams of generic open section with a spatially curved and twisted longitudinal axes proposed in the previous chapter with due account for the axial precompression is utilized here for the analysis of the impact response, since the derived hyperbolic system of recurrent equations together with the ray expansions allow one to describe the short-time processes, in particular, the processes of shock interaction, because the convergence of the ray series essentially depends on the rapidity of the duration of the process under consideration. Examples of using the ray expansions for analyzing the impact response of spatially curved thin-walled beams of open cross-section are demonstrated by solving the problem about the normal impact of a steel rod with a rounded end upon a steel arch, representing itself a channel-beam curved along an arc of the circumference. The time-dependence of the contact force is found and analyzed graphically at different levels of the axial compression force.

Keywords Impact · Shock interaction · Linear contact force · Hertz's contact law · Axial compression force

4.1 Introduction

During the past two decades foreign object impact damage to structures has received a great deal of attention, since thin-walled structures are known to be susceptible to damage resulting from accidental impact by foreign objects. Impact on aircraft structures or civil engineering structures, for instance, from dropped tools, hail, and debris thrown up from the runway, poses a problem of great concern to designers. Since the impact response is not purely a function of material's properties and depends also on the dynamic structural behaviour of a

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target, it is important to have a basic understanding of the structural response and how it is affected by different parameters [1]. From this point of view, analytical models are useful as they allow systematic parametric investigation and provide a foundation for prediction of impact damage.

An impact response analysis requires a good estimate of contact force throughout the impact duration. Low velocity impact problems, which also took the local indentation into account, have been solved by many authors. Reference to the state-of-the-art paper [1] shows that in most studies it was assumed that the impacted structure was free of any initial stresses. But this does not adequately reflect the real multidirectional complex loading states that the materials experience during their service life. In practice, the composite facing of a structure may be under a preload, e.g. a sandwich structure with laminate facing under bending loads, jet engine fan blades subjected to centrifugal forces. Even when stationary on the runway a composite airframe is under pre-stress. The other example of great practical interest is the analysis of impact response of pipes pressurized for hydrotests subjected to dropped tools.

To the authors knowledge, the first attempts to investigate the dynamic response of straight thin-walled beams of bisymmetric and monosymmetric open profile impacted by an elastic sphere were made in [2] and [3], respectively. The technical theory by Korbut and Lazarev [4], which takes the rotary inertia and transverse shear deformations into account, was adopted to describe the dynamic behaviour of thin-walled beams, and the local bearing of the material in the place of contact was considered according to the Hertz's contact theory. One-term ray expansions [1] were used to construct the desired stress and velocity fields what allowed the authors to find an analytical solution for the maximal contact force and to estimate the duration of contact. Later this theory was generalized by Rossikhin and Shitikova [5] taking the extension of the thin-walled beam's middle surface into account, and the dynamic response of a Timoshenko-type thin-walled beam of general open cross-section to the impact of a sphere was investigated in [5] using the wave approach [1]. But, as it has been demonstrated in Introduction, the main disadvantage of all technical theories of thin-walled beams, among which is the theory by Korbut and Lazarev [4] and its generalization [5], is that they produce three or even four transient shear-torsional waves propagating with the velocities dependent on the geometrical parameters of the beam. Thus, it has been shown that all existing technical theories of thin-walled beams could not capture the general relationships of the transient shear-torsional waves, since each thin-walled beam possesses its own velocities, and thus, each time it is needed to study a concrete object with its concrete dimensions (some examples could be found in [5]).

Except papers cited above, i.e. [2, 3] and [5], these authors have found in literature only one paper by Taiwanese researchers Lin et al. [6] suggesting a numerical approach to determining the transient response of straight nonrectangular bars subjected to transverse elastic impact. To our great surprise, this paper is free from any formulas, although it is devoted to 'transverse impact response' of straight thin-walled beams with channel and tee profiles. The results obtained in [6] via finite element method (but it is impossible to understand what theory was adopted during solution, as well as what numerical algorithms were implemented) were compared graphically via numerous figures with experimental data obtained by the same authors themselves. As this takes place, only longitudinal waves were taken into account. But numerous data on impact analysis of structures [1] shows that during transverse impact the transverse forces and, thus, the shear waves predominate in the wave phenomena. That is why, despite the fact that the authors of the cited paper [6] declared the good agreement between their numerical and experimental investigations, it is hard to believe in such perfect matching.

Below in order to analyze the impact response of spatially curved thin-walled beams of open profile we shall implement the theoretical results which have already been described in Chap. 3, since the derived hyperbolic system of recurrent equations together with the ray expansions allow one to describe the short-time processes, in particular, the processes of shock interaction. The theory suggested in Sect. 3.1 is free from new additional constants such as the shear coefficients in Timoshenko-like theories, and it involves only elastic moduli and Poisson's coefficient, resulting in the fact that only two transient waves, quasi-longitudinal and quasi-transverse, propagate in the thin-walled beam. Besides, the velocity of the transient shear wave coincides with that of the transverse wave in three-dimensional elastic media, in contrast to the Timoshenko-like theories, according to which several transverse waves propagate, as a rule, with the velocities dependent on the geometrical parameters of the beam. As for the longitudinal wave, then in both theories it propagates with the velocity of the longitudinal wave in a thin elastic rod.

The problems of shock interaction fully conform to the requirements of the theory presented in Chap. 3, wherein the convergence of the ray series essentially depends on the rapidity of the duration of the process under consideration.

4.2 Impact of a Hemispherical-Nosed Rod Against a Thin-Walled Beam of Open Profile

Below utilizing the theory developed in Chap. 3 and using one-term and multiple-term ray expansions, the normal impact of an elastic thin spherically-headed rod of circular cross-section against a lateral surface of the initially stressed thin-walled beam of generic open section will be investigated.

4.2.1 The Wave Theory of Impact with Due Account for One-Term Ray Expansions

Let us consider the normal impact of an elastic thin rod of circular cross section upon a lateral surface of a thin-walled elastic beam of open section (Fig. 4.1). At the moment of impact, which occurs at the point $n = 0$, the velocity of the

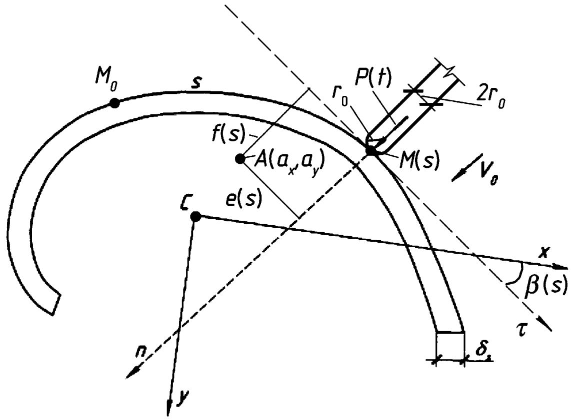

Fig. 4.1 Scheme of shock interaction

impacting rod is equal to V_0, and the longitudinal shock wave begins to propagate along the rod with the velocity $G_0 = \sqrt{E_0\rho_0^{-1}}$, where E_0 is its elastic modulus, and ρ_0 is its density.

The interacting bodies are considered to be of rather long extent in order to ignore reflected waves, since they arrive at the place of contact after the rebounce of the impactor from the target.

Behind the wave front the stress σ^- and velocity v^- fields can be represented using the ray series [1]

$$\sigma^- = -\sum_{k=0}^{\infty} \frac{1}{k!} \left[\frac{\partial^k \sigma}{\partial t^k}\right] \left(t - \frac{n}{G_0}\right)^k, \tag{4.1}$$

$$v^- = V_0 - \sum_{k=0}^{\infty} \frac{1}{k!} \left[\frac{\partial^k v}{\partial t^k}\right] \left(t - \frac{n}{G_0}\right)^k, \tag{4.2}$$

where n is the coordinate directed along the rod's axis with the origin in the place of contact.

Considering that the discontinuities in the elastic rod remain constant during the process of the wave propagation and utilizing the condition of compatibility, we have

$$\left[\frac{\partial^{k+1} u}{\partial n \partial t^k}\right] = -G_0^{-1} \left[\frac{\partial^{k+1} u}{\partial t^{k+1}}\right] = -G_0^{-1} \left[\frac{\partial^k v}{\partial t^k}\right], \tag{4.3}$$

where u is the displacement.

With due account of (4.3) the Hook's law on the wave surface can be rewritten as

$$\left[\frac{\partial^k \sigma}{\partial t^k}\right] = -\rho_0 G_0 \left[\frac{\partial^k v}{\partial t^k}\right]. \tag{4.4}$$

Substituting (4.4) in (4.1) yields

$$\sigma^{-} = \rho_0 G_0 \sum_{k=0}^{\infty} \frac{1}{k!} \left[\frac{\partial^k v}{\partial t^k} \right] \left(t - \frac{n}{G_0} \right)^k . \tag{4.5}$$

Comparison of relationships (4.5) and (4.2) gives

$$\sigma^{-} = \rho G_0 (V_0 - v^{-}). \tag{4.6}$$

When $n = 0$, (4.6) takes the form

$$\sigma_{\text{cont}} = \rho G_0 (V_0 - v_n), \tag{4.7}$$

where $\sigma_{\text{cont}} = \sigma^{-}|_{n=0}$ is the contact stress, and $v_n = v^{-}|_{n=0}$ is the normal velocity of the beam's points within the contact domain.

Formula (4.7) allows one to find the contact force

$$P = \pi r_0^2 \rho_0 G_0 (V_0 - v_n), \tag{4.8}$$

where r_0 is the radius of the rod's cross section.

However, the contact force can be determined not only via Eq. 4.8, but using the Hertz's law as well [1, 7]

$$P = k\alpha^{3/2}, \tag{4.9}$$

where α is the value governing the local bearing of the target's material during the process of its contact interaction with the impactor, and k is the contact stiffness coefficient depending on the geometry of colliding bodies, as well as on their elastic constants

$$k = \frac{4}{3} E^{*} \sqrt{R'}\ , \quad \frac{1}{R'} = \frac{1}{r_0} + \frac{1}{R_{\text{t}}}, \quad \frac{1}{E^{*}} = \frac{1 - \nu_0^2}{E_0} + \frac{1 - \nu_{\text{t}}^2}{E_{\text{t}}}. \tag{4.10}$$

Here, r_0 is the radius of the rod's rounded end (Fig. 4.1), R_{t} is the radius of curvature of the target in the place of impact, and E_0, ν_0 and E_{t}, ν_{t} are the elastic modulus and the Poisson's ratio of the indenter and the target, respectively.

Eliminating the force P from (4.8) and (4.9), we are led to the equation for determining the value $\alpha(t)$

$$v_n + \frac{k}{\pi r_0^2 \rho_0 G_0} \alpha^{3/2} = V_0. \tag{4.11}$$

In order to express the velocity v_n in terms of α

$$v_n = \dot{\alpha} + \theta^0_{(0)} \sin \beta(s_1) + \eta^0_{(0)} \cos \beta(s_1) + e(s_1) \omega^{1\lambda}_{(0)}, \tag{4.12}$$

let us analyze the wave processes occurring in the thin-walled beam of open section.

Since the contours of the beam's cross sections remain rigid during the process of impact, then all sections involving by the contact domain form a layer which moves as rigid whole. Let us name it as a contact layer (Fig. 4.1). If we neglect the inertia forces due to the smallness of this layer, then the equations describing its motion take the form

$$2Q_{\lambda x} + P\sin\beta(s_1) = 0, \tag{4.13}$$

$$2Q_{\lambda y} + P\cos\beta(s_1) = 0, \tag{4.14}$$

$$2M_A + Pe(s_1) = 0, \tag{4.15}$$

where $\beta(s_1)$ is the angle between the x-axis and the tangent to the contour at the point M with the $s_1$coordinate, and $e(s_1)$ is the length of the perpendicular erected from the flexural center to the rod's axis (Fig. 4.1).

The values $Q_{\lambda x}$, $Q_{\lambda y}$ and M_A entering in (4.13)–(4.15) are calculated as follows: behind the wave fronts of the quasi-longitudinal and quasi-transverse waves upto the boundary planes of the contact layer, the ray series (3.135) are written on the unknown moving boundary *a(t)* of the contact domain with due account for the effect of 'retardation' [8] implying in the fact that the transient waves detach from the boundary of the contact domain not immediately at the moment of impact $t = 0$, but after some time duration $t = t^* = r_0 V_0 / 2G_{II}^2$ elapsed from the moment of impact [9]. Thus as a result we obtain

$$Z(t) = \sum_{\alpha = I, II} \sum_{k=0}^{\infty} \frac{1}{k!} [Z^{\alpha}_{,(k)}]|_{s=a-a^*} \left(t - t^* - \frac{a - a^*}{G_\alpha} \right)^k, \tag{4.16}$$

where $s = a^*$ is the location of the contact region boundary at $t = t^*$,

$$a = a^* + a_0(t - t^*) + a_1(t - t^*)^2 + \cdots, \tag{4.17}$$

$$t - t^* - (a - a^*)G_\alpha^{-1} = (1 - a_0 G_\alpha^{-1})(t - t^*) - a_1 G_\alpha^{-1}(t - t^*)^2 + \cdots, \tag{4.18}$$

and a_0, a_1, ... are yet unknown constant values, and $a^* = r_0 V_0 G_{II}^{-1}$.

The series (4.16)–(4.18) can be employed due to the smallness of the duration of contact of the rod with the beam. If we restrict ourselves only by the first terms of the series, then it is possible to find them from (3.56), (3.57) and (3.67) at $k = 0$. As a result, we obtain the following expressions for the values $Q_{\lambda x} = Q_{\lambda x(0)}$, $Q_{\lambda y} = Q_{\lambda y(0)}$ and $M_A = M_{A(0)}$:

$$Q_{\lambda x(0)} = -G_{II}^{-1} \rho G_2^2 F \theta^0_{(0)}, \tag{4.19}$$

$$Q_{\lambda y(0)} = -G_{II}^{-1} \rho G_2^2 F \eta^0_{(0)}, \tag{4.20}$$

$$M_{A(0)} = -G_{II}^{-1} \rho G_2^2 I_p^A \omega^{1\lambda}_{(0)}. \tag{4.21}$$

Substituting (4.19)–(4.21) in (4.13)–(4.15) with due account for (4.9), we have

$$\theta^0_{(0)} = \frac{1}{2}\, G_{II}(\rho G_2^2)^{-1}F^{-1}k\alpha^{3/2}\sin\beta, \tag{4.22}$$

$$\eta^0_{(0)} = \frac{1}{2}\, G_{II}(\rho G_2^2)^{-1}F^{-1}k\alpha^{3/2}\cos\beta, \tag{4.23}$$

$$\omega^{1\lambda}_{(0)} = \frac{1}{2}\, G_{II}(\rho G_2^2)^{-1}(I_p^A)^{-1}ek\alpha^{3/2}. \tag{4.24}$$

Considering (4.22)–(4.24), we can rewrite (4.12) as

$$v_n = \dot{\alpha} + \frac{1}{2}\, G_{II}(\rho G_2^2)^{-1}k\alpha^{3/2}\left\{F^{-1} + (I_p^A)^{-1}e^2\right\}. \tag{4.25}$$

Substituting (4.25) in (4.11), we obtain the equation for defining α

$$\dot{\alpha} + \kappa\alpha^{3/2} = V_0, \tag{4.26}$$

where

$$\kappa = k\left\{\frac{1}{\pi r_0^2 \rho_0 G_0} + \frac{1}{2}\,\frac{G_{II}}{\rho G_2^2}\left[F^{-1} + (I_p^A)^{-1}e^2\right]\right\}.$$

An approximate solution of (4.26) can be written as

$$\alpha \approx V_0 t\left(1 - \frac{2}{5}\, V_0^{1/2}\kappa t^{3/2}\right), \tag{4.27}$$

whence it follows the contact duration

$$t_{\text{cont}} = \left(\frac{5}{2}\, V_0^{-1/2}\kappa^{-1}\right)^{2/3}. \tag{4.28}$$

The maximum deformation $\alpha_{\max}$ is reached at $\dot{\alpha} = 0$ and, due to (4.26), is equal to

$$\alpha_{\max} = \left(\frac{V_0}{\kappa}\right)^{2/3}. \tag{4.29}$$

From (4.29) it follows that

$$P_{\max} = k\alpha_{\max}^{3/2} = kV_0\kappa^{-1}. \tag{4.30}$$

If a thin-walled open section beam is subjected to the initial extension stress $\sigma^0_{\lambda\lambda} > 0$, then the velocity G_{II} increases with the increase in $\sigma^0_{\lambda\lambda}$, and hence $\alpha_{\max}$ and $P_{\max}$ decrease. If the initial pre-stress $\sigma^0_{\lambda\lambda}$ is compressional, then with the increase in $\left|\sigma^0_{\lambda\lambda}\right|$ the velocity G_{II} decreases, and hence $\alpha_{\max}$ and $P_{\max}$ increase.

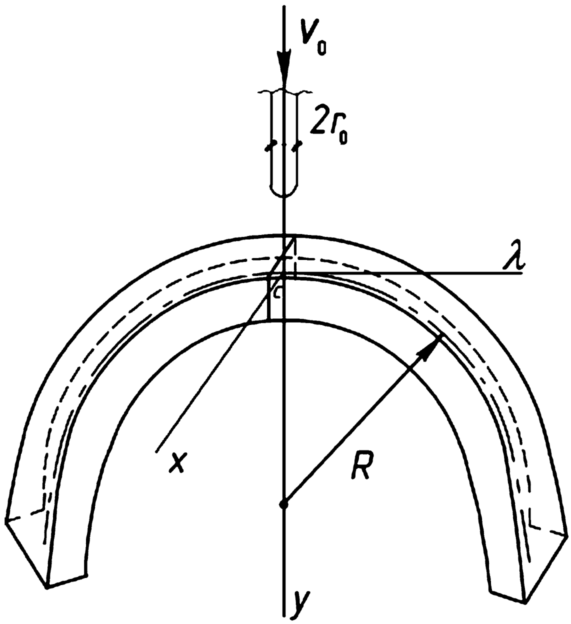

Fig. 4.2 Scheme of shock interaction

The values $\alpha_{\max}$ and $P_{\max}$ attain their maximal magnitudes at $G_{II} = 0$ $\left(\sigma^0_{\lambda\lambda} = \rho G_2^2\right)$ and have the form

$$\alpha_{\max} = \left(\frac{\pi r_0^2 \rho_0 G_0 V_0}{k}\right)^{2/3}, \tag{4.31}$$

$$P_{\max} = \pi r_0^2 \rho_0 G_0 V_0. \tag{4.32}$$

4.2.2 Wave Theory of Impact with Due Account for Multiple-Term Ray Expansions

Let us now consider the problem stated in Sect. 4.2.1, but instead of one-term ray expansions for the desired values we shall use two- and three-term ray expansions (3.135) and (4.16)–(4.18). Besides, here we shall take the inertia of the contact domain into account.

For simplicity, we shall suppose that the thin-walled beam has the constant curvature $æ = R_t^{-1} = R^{-1}$, while the torsion $\tau = 0$. Moreover, the beam possesses one axis of symmetry, along which the impactor moves. At the point of impact, the value $\beta(s_1) = 0$ (Fig. 4.2).

With the assumptions made, the motion of the contact domain is described by one equation

$$M\dot{\eta}^0 = 2Q_{\lambda y} + P\ , \tag{4.33}$$

where $M = 2\rho Fa(t - t^*)$ and $2a(t - t^*)$ are the mass and the width of the contact spot, respectively, while the value α is connected with the value a according to the Hertz's theory by the following relationship:

$$\alpha = R'^{-1}a^2, \tag{4.34}$$

in so doing in view of (4.34). Equation 4.11 takes the form

$$\dot{\alpha} + \eta^0 + \frac{k}{\pi\rho_0 G_0 R'}\ \alpha^{1/2} = V_0. \tag{4.35}$$

Equation 4.9, the initial condition

$$\dot{\alpha} = V_0, \tag{4.36}$$

which may be written at $t = t^*$ due the smallness of the time t^*, as well as the boundary conditions

$$\theta^0 = \omega^{1y} = \omega^{1\lambda} = \psi = 0, \tag{4.37}$$

$$\omega^{1x} = -\dot{a}R^{-1}, \tag{4.38}$$

$$\omega^0 = \dot{a} \tag{4.39}$$

should be added to Eqs. 4.33–4.35.

Representing the value α in terms of a series

$$\alpha = \alpha^* + \alpha_1(t - t^*) + \alpha_2(t - t^*)^2 + \alpha_3(t - t^*)^3 + \cdots, \tag{4.40}$$

where $\alpha^* \approx 2V_0t^* = r_0V_0^2G_{II}^{-2}$, and $\alpha_1, \alpha_2, \alpha_3, \ldots$ are yet unknown constants, and substituting (4.17) and (4.40) in (4.34), we find with due account for the initial condition (4.36)

$$\begin{aligned}
&\alpha_1 = V_0\ , \quad a^* = (\alpha^*R')^{1/2}, \quad a_0 = \frac{1}{2}\ V_0\alpha^{*-1/2}R'^{1/2},\\
&a_1 = \frac{1}{2}\ \alpha^{*1/2}R'^{1/2}\left(\alpha_2\alpha^{*-1} - \frac{1}{4}\ V_0^2\alpha^{*-2}\right),\\
&a_2 = \frac{1}{2}\ \alpha^{*1/2}R'^{1/2}\left[\alpha_3\alpha^{*-1} - \frac{1}{2}\ \alpha_1\alpha^{*-1}\left(\alpha_2\alpha^{*-1} - \frac{1}{4}\ V_0^2\alpha^{*-2}\right)\right].
\end{aligned} \tag{4.41}$$

Since the solution behind the wave fronts upto the contact domain is constructed in terms of the ray series (4.16), then we should first determine the ray series coefficients for the desired values. Thus, putting $k = 0$ and 1 in (3.80)–(3.82) and (3.83)–(3.86) with due account for $K = \tau = 0$, $æ = R^{-1}$, and $\varphi = \chi = 0$ (Fig. 4.2), we obtain for the quasi-longitudinal wave

$$\omega^0_{(0)I} = c_{0(0)}, \quad \omega^{1x}_{(0)I} = c_{1(0)} - \frac{1}{R}\,\omega^0_{(0)I} = c_{1(0)} - \frac{1}{R}\,c_{0(0)}, \tag{4.42}$$

$$\omega^{1y}_{(0)I} = c_{2(0)}, \quad \psi_{(0)I} = c_{3(0)} - \frac{1}{R}\,\omega^{1y}_{(0)I} = c_{3(0)} - \frac{1}{R}\,c_{2(0)},$$

$$\eta^0_{(0)I} = \theta^0_{(0)I} = \omega^{1\lambda}_{(0)I} = 0,$$

$$\omega^0_{(1)I} = c_{0(1)} + \frac{G_2^2}{2RG_I}\left[\frac{G_I^2 + G_{II}^2 + G_2^2}{G_I^2 - G_{II}^2}\left(c_{1(0)} - \frac{G_I^2 + G_{II}^2}{RG_2^2}\,c_{0(0)}\right) - c_{1(0)}\right]s,$$

$$\eta^0_{(1)I} = \frac{G_I G_2^2}{G_I^2 - G_{II}^2}\left[c_{1(0)} - \frac{G_I^2 + G_{II}^2}{G_2^2 R}\left(c_{0(0)} - \frac{a_x I_x}{I_p^C}\,c_{2(0)}\right)\right],$$

$$\theta^0_{(1)I} = \frac{G_I G_2^2}{G_I^2 - G_{II}^2}\left[1 - \frac{(G_I^2 + G_{II}^2)}{G_2^2}\,\frac{a_y I_x}{RI_p^C}\right]\omega^{1y}_{(0)I}, \quad \omega^{1\lambda}_{(1)I} = \frac{G_I(G_I^2 + G_{II}^2)}{G_I^2 - G_{II}^2}\,\frac{I_x}{RI_p^C}\,\omega^{1y}_{(0)I},$$

$$\omega^{1x}_{(1)I} = c_{1(1)} - \frac{1}{R}\,\omega^0_{(1)I} + \frac{G_2^2}{R^2 G_I}\,c_{1(0)}\,s,$$

$$\omega^{1y}_{(1)I} = c_{2(1)} + \frac{G_2^2}{R^2 G_I}\left\{1 - \frac{(G_I^2 + G_{II}^2 + G_2^2)(G_I^2 + G_{II}^2)}{2(G_I^2 - G_{II}^2)G_2^2}\,\frac{I_x}{I_p^C}\right\}c_{2(0)}\,s,$$

$$\psi_{(1)I} = c_{3(1)} - \frac{1}{R}\,\omega^{1y}_{(1)I} + \frac{3}{2}\,\frac{G_2^2}{R^2 G_I}c_{3(0)}\,s,$$

$$\eta^0_{(2)I} = \frac{G_I G_2^2}{G_I^2 - G_{II}^2}\left\{\omega^{1x}_{(1)I} + \frac{1}{R}\,\omega^0_{(1)I} - \frac{G_I^2 + G_{II}^2}{RG_2^2}\left(\omega^0_{(1)I} - \frac{a_x I_x}{I_p^C}\,\omega^{1y}_{(1)I}\right)\right\},$$

$$\theta^0_{(2)I} = \frac{G_I G_2^2}{G_I^2 - G_{II}^2}\left[1 - \frac{(G_I^2 + G_{II}^2)}{G_2^2}\,\frac{a_y I_x}{RI_p^C}\right]\omega^{1y}_{(1)I}, \quad \omega^{1\lambda}_{(2)I} = \frac{G_I(G_I^2 + G_{II}^2)}{G_I^2 - G_{II}^2}\,\frac{I_x}{RI_p^C}\,\omega^{1y}_{(1)I},$$

$$\begin{aligned}\omega^0_{(2)I} = {} & c_{0(2)} + \frac{G_2^2}{2RG_I}\left[\frac{G_I^2 + G_{II}^2 + G_2^2}{G_I^2 - G_{II}^2}\left(c_{1(1)} - \frac{G_I^2 + G_{II}^2}{RG_2^2}\,c_{0(1)}\right) - c_{1(1)}\right]s \\ & + \frac{G_2^4}{2R^3 G_I^2}\left\{\frac{G_I^2 + G_{II}^2 + G_2^2}{G_I^2 - G_{II}^2}\left[c_{1(0)} - \frac{G_I^2 + G_{II}^2}{2G_2^2}\left(\frac{G_I^2 + G_{II}^2 + G_2^2}{G_I^2 - G_{II}^2}\right.\right.\right. \\ & \left.\left.\left.\times\left(c_{1(0)} - \frac{G_I^2 + G_{II}^2}{RG_2^2}\,c_{0(0)}\right) - c_{1(0)}\right)\right] - c_{1(0)}\right\}\frac{s^2}{2},\end{aligned}$$

$$\omega^{1x}_{(2)I} = c_{1(2)} - \frac{1}{R}\,\omega^0_{(2)I} + \frac{G_2^2}{R^2 G_I}\,c_{1(1)}\,s + \frac{G_2^4}{R^4 G_I^2}\,c_{1(0)}\,\frac{s^2}{2},$$

$$\omega^{1y}_{(2)I} = c_{2(2)} + \frac{G_2^2}{R^2 G_I}\left(1 - \frac{(G_I^2 + G_{II}^2 + G_2^2)(G_I^2 + G_{II}^2)}{2(G_I^2 - G_{II}^2)G_2^2}\,\frac{I_x}{I_p^C}\right) c_{2(1)}\ s$$
$$+ \frac{G_2^4}{R^4 G_I^2}\left(1 - \frac{(G_I^2 + G_{II}^2 + G_2^2)(G_I^2 + G_{II}^2)}{2(G_I^2 - G_{II}^2)G_2^2}\,\frac{I_x}{I_p^C}\right)^2 c_{2(0)}\ \frac{s^2}{2},$$

$$\psi_{(2)I} = c_{3(2)} - \frac{1}{R}\,\omega^{1y}_{(2)I} + \frac{G_2^2}{R^2 G_I}\left(\frac{3}{2}\ c_{3(1)} - G_2^2 c_{1(0)}\right)s + \frac{9}{4}\ \frac{G_2^4}{R^4 G_I^2}\, c_{3(0)}\ \frac{s^2}{2}$$
$$+ \frac{G_2^2}{2RG_I}\left[\frac{G_I^2 + G_{II}^2 + G_2^2}{G_I^2 - G_{II}^2}\left(c_{1(0)} - \frac{G_I^2 + G_{II}^2}{RG_2^2}\, c_{0(0)}\right) - c_{1(0)}\right]s$$
$$- \frac{G_2^2(G_I^2 + G_{II}^2)}{R^3 G_I}\left(1 - \frac{(G_I^2 + G_{II}^2 + G_2^2)(G_I^2 + G_{II}^2)}{2(G_I^2 - G_{II}^2)G_2^2}\,\frac{I_x}{I_p^C}\right)\frac{a_x I_x}{I_p^C}\, c_{2(0)}\ s,$$

and for the quasi-transverse wave

$$\omega^{1\lambda}_{(0)II} = k^0_{(0)}, \quad \omega^{1x}_{(0)II} = \omega^{1y}_{(0)II} = \omega^0_{(0)II} = \psi_{(0)II} = 0, \tag{4.43}$$

$$\theta^0_{(0)II} = h^0_{(0)} - a_y \omega^{1\lambda}_{(0)II} = h^0_{(0)} - a_y k^0_{(0)}, \quad \eta^0_{(0)II} = g^0_{(0)} + a_x \omega^{1\lambda}_{(0)II} = g^0_{(0)} + a_x k^0_{(0)},$$

$$\omega^0_{(1)II} = \frac{G_{II}(G_I^2 + G_{II}^2 + G_2^2)}{R(G_2^2 - G_1^2)}\, g^0_{(0)}, \quad \psi_{(1)II} = \frac{G_{II}(G_I^2 + G_{II}^2 + G_2^2)}{R^2(G_2^2 - G_1^2)}\, k^0_{(0)},$$

$$\omega^{1x}_{(1)II} = -\frac{G_{II}(G_I^2 + G_{II}^2 + G_2^2)}{R^2(G_2^2 - G_1^2)}\, g^0_{(0)}, \quad \omega^{1y}_{(1)II} = -\frac{G_{II}(G_I^2 + G_{II}^2 + G_2^2)}{R(G_2^2 - G_1^2)}\, k^0_{(0)},$$

$$\omega^{1\lambda}_{(1)II} = k^0_{(1)} + \frac{1}{2R^2 G_{II} I_p^C}\left[G_1^2\left(a_x^2 F - I_x\right)k^0_{(0)} - \frac{\sigma^0_{\lambda\lambda}}{\rho}\left(I_y + 2a_y RF + a_x^2 F\right)k^0_{(0)}\right.$$
$$\left. - G_1^2 a_x F g^0_{(0)} - \frac{(G_I^2 + G_{II}^2)(G_I^2 + G_{II}^2 + G_2^2)}{G_2^2 - G_1^2}\, I_x k_{(0)}\right]s,$$

$$\theta^0_{(1)II} = h^0_{(1)} - a_y \omega^{1\lambda}_{(1)II} - \frac{1}{2RG_{II}}\left[\frac{G_2^2(G_I^2 + G_{II}^2 + G_2^2)}{G_2^2 - G_1^2} - \frac{\sigma^0_{\lambda\lambda}}{\rho}\right]k^0_{(0)}\ s,$$

$$\eta^0_{(1)II} = g^0_{(1)} + a_x \omega^{1\lambda}_{(1)II} - \frac{1}{2R^2 G_{II}}\left[\frac{(G_I^2 + G_{II}^2)(G_I^2 + G_{II}^2 + G_2^2)}{G_2^2 - G_1^2}\, g^0_{(0)}\right.$$
$$\left. + G_1^2 g^0_{(0)} + \frac{\sigma^0_{\lambda\lambda}}{\rho}\left(g^0_{(0)} + a_x k^0_{(0)}\right)\right]s,$$

$$\omega^0_{(2)II} = -\frac{G_{II}(G_I^2 + G_{II}^2 + G_2^2)}{R(G_2^2 - G_1^2)}\left(a_x \omega^{1\lambda}_{(1)II} - \eta^0_{(1)II}\right), \quad \omega^{1x}_{(2)II} = -\frac{1}{R}\,\omega^0_{(2)II},$$

$$\omega^{1y}_{(2)II} = -\frac{G_{II}(G_I^2 + G_{II}^2 + G_2^2)}{R(G_2^2 - G_1^2)}\,\omega^{1\lambda}_{(1)II}\,, \quad \psi_{(2)II} = -\frac{1}{R}\,\omega^{1y}_{(2)II},$$

$$\begin{aligned}
\omega^{1\lambda}_{(2)II} = k^0_{(2)} &+ \frac{1}{2R^2G_{II}I_p^C}\Bigg[G_1^2\left(a_x^2F - I_x\right)k^0_{(1)} - \frac{\sigma^0_{\lambda\lambda}}{\rho}\left(I_y + 2a_yRF + a_x^2F\right)k^0_{(1)} \\
&- G_1^2a_xFg^0_{(1)} - \frac{(G_I^2 + G_{II}^2)(G_I^2 + G_{II}^2 + G_2^2)}{G_2^2 - G_1^2}\,I_xk^0_{(1)}\Bigg]s \\
&+ \frac{1}{\left(2R^2G_{II}I_p^C\right)^2}\Bigg\{\Bigg[G_1^2\left(a_x^2F - I_x\right) - \frac{\sigma^0_{\lambda\lambda}}{\rho}\left(I_y + 2a_yRF + a_x^2F\right) \\
&- \frac{(G_I^2 + G_{II}^2)(G_I^2 + G_{II}^2 + G_2^2)}{G_2^2 - G_1^2}\,I_x\Bigg]^2 k^0_{(0)} - G_1^2a_xF\Bigg[G_1^2\left(a_x^2F - I_x\right) \\
&- \frac{\sigma^0_{\lambda\lambda}}{\rho}\left(I_y + 2a_yRF + a_x^2F\right) - \frac{(G_I^2 + G_{II}^2)(G_I^2 + G_{II}^2 + G_2^2)}{G_2^2 - G_1^2}\,I_x\Bigg]g^0_{(0)}\Bigg\}\frac{s^2}{2} \\
&+ \frac{G_1^2a_xF}{4R^4G_{II}^2I_p^C}\Bigg[\frac{(G_I^2 + G_{II}^2)(G_I^2 + G_{II}^2 + G_2^2)}{G_2^2 - G_1^2}\,g^0_{(0)} + G_1^2g^0_{(0)} \\
&+ \frac{\sigma^0_{\lambda\lambda}}{\rho}\left(g^0_{(0)} + a_xk^0_{(0)}\right)\Bigg]\frac{s^2}{2},
\end{aligned}$$

$$\begin{aligned}
\theta^0_{(2)II} = h^0_{(2)} - a_y\omega^{1\lambda}_{(2)II} &- \frac{1}{2RG_{II}}\left[\frac{G_2^2(G_I^2 + G_{II}^2 + G_2^2)}{G_2^2 - G_1^2} - \frac{\sigma^0_{\lambda\lambda}}{\rho}\right]k^0_{(1)}\;s \\
&- \frac{1}{4R^3G_{II}^2I_p^C}\left[\frac{G_2^2(G_I^2 + G_{II}^2 + G_2^2)}{G_2^2 - G_1^2} - \frac{\sigma^0_{\lambda\lambda}}{\rho}\right]\Bigg[G_1^2\left(a_x^2F - I_x\right)k^0_{(0)} - G_1^2a_xFg^0_{(0)} \\
&- \frac{\sigma^0_{\lambda\lambda}}{\rho}\left(I_y + 2a_yRF + a_x^2F\right)k^0_{(0)} - \frac{(G_I^2 + G_{II}^2)(G_I^2 + G_{II}^2 + G_2^2)}{G_2^2 - G_1^2}\,I_xk^0_{(0)}\Bigg]\frac{s^2}{2},
\end{aligned}$$

$$\begin{aligned}
\eta^0_{(2)II} = g^0_{(2)} + a_x\omega^{1\lambda}_{(2)II} &- \frac{1}{2R^2G_{II}}\Bigg[\frac{(G_I^2 + G_{II}^2)(G_I^2 + G_{II}^2 + G_2^2)}{G_2^2 - G_1^2}\,g^0_{(1)} + G_1^2g^0_{(1)} \\
&+ \frac{\sigma^0_{\lambda\lambda}}{\rho}\left(g^0_{(1)} + a_xk^0_{(1)}\right)\Bigg]s - \frac{1}{4R^4G_{II}^2}\Bigg\{\left[\frac{(G_I^2 + G_{II}^2)(G_I^2 + G_{II}^2 + G_2^2)}{G_2^2 - G_1^2} + G_1^2\right]^2 g^0_{(0)} \\
&- \left(\frac{\sigma^0_{\lambda\lambda}}{\rho}\right)^2\left(g^0_{(0)} + a_xk^0_{(0)}\right) + a_x\left(\frac{(G_I^2 + G_{II}^2)(G_I^2 + G_{II}^2 + G_2^2)}{G_2^2 - G_1^2} + G_1^2\right)k^0_{(0)}\Bigg\}\frac{s^2}{2}.
\end{aligned}$$

Substituting the found ray series coefficients (4.42) and (4.43) in (4.16), we can write the three-term ray expansions for the desired fields

$$\begin{aligned}\omega^0 = \omega^0_{(0)I} &+ \left\{\left(1-\frac{a_0}{G_I}\right)\omega^0_{(1)I} + \left(1-\frac{a_0}{G_{II}}\right)\omega^0_{(1)II}\right\}(t-t^*)\\ &+ \left\{\frac{1}{2}\left(1-\frac{a_0}{G_I}\right)^2\omega^0_{(2)I} + \frac{1}{2}\left(1-\frac{a_0}{G_{II}}\right)^2\omega^0_{(2)II}\right.\\ &\left. -\frac{a_1}{G_I}\,\omega^0_{(1)I} - \frac{a_1}{G_{II}}\,\omega^0_{(1)II}\right\}(t-t^*)^2,\end{aligned} \tag{4.44}$$

$$\begin{aligned}\omega^{1x} = \omega^{1x}_{(0)I} &+ \left\{\left(1-\frac{a_0}{G_I}\right)\omega^{1x}_{(1)I} + \left(1-\frac{a_0}{G_{II}}\right)\omega^{1x}_{(1)II}\right\}(t-t^*)\\ &+ \left\{\frac{1}{2}\left(1-\frac{a_0}{G_I}\right)^2\omega^{1x}_{(2)I} + \frac{1}{2}\left(1-\frac{a_0}{G_{II}}\right)^2\omega^{1x}_{(2)II}\right.\\ &\left. -\frac{a_1}{G_I}\,\omega^{1x}_{(1)I} - \frac{a_1}{G_{II}}\,\omega^{1x}_{(1)II}\right\}(t-t^*)^2,\end{aligned} \tag{4.45}$$

$$\begin{aligned}\omega^{1y} = \omega^{1y}_{(0)I} &+ \left\{\left(1-\frac{a_0}{G_I}\right)\omega^{1y}_{(1)I} + \left(1-\frac{a_0}{G_{II}}\right)\omega^{1y}_{(1)II}\right\}(t-t^*)\\ &+ \left\{\frac{1}{2}\left(1-\frac{a_0}{G_I}\right)^2\omega^{1y}_{(2)I} + \frac{1}{2}\left(1-\frac{a_0}{G_{II}}\right)^2\omega^{1y}_{(2)II}\right.\\ &\left. -\frac{a_1}{G_I}\,\omega^{1y}_{(1)I} - \frac{a_1}{G_{II}}\,\omega^{1y}_{(1)II}\right\}(t-t^*)^2,\end{aligned} \tag{4.46}$$

$$\begin{aligned}\psi = \psi_{(0)I} &+ \left\{\left(1-\frac{a_0}{G_I}\right)\psi_{(1)I} + \left(1-\frac{a_0}{G_{II}}\right)\psi_{(1)II}\right\}(t-t^*)\\ &+ \left\{\frac{1}{2}\left(1-\frac{a_0}{G_I}\right)^2\psi_{(2)I} + \frac{1}{2}\left(1-\frac{a_0}{G_{II}}\right)^2\psi_{(2)II}\right.\\ &\left. -\frac{a_1}{G_I}\,\psi_{(1)I} - \frac{a_1}{G_{II}}\,\psi_{(1)II}\right\}(t-t^*)^2,\end{aligned} \tag{4.47}$$

$$\begin{aligned}\eta^0 = \eta^0_{(0)II} &+ \left\{\left(1-\frac{a_0}{G_I}\right)\eta^0_{(1)I} + \left(1-\frac{a_0}{G_{II}}\right)\eta^0_{(1)II}\right\}(t-t^*)\\ &+ \left\{\frac{1}{2}\left(1-\frac{a_0}{G_I}\right)^2\eta^0_{(2)I} + \frac{1}{2}\left(1-\frac{a_0}{G_{II}}\right)^2\eta^0_{(2)II}\right.\\ &\left. -\frac{a_1}{G_I}\,\eta^0_{(1)I} - \frac{a_1}{G_{II}}\,\eta^0_{(1)II}\right\}(t-t^*)^2,\end{aligned} \tag{4.48}$$

$$\theta^0 = \theta^0_{(0)II} + \left\{ \left(1 - \frac{a_0}{G_I}\right) \theta^0_{(1)I} + \left(1 - \frac{a_0}{G_{II}}\right) \theta^0_{(1)II} \right\} (t - t^*)$$
$$+ \left\{ \frac{1}{2}\left(1 - \frac{a_0}{G_I}\right)^2 \theta^0_{(2)I} + \frac{1}{2}\left(1 - \frac{a_0}{G_{II}}\right)^2 \theta^0_{(2)II} \right.$$
$$\left. - \frac{a_1}{G_I}\, \theta^0_{(1)I} - \frac{a_1}{G_{II}}\, \theta^0_{(1)II} \right\} (t - t^*)^2, \tag{4.49}$$

$$\omega^{1\lambda} = \omega^{1\lambda}_{(0)II} + \left\{ \left(1 - \frac{a_0}{G_I}\right) \omega^{1\lambda}_{(1)I} + \left(1 - \frac{a_0}{G_{II}}\right) \omega^{1\lambda}_{(1)II} \right\} (t - t^*)$$
$$+ \left\{ \frac{1}{2}\left(1 - \frac{a_0}{G_I}\right)^2 \omega^{1\lambda}_{(2)I} + \frac{1}{2}\left(1 - \frac{a_0}{G_{II}}\right)^2 \omega^{1\lambda}_{(2)II} \right.$$
$$\left. - \frac{a_1}{G_I}\, \omega^{1\lambda}_{(1)I} - \frac{a_1}{G_{II}}\, \omega^{1\lambda}_{(1)II} \right\} (t - t^*)^2, \tag{4.50}$$

where the discontinuities of all values are taken at $s = a - a^*$.

Substituting relationship for $Q_{y\lambda(k)}$ defined by (3.57), as well as (4.9), (4.10) and (4.44)–(4.50) with due account for (4.17), (4.18) and (4.40)–(4.43) in Eqs. 4.33, 4.35 and 4.37–4.39, and equating the coefficients at equal powers of $t - t^*$ yields

$$k^0_{(0)} = h^0_{(0)} = c_{1(0)} = c_{2(0)} = c_{3(0)} = 0, \tag{4.51}$$

$$c_{0(0)} = \frac{1}{2}\, V_0 R'^{1/2} \alpha^{*-1/2}\,, \quad g^0_{(0)} = -\frac{k\alpha^{*1/2}}{\pi\rho_0 G_0 R'},$$

$$2\alpha_2 = -\frac{1}{2}\frac{k}{\pi\rho_0 R'\alpha^{*1/2}}\left(\frac{V_0}{G_0} + \frac{\pi\rho_0 R'^{1/2}\alpha^{*3/2}}{\rho F} + \frac{2G_2^2\alpha^{*1/2}}{G_0 G_{II} R'^{1/2}}\right) < 0,$$

$$k^0_{(1)} = h^0_{(1)} = c_{1(1)} = c_{2(1)} = c_{3(1)} = 0,$$

$$c_{0(1)} = \left(1 - \frac{a_0}{G_{II}}\right)\left(1 - \frac{a_0}{G_I}\right)^{-1} \frac{G_{II}(G_I^2 + G_{II}^2 + G_2^2)}{R(G_I^2 - G_{II}^2)}\, g^0_{(0)}$$
$$+ R'^{1/2}\alpha^{*1/2}\left(\alpha_2\alpha^{*-1} - \frac{1}{4}\, V_0^2\alpha^{*-2}\right)\left(1 - \frac{a_0}{G_I}\right)^{-1},$$

$$g^0_{(1)} = \left(1 - \frac{a_0}{G_I}\right)\left(1 - \frac{a_0}{G_{II}}\right)^{-1} \frac{G_I(G_I^2 + G_{II}^2)}{R(G_I^2 - G_{II}^2)}\, c_{0(0)}$$
$$+ k\left(\frac{\alpha^*}{2\rho F R'^{1/2}} + \frac{G_2^2}{\pi\rho_0 G_0 G_{II} R'^{3/2}}\right)\left(1 - \frac{a_0}{G_{II}}\right)^{-1},$$

$$3\alpha_3 = -\frac{1}{4}\,\frac{k}{\pi\rho_0 G_0 R' \alpha^{*1/2}}\left(2\alpha_2 - \frac{V_0^2}{2\alpha^*}\right) + \frac{1}{4}\,\frac{kV_0}{R'^{1/2}}\left(\frac{G_2^2}{\pi\rho_0 G_0 G_{II} R' \alpha^*} - \frac{1}{\rho F}\right)$$
$$+\frac{1}{2}\,\frac{kG_2^2\alpha^{*1/2}}{G_{II}R'}\left(\frac{G_2^2}{\pi\rho_0 G_0 G_{II} R'\alpha^*} + \frac{1}{\rho F}\right)$$
$$+\frac{1}{4}\,\frac{V_0 G_2^2}{\alpha^*}\left(1 - \frac{a_0}{G_I}\right)\left(\frac{G_I}{G_{II}} - 1\right)\frac{G_I^2 + G_{II}^2}{R(G_I^2 - G_{II}^2)}.$$

Substituting the found arbitrary constants (4.51) in the ray series (4.44)–(4.49), we obtain the final expressions for the desired fields. Thus, for example, knowing the values α_2 and α_3 (4.51), it is possible to determine $\alpha(t)$ (4.40) and *a(t)* (4.17), and therefore to obtain the typical time-dependence of the contact force (4.9) within an accuracy of $(t - t^*)^3$, since α_2 is a negative value:

$$P(t - t^*) - P^* = k\left\{V_0(t - t^*) + \alpha_2(t - t^*)^2 + \alpha_3(t - t^*)^3\right\}^{3/2}, \tag{4.52}$$

where $P^* = P|_{t=t^*} = k\alpha^{*3/2}$.

Equating to zero the expression for the contact force (4.52), we obtain the approximate formula for the duration of contact of the impacting rod with the thin-walled beam of open section.

Note that the solution for a particular case of a straight thin-walled beam of open profile could be obtained by putting $R_t = R \to \infty$ and $R' = r_0$, as it follows from (4.10), in Eqs. 4.42–4.52.

4.2.3 Numerical Example

As an example, let us consider the impact of a steel rod with a rounded end upon a steel arch with a constant radius of curvature and zero torsion, the cross section of which represents a channel (Fig. 4.2). The dimensionless time $\tilde{t} - \tilde{t}^* = (t - t^*)\left(\frac{2}{5}\right)^{2/3} V_0^{1/3} k^{2/3} \left(\pi r_0^2 \rho G_0\right)^{-2/3}$ dependence of the dimensionless contact force $\widetilde{P} - \widetilde{P}^* = (P - P^*)\left(\pi r_0^2 \rho G_0 V_0\right)^{-1}$ is presented in Fig. 4.3 for different levels of the initial axial compression $\tilde{\sigma}_{\lambda\lambda} = \sigma_{\lambda\lambda}^0 (\rho G_2^2)^{-1}$. Reference to Fig. 4.3 shows that the increase in the initial axial compression results in the increase of both the maximal magnitude of the contact force and the duration of contact.

The curve of the $\tilde{\sigma}_{\lambda\lambda}$-dependence of the dimensionless initial velocity of impact $\widetilde{V}_0 = V_0 G_0^{-1}$, resulting in the local damage of the thin-walled open-section beam in the place of contact is shown in Fig. 4.4 at the given magnitude of the dimensionless yield limit $\tilde{\sigma}_y = \sigma_y (\rho_0 G_0^2)^{-1}$. From Fig. 4.4 it is evident that with the increase in the initial axial compression the initial velocity of impact, which may lead to the local damage of the structure, decreases.

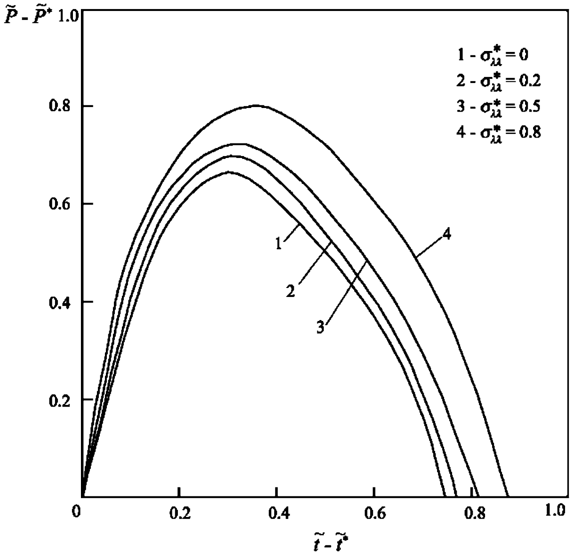

Fig. 4.3 The dimensionless time-dependence of the dimensionless contact force

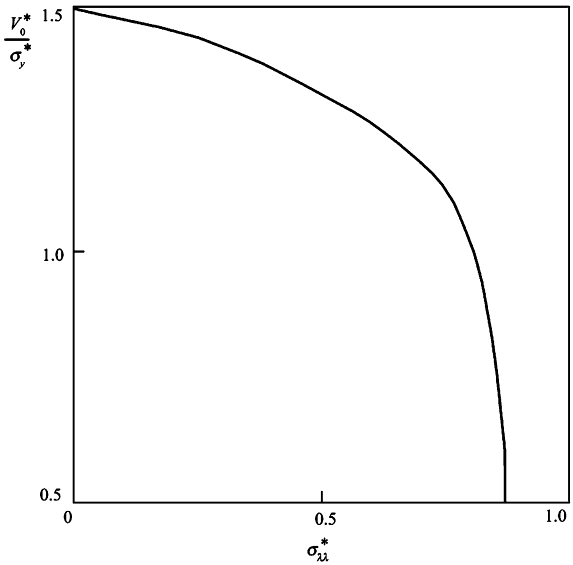

Fig. 4.4 The $\widetilde{\sigma}_{\lambda\lambda}$-dependence of the dimensionless initial velocity of impact in the case when the contact stress is equal to the yield limit

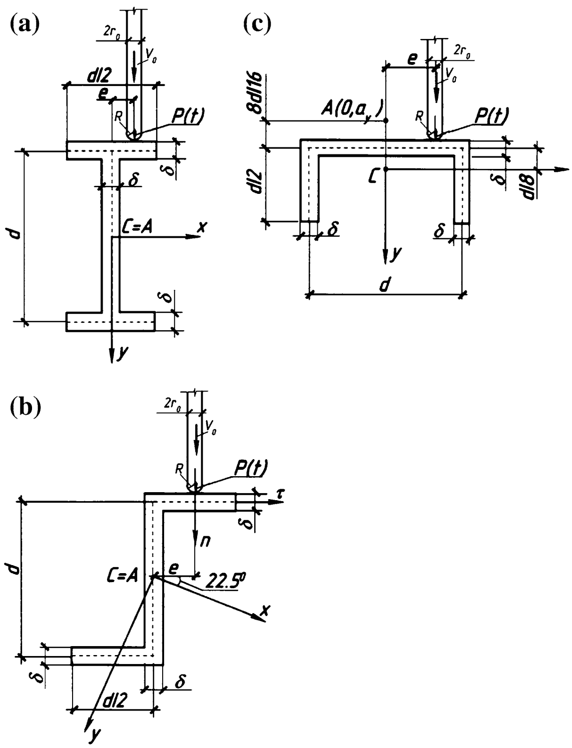

Fig. 4.5 The scheme of the shock interaction of a thin rod with a thin-walled beam of open profile: **a** I-beam, **b** *Z*-shape beam, and **c** channel beam

Appendix

Numerical Example of the Application of the Korbut-Lazarev Technical Theory

As an example of the application of the Korbut-Lazarev technical theory, let us consider the impact of a steel thin cylindrical rod of radius $r_0 = 0.5\,\text{cm}$ with one rounded end of the same radius upon steel thin-walled beams of open profile with different cross-section: I-beam (Fig. 4.5a), *Z*-shape beam (Fig. 4.5b), and channel beam (Fig. 4.5c), but with the equal cross-section area and with the following dimensions: $d = 20$ cm, and $\delta_s = \delta = 2\,\text{cm}$.

The following characteristics of the material have been adopted: $\rho = 7950\,\text{kg/m}^3$, $E = 210\,\text{GPa}$, $\mu = E/2.6$, and $v = 0.3$. The impact occurs at the distance e = 4 cm from the flexural center of the thin-walled beam with different initial velocities.

To determine the geometrical characteristics of the beam cross section with the cross-section area $F = 2d\delta = 0.008\,\text{m}^2$, we should construct the epures of

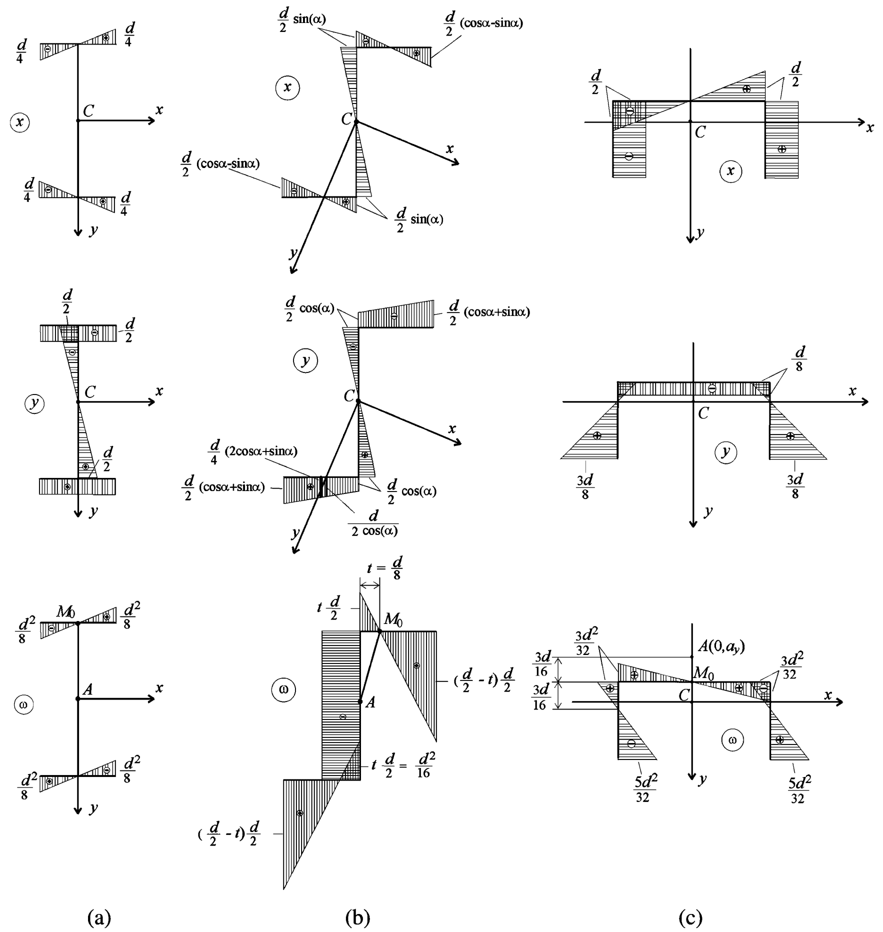

Fig. 4.6 Epures of the principal coordinates of the points lying along the medial line for **a** I-beam, **b** *Z*-shape beam, and **c** channel beam

the following values: the x and y principal coordinates of the points possessed by the medial line and the main sectorial coordinate ω, which are presented in Fig. 4.6a–c for an I-beam, a channel, and a Z-shaped beam, respectively.

With the help of these diagrams one can construct the epures of the axial static moments S_x and S_y and the sectorial static moment S_ω

$$S_x = \int_F ydF, \quad S_y = \int_F xdF, \quad S_\omega = \int_F \omega dF,$$

which are presented in Fig. 4.7a–c for an I-beam, a channel, and a Z-shaped beam, respectively, as well as one can calculate the centroidal moments of inertia

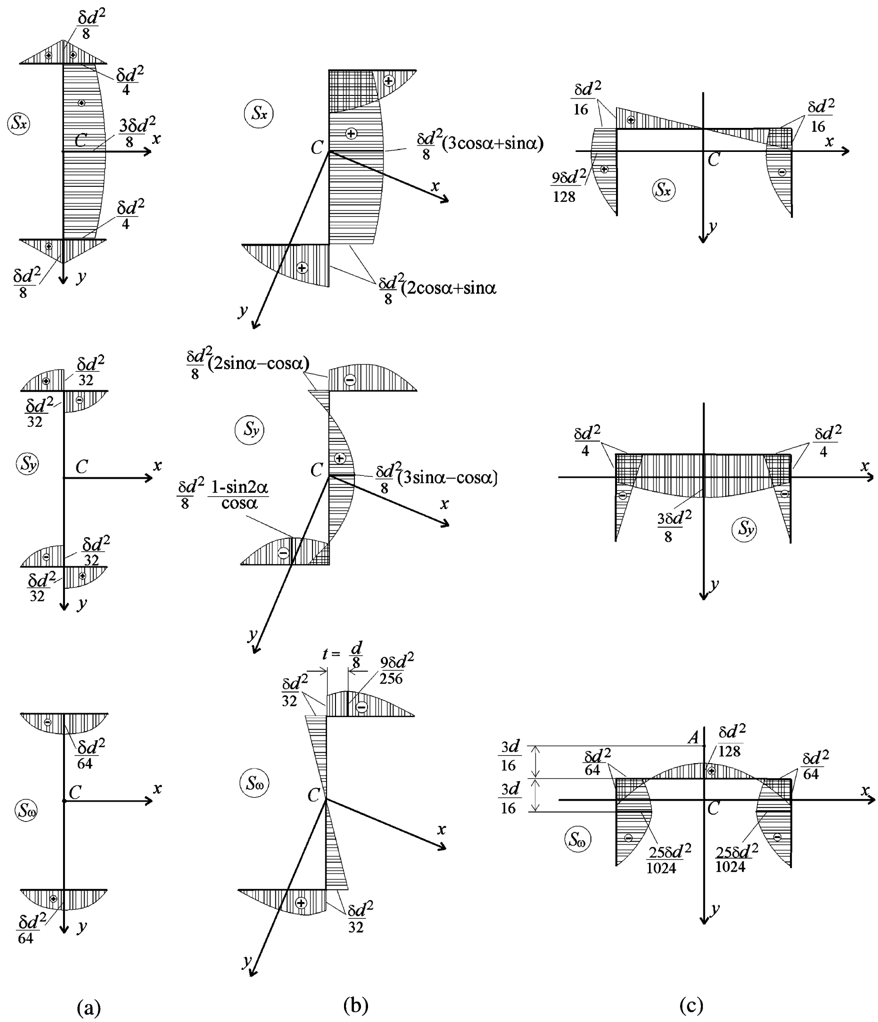

Fig. 4.7 Diagrams of the axial and sectorial static moments for **a** I-beam, **b** *Z*-shape beam, and **c** channel beam

$$I_x = \int_F y^2 dF, \quad I_y = \int_F x^2 dF,$$

the polar moment of inertia

$$I_p = I_z + F(a_x^2 + a_y^2),$$

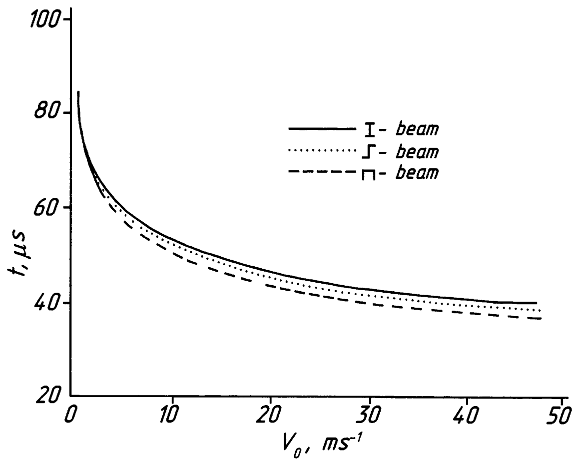

Fig. 4.8 The initial velocity dependence of the contact duration

the sectorial moment of inertia

$$I_\omega = \int_F \omega^2 \mathrm{d}F,$$

and the moment of inertia due to pure torsion

$$I_k = \frac{1}{3}\left(\delta^3 d + 2\frac{d}{2}\delta^3\right).$$

By virtue of the diagrams for the values S_x, S_y, and S_ω (Fig. 4.7) we calculate the magnitudes of the coefficients which take into consideration shears by the formulas (2.6). Now knowing all geometrical cross-sectional characteristics, we calculate the speeds of the waves propagating in the thin-walled beam by formulas (2.15)–(2.22). The geometrical characteristics for the beams under consideration and the wave speed data are presented in Table 4.1.

From the above example it is seen that in order to obtain the wave speed data according to the Korbut–Lazarev technical theory of thin-walled beams of open section, it is necessary to carry out rather cumbersome constructions of epures for calculating the corresponding shear coefficients (2.6), while the theory suggested by the authors in Chap. 3 is free from this disadvantage.

The curves describing the initial velocity of impact V_0 dependence of the contact duration are given in Fig. 4.8. Reference to Fig. 4.8 shows that the duration of contact decreases with increase in the initial velocity of impact. As it takes place, the duration of contact for the I-beam is greater than that for the Z-shaped beam, but the latter, in its turn, is greater than that for the channel beam at common magnitudes of the initial velocity of impact.

Table 4.1 Geometrical characteristics and wave velocities

Geometrical characteristics and wave velocities	The type of the thin-walled beam cross section		
	I-beam	*Z*-shape beam	Channel beam
F, m^2	0.008	0.008	0.008
a_x, m	0	0	0
a_y, m	0	0	−0.0665
I_x, m^4	5.33×10^{-5}	6.16×10^{-5}	8.33×10^{-6}
I_y, m^4	3.33×10^{-6}	5.06×10^{-6}	5.33×10^{-5}
I_p, m^4	5.667×10^{-5}	6.667×10^{-5}	9.292×10^{-5}
I_ω, m^6	3.33×10^{-8}	8.33×10^{-8}	5.833×10^{-8}
I_k, m^4	1.067×10^{-6}	1.067×10^{-6}	1.067×10^{-6}
k_x, m^{-2}	265.0	263.0	408.0
k_y, m^{-2}	300.0	257.5	300.0
k_ω, m^{-4}	3.0×10^4	4.08×10^4	3.184×10^4
$k_{x\omega}$, m^{-3}	0	0	0
$k_{y\omega}$, m^{-3}	0	0	964.25
k_{xy}, m^{-2}	0	196.925	0
G_1, m/s	2559.23	1974.24	1873.13
G_2, m/s	2057.48	4478.94	2674.24
G_3, m/s	2189.14	1666.66	1764.27
G_4, m/s	5139.56	5139.56	5139.56

Since the impact occurs with an eccentricity with respect to the flexural center in all considered cases, then the twisting motions dominate for the sections contacting with a striker. The inertia of area at the twisting motions is determined by the polar moment of inertia, which magnitudes for the three types of thin-walled beams are presented in Table 4.1. Reference to Table 4.1 shows that the channel beam and the I-beam have the largest and the smallest magnitudes of the polar moment of inertia, respectively, and the Z-shaped beam is sandwiched between them. It is obvious that during the impact of a sphere upon the channel beam the duration of contact will be the smallest, since this type of the section possesses the largest inertia under twisting, but the duration of contact of the striker with the I-beam will be the largest, since the I-beam has the smallest moment of inertia. In other words, the greater the magnitude of polar moment of inertia, the smaller the duration of contact at the same magnitude of the initial velocity of impact. However, the magnitude of the contact duration may not exceed the value calculated by the Hertz's contact theory for a semi-infinite medium at the same initial velocity of impact. Such a conclusion is supported by the experimental investigations reported in [7] and [10] for beams of continuous cross section. When $V_0 < 5\,\mathrm{m/s}$, the duration of contact practically coincides for all three thin-walled systems, since for small velocities the duration of contact is governed by the quasistatic process, which is common for all thin-walled systems under consideration.

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Chapter 5
Conclusion

Starting from the three-dimensional dynamic theory of linear elasticity and the Vlasov and Gol'denveizer theories, the dynamic theory of thin-walled beams of open section has been proposed with due account for the axial precompression. As this takes place, the derived hyperbolic system of recurrent equations together with the ray expansions allow one to describe the short-time processes, in particular, the processes of shock interaction.

The theory suggested in this book is free from new additional constants such as the shear coefficients in Timoshenko-like theories, which are dependent on the geometry of a beam, but it involves only elastic moduli and Poisson's coefficient, resulting in the fact that the velocity of the transient shear wave coincides with that of the transverse wave in a three-dimensional elastic media, in contrast to the Timoshenko-like theories, according to which several transverse waves propagate, as a rule, with the velocities dependent on the geometrical parameters of the beam. As for the longitudinal wave, then in both theories it propagates with the velocity of the longitudinal wave in a thin elastic rod.

The new theory proposed is of great engineering importance, since precisely curved members in modern bridges and architectural structures continue to predominate over the straight ones because of emphasis on aesthetics and transportation alignment restrictions in metropolitan areas. Thus, the increasing use of curved thin-walled beams in highway bridges, civil engineering and aircraft has resulted in considerable effort that should be directed toward developing accurate methods for analyzing the dynamic behaviour of such structures.

The analytical method for investigating the transient wave propagation in spatially curved thin-walled beams of generic open profile, which has been utilized in this paper, possesses unique features, because it allows one to show more emphatically and more informatively the whole process of wave propagation. It has been shown that all existing technical theories of thin-walled beams could not capture the general relationships of the transient shear-torsional waves, since each thin-walled beam possesses its own velocities, and thus, each time it is

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needed to study a concrete object with its concrete dimensions. As for numerical approaches, then neither of existing numerical methods enables one to understand the whole dynamics of transient wave propagation in such a complex mechanical structure as a thin-walled beam of general open profile which is spatially curved and twisted, and neither of numerical procedures could reveal new effects that have been discovered by the proposed analytical treatment.

The dynamic stability with respect to small perturbations, as well as the local damage of geometrically nonlinear elastic spatially curved open section beams with axial precompression have been analyzed. Transient waves, which are the surfaces of strong discontinuity and wherein the stress and strain fields experience discontinuities, have been used as small perturbations, in so doing the discontinuities have been considered to be of small values. Such waves could be initiated during low-velocity impacts upon thin-walled beams.

Thus, the following new results have been found. On the quasi-longitudinal-flexural-warping wave, all primary values which define the type of this wave are interconnected with each other, and they are expressed in terms of the different-order discontinuities in the longitudinal component of the velocity. This coupling is due to the presence of the rod's curvature and of the angle between two reference systems locating in the plane of the strong discontinuity for the case when the natural axes, the main normal and binormal, of the thin-walled beam with a spatially curved and twisted longitudinal axis do not coincide with the main central axes of its cross-section. To the authors' knowledge, the influence of this angle on the character of the quasi-longitudinal-flexural-warping wave has been studied for the first time. Moreover, it has been found that on this wave there exist 'admixed' secondary shear-torsional components of higher order than the primary longitudinal-flexural-warping components.

On the quasi-shear-torsional wave, the main values which define the type of this wave, are interconnected with each other, and they are expressed in terms of the different-order discontinuities in the angular velocity of rotational motion around spatially curved and twisted longitudinal axis of the thin-walled beam via the shear center coordinates which, in the general case of the beam's cross-section, are different from those of the center of gravity. It has been discovered that on this wave there also occur 'admixed' secondary flexural motions of the same order as the primarily motions, as well as secondary longitudinal and warping motions of the higher order than the primarily components. It has been shown that this coupling is supported by the characteristics of the thin-walled beam's longitudinal axis: its torsion, curvature and the angle between two reference systems in the case when the natural axes, the main normal and binormal, of the thin-walled beam with a spatially curved and twisted longitudinal axis do not coincide with the main central axes of its cross-section.

It is reported that after finding the discontinuities of the desired stress and velocity fields determined within an accuracy of arbitrary constants on the two waves of strong discontinuity, quasi-longitudinal and quasi-transverse, propagating in the thin-walled beam of open profile, the fields of the desired functions could be constructed also with an accuracy of the arbitrary constants utilizing the ray series, which are the power series with variable coefficients and which allow one to obtain the analytical solution behind the wave fronts of strong discontinuity.

The approach proposed permits to solve analytically for the first time, to the authors' knowledge, the problem of normal impact of a rod against an elastic thin-walled beam of open section making allowance for the beam's translatory and rotational motions, warping, rotary inertia, shear deformation, the local bearing due to the Hertz's theory, and the initial axial compression of the beam. The method of expansion along the rays behind the wave front is valid for short times after the wave front has passed. That is why it is well suited for solving the problems of shock interactions, since the duration of contact is very short, and thus a small number of terms in ray expansions is sufficient to achieve reasonable accuracy of the solution.

The example of using the ray expansions for analyzing the impact response of spatially curved thin-walled beams of open cross-section is demonstrated by solving the problem about the normal impact of an elastic hemispherical-nosed rod upon an elastic arch, representing itself a channel-beam curved along an arc of the circumference.

The theory constructed allows one to investigate the influence of the initial stresses on the dynamic fields and to answer a set of major questions, among them: (1) What magnitude should the initial velocity of impact take on at the given axial precompression of the thin-walled open section beam in order to produce its local damage at the place of shock interaction between the target and the impactor? (2) How does the level of the initial axial compression influence the maximal contact force and the duration of contact? The graphs presented in Figs. 4.3 and 4.4 are the answers to the above questions.

Chapter 6
Peculiarities of Transient Wave Propagation in Thin-Walled Beams of Open Section

We will interpret *a transient wave (a surface of strong discontinuity) in a thin-walled beam of open section* as a plane wave taking a pattern of the beam's profile and propagating perpendicular to its cetroidal axis. Such a wave could be named as a 'beam-like wave'. During the transition through the wave surface the displacement fields remain continuous, while the fields of stresses, deformations and velocities of particle's displacements are discontinuous across this surface.

For the fields, which experience the discontinuities, there exist the geometrical, kinematic and dynamic conditions of compatibility [1].

6.1 Conditions of Compatibility

On the wave surface of strong discontinuity, the following conditions of compatibility for the desired values could be written according to Thomas [1]: the kinematic condition of compatibility

$$\frac{\delta[f]}{\delta t} = \left[\frac{\partial f}{\partial t}\right] + \left[\frac{\mathrm{d}f}{\mathrm{d}n}\right] G, \tag{6.1}$$

and the geometric condition of compatibility

$$\left[\frac{\partial f}{\partial x_j}\right] = \left[\frac{\mathrm{d}f}{\mathrm{d}n}\right] \lambda_j + [f]_{,\alpha} g^{\alpha\beta} x_{j,\beta}, \tag{6.2}$$

where $[f] = f^+ - f^-$ is the jump of the value f, the signs "+" and "−" refer to the magnitudes of f calculated ahead and behind of the wave front, respectively, $\delta/\delta t$ is the Thomas δ-derivative, $\mathrm{d}/\mathrm{d}n$ is the derivative with respect to the normal to the wave surface, G is the normal velocity of the wave surface, λ_j are the components of the unit vector normal to the wave surface, $g_{\alpha,\beta} = x_{i,\alpha} x_{i,\beta}$ are the covariant

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components of the metric tensor of the wave surface, $x_{i,\alpha} = \partial x_i/\partial u^\alpha$, u^α $\alpha = 1, 2$ are the coordinates on the wave surface, $g^{\alpha\beta}$ are the contravariant components of the metric tensor of the wave surface, in so doing $g^{\alpha\gamma} g_{\beta\gamma} = \delta^\alpha_\beta$, where δ^α_β is the Kroneker's symbol, and $[f]_{,\alpha}$ is the covariant derivative of the discontinuity in the desired function with respect to the surface coordinates u^α.

Formula (6.1) is the definition of the Thomas δ-derivative written in jumps. The validity of (6.2) can be shown by sequential multiplication of its right- and left-hand sides by λ_j and $x_{j,\gamma}$ at a time and considering that $\lambda_j x_{j,\gamma} = 0$.

Excluding the value $[\mathrm{d}f/\mathrm{d}n]$ from (6.1) and (6.2) yields

$$\left[\frac{\partial f}{\partial x_j}\right] = -\left[\frac{\partial f}{\partial t}\right] \lambda_j G^{-1} + \frac{\delta[f]}{\delta t} \lambda_j G^{-1} + [f]_{,\alpha} g^{\alpha\beta} x_{j,\beta}. \tag{6.3}$$

Let us chose as the surface coordinates u^1 and u^2, respectively, x and y (the main central axes of the cross section, see Fig. 3.1), and consider that $g_{11} = g_{22} = 1$ and

$$\frac{\delta[f]}{\delta t} = \frac{\mathrm{d}[f]}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}t} = \frac{\mathrm{d}[f]}{\mathrm{d}s} G,$$

where s is the arc length measured along the ray.

Considering the above said, let us rewrite formula (6.3) in the form

$$\left[\frac{\partial f}{\partial x_j}\right] = -\left[\frac{\partial f}{\partial t}\right] \lambda_j G^{-1} + \frac{\mathrm{d}[f]}{\mathrm{d}s} \lambda_j + \frac{\partial[f]}{\partial x} k_j + \frac{\partial[f]}{\partial y} s_j. \tag{6.4}$$

Substituting the function f in (6.4) by the k-order time-derivative of the desired function, we obtain the required conditions of compatibility.

6.2 Conditions of Compatibility for Engineering Theories

Let us prove the validity of formula (2.1) by the method of mathematical induction. At $n = 1$, the known formula is obtained, which is the basis for the definition of the Thomas δ-derivative [1],

$$G\frac{\partial Z}{\partial z} = -Z_{,(1)} + \frac{\delta Z}{\delta t}. \tag{6.5}$$

Note that the kinematic condition of compatibility (6.5) is valid both before and after the wave front, that is why it is written without jumps.

Now we suppose that formula (2.1) is valid for $n - 1$, i.e.

$$G^{n-1}\frac{\partial^{n-1} Z}{\partial z^{n-1}} = \sum_{m=0}^{n-1} (-1)^m \frac{(n-1)!}{m!(n-1-m)!} \frac{\delta^{n-1-m} Z_{,(m)}}{\delta t^{n-1-m}}. \tag{6.6}$$

To prove the validity of (2.1), let us multiply (6.6) by G, differentiate over z, and apply formula (6.5). As a result we obtain

$$G^n \frac{\partial^n Z}{\partial z^n} = \sum_{m=0}^{n-1} (-1)^{m+1} \frac{(n-1)!}{m!(n-1-m)!} \frac{\delta^{n-1-m} Z_{,(m+1)}}{\delta t^{n-1-m}} + \sum_{m=0}^{n-1} (-1)^m \frac{(n-1)!}{m!(n-1-m)!} \frac{\delta^{n-m} Z_{,(m)}}{\delta t^{n-m}}. \tag{6.7}$$

In the first sum of (6.7), we substitute $m+1$ by m, in so doing its low limit becomes equal to unit, while the upper limit is equal to n.

Let us separate out the term at $m = n$ in the newly obtained sum and the term at $m = 0$ in the second sum of (6.7), and add together all remained sums. As a result, we obtain

$$G^n \frac{\partial^n Z}{\partial z^n} = (-1)^n Z_{,(n)} + \frac{\delta^n Z}{\delta t^n} + \sum_{m=1}^{n-1} (-1)^m \left[\frac{(n-1)!}{(n-m)!(m-1)!} + \frac{(n-1)!}{(n-1-m)!m!} \right] \frac{\delta^{n-m} Z_{,(m)}}{\delta t^{n-m}},$$

or

$$G^n \frac{\partial^n Z}{\partial z^n} = (-1)^n Z_{,(n)} + \frac{\delta^n Z}{\delta t^n} + \sum_{m=1}^{n-1} (-1)^m \frac{n!}{m!(n-m)!} \frac{\delta^{n-m} Z_{,(m)}}{\delta t^{n-m}}. \tag{6.8}$$

If we include the second term standing in the right-hand side of (6.8) into the sum, and express the value $(-1)^n Z_{,(n)}$, then we are led to relationship (2.1).

6.3 The Main Kinematic and Dynamic Characteristics of the Wave Surface

Now choosing the function u_i as the function f in formula (6.4) we have

$$[u_{i,j}] = -G^{-1}[v_i]\lambda_j + \left[\frac{\partial(u_i k_j)}{\partial x}\right] + \left[\frac{\partial(u_i s_j)}{\partial y}\right], \tag{6.9}$$

where u_i are the displacement vector components, $[u_{i,j}] = [\partial u_i/\partial x_j], x_j$ are the spatial rectangular Cartesian coordinates, x- and y- are the main axes of the cross section of the beam (Fig. 3.1), $[u_{i,(k)}] = [\partial^k u_i/\partial t^k]$, t is the time, $v_i = u_{i,(1)}$, λ_i, k_i, and s_i are the components of the unit vectors of the tangential to the centroid axis, and directed along the main axes, respectively, and Latin indices take on the values 1, 2, 3.

Writing the Hook's law for a three-dimensional medium in terms of discontinuities and using the condition of compatibility (6.9), we find

$$[\sigma_{ij}] = -G^{-1}\lambda[v_\lambda]\delta_{ij} - G^{-1}\mu\big([v_i]\lambda_j + [v_j]\lambda_i\big) + \lambda\big([u_{x,x}] + [u_{y,y}]\big)\delta_{ij}$$
$$+ \mu\left(\left[\frac{\partial(u_i k_j)}{\partial x}\right] + \left[\frac{\partial(u_j k_i)}{\partial x}\right] + \left[\frac{\partial(u_i s_j)}{\partial y}\right] + \left[\frac{\partial(u_j s_i)}{\partial y}\right]\right), \tag{6.10}$$

where $[v_\lambda] = [v_i]\lambda_i$,

$$[u_{x,x}] = \left[\frac{\partial(u_i k_i)}{\partial x}\right] = \left[\frac{\partial u_x}{\partial x}\right] = [\varepsilon_x], \quad [u_{y,y}] = \left[\frac{\partial(u_i s_i)}{\partial y}\right] = \left[\frac{\partial u_y}{\partial y}\right] = [\varepsilon_y],$$

λ and μ are Lame constants, and δ_{ij} is the Kroneker's symbol.

Multiplying relationship (6.10) from the right and from the left by $k_i k_j$ and $s_i s_j$ and considering equations

$$[\sigma_{xx}] = [\sigma_{ij}]k_i k_j = 0, \quad [\sigma_{yy}] = [\sigma_{ij}]s_i s_j = 0,$$

what corresponds to the assumption that the normal stresses on the cross-sections parallel to the middle surface could be neglected with respect to other stresses, we obtain

$$[\sigma_{xx}] = -G^{-1}\lambda[v_\lambda] + \lambda\big([u_{x,x}] + [u_{y,y}]\big) + 2\mu[u_{x,x}] = 0,$$

$$[\sigma_{yy}] = -G^{-1}\lambda[v_\lambda] + \lambda\big([u_{x,x}] + [u_{y,y}]\big) + 2\mu[u_{y,y}] = 0,$$

whence it follows that

$$[u_{x,x}] = [u_{y,y}] = \frac{\lambda}{2G(\lambda+\mu)}[v_\lambda],$$

or

$$[u_{x,x}] = [u_{y,y}] = \nu G^{-1}[v_\lambda], \tag{6.11}$$

since $\nu = \frac{\lambda}{2(\lambda+\mu)}$ is the Poisson's ratio.

Multiplying relationship (6.10) from the right and from the left by $\lambda_i\lambda_j$, we are led to the equation

$$[\sigma_{\lambda\lambda}] = [\sigma_{ij}]\lambda_i\lambda_j = -G^{-1}(\lambda + 2\mu)[v_\lambda] + 2\lambda[u_{x,x}]. \tag{6.12}$$

Substituting (6.11) in (6.12) and considering that $E = \frac{(3\lambda+2\mu)\mu}{\lambda+\mu}$ yields

$$[\sigma_{\lambda\lambda}] = -G^{-1}E[v_\lambda], \tag{6.13}$$

where E is the elastic modulus.

Alternatively, multiplying the dynamic condition of compatibility, i.e. the equations of motion rewritten in jumps,

$$[\sigma_{ij}]\lambda_j = -\rho G[v_i], \tag{6.14}$$

by λ_i, we obtain

$$[\sigma_{\lambda\lambda}] = -\rho G[v_\lambda], \tag{6.15}$$

where ρ is the density of the beam's material.

Eliminating the value $[\sigma_{\lambda\lambda}]$ from (6.13) and (6.15), we find the velocity of the quasi-longitudinal wave propagating in the thin-walled beam of open section

$$G_1 = \sqrt{\frac{E}{\rho}}. \tag{6.16}$$

Relationship (6.13) with due account for (6.16) takes the form

$$[\sigma_{\lambda\lambda}] = -\rho G_1[v_\lambda]. \tag{6.17}$$

Now multiplying (6.9) by $\lambda_i\lambda_j$ and considering that on the quasi-longitudinal wave $G = G_1$, we obtain

$$[v_\lambda] = -G_1[u_{\lambda,\lambda}]. \tag{6.18}$$

Substituting (6.18) in (6.11), we are led to the following equations valid on the quasi-longitudinal wave:

$$[u_{x,x}] = [u_{y,y}] = -\nu[u_{\lambda,\lambda}]. \tag{6.19}$$

Note that in the three-dimensional medium only one value, i.e. $[u_{\lambda,\lambda}]$, is nonzero on the quasi-longitudinal wave, while in the one-dimensional medium, where the 'beam-wave' propagates, on the quasi-longitudinal wave there are two nonvanishing values, namely, $[u_{x,x}]$ and $[u_{y,y}]$, resulting to the fact that, as distinct to the statics of the thin-walled beams of open section where the contour of the cross-section remains to be rigid during the process of its deformation, in the dynamic problems this contour experiences the deformation on the quasi-longitudinal wave.

It is interesting to emphasize that relationships (6.19), which are valid only on the front of the quasi-longitudinal wave propagating in the thin-walled beam of open profile, coincide by their form with the well-known formulas for simple beams of solid cross-section being under simple tension-compression, i.e.

$$u_{x,x} = u_{y,y} = -\nu u_{z,z},$$

which are valid for all cross sections of the simple beam with the longitudinal z-axis.

Multiplying (6.10) by $\lambda_i k_j$ and by $\lambda_i s_j$ and (6.14) by k_i and by s_i, respectively, we have

$$[\sigma_{\lambda x}] = [\sigma_{ij}]\lambda_i k_j = -\mu G^{-1}[v_x], \quad [\sigma_{\lambda y}] = [\sigma_{ij}]\lambda_i s_j = -\mu G^{-1}[v_y], \tag{6.20}$$

and

$$[\sigma_{\lambda x}] = -\rho G[v_x], \quad [\sigma_{\lambda y}] = -\rho G[v_y], \tag{6.21}$$

where $[v_x] = [v_i]k_i$ and $[v_y] = [v_i]s_i$.

Eliminating the values $[\sigma_{\lambda x}]$ and $[\sigma_{\lambda y}]$ from (6.20) and (6.21), we find the velocity of the quasi-transverse wave

$$G_2 = \sqrt{\frac{\mu}{\rho}}. \tag{6.22}$$

Considering (6.22), relationships (6.20) take the form

$$[\sigma_{\lambda x}] = -\rho G_2[v_x], \quad [\sigma_{\lambda y}] = -\rho G_2[v_y]. \tag{6.23}$$

Thus, we have just prove that the dynamic theory of thin-walled beams of open profile proposed in this book admits only two transient waves, quasi-longitudinal and quasi-transverse, propagating with the velocities (6.16) and (6.22), in so doing the velocity of the quasi-longitudinal wave is equal to that of longitudinal wave in a thin elastic rod of solid cross section, while the velocity of the quasi-transverse wave coincides with the velocity of the shear wave propagating in a three-dimensional elastic medium.

Reference

1. T.Y. Thomas, *Plastic flow and fracture in solids*, (Academic Press, New York, 1961)